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Quantum interferometry with and without an external phase reference

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We discuss the role of an external phase reference in quantum interferometry. We point out inconsistencies in the literature with regard to the use of the quantum Fisher information (QFI) in phase estimation interferometric schemes. We discuss the interferometric schemes with and without an external phase reference and show a proper way to use QFI in both situations.

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Laws of quantum mechanics impose fundamental bounds on measurement precisions of basic physical quantities such as position, momentum, energy, time, phase, etc. Theses bounds follow from the structure of the theory itself, which contrasts the situation encountered in classical physics where measurement uncertainties are due to factors that, in principle, may be eliminated by improving the quality of measurement procedures. One of the most important measurement techniques where such bounds have been analyzed is optical interferometry [1].

In a generic interferometric measurement using a Mach-Zehnder setup and classical light sources the precision of estimating the relative phase delay φ inside the interferometer is bounded by the so-called standard quantum limit (SQL) $\delta\varphi \geqslant 1/\sqrt{N}$, where N is average number of photon-counts. At the fundamental quantum level, the bound is a result of an independent probabilistic behavior of individual photons propagating through the interferometer.

Breaching the SQL requires the use of special nonclassical states of light where photons can no longer be regarded as independent. One of the first proposals in this direction was the idea to mix coherent light with the squeezed vacuum at the input beam splitter of the Mach-Zehnder interferometer [2]. Thanks to the reduced vacuum fluctuations in one of the quadratures of the squeezed state, it is possible to improve the precision beyond the SQL. This observation prompted the search for more fundamental bounds on achievable precision, which would be obeyed by all quantum states [3].

In general, looking for the optimal phase estimation protocols is difficult since one needs to optimize over the input state $|\psi_{\rm in}\rangle$ that is fed into the interferometer, the measurement $\{\Pi_n\}$ that is performed at the output, and the estimator $\varphi(n)$—a function that assigns a phase value to a given measurement outcome. One of the popular ways to obtain useful bounds in quantum metrology, without the need for cumbersome optimization, is to use the concept of the quantum Fisher information (QFI) [4] (for alternative approaches see, e.g., Ref. [5]).

The purpose of this paper is to give a proper interpretation to the bounds obtained via the QFI and point out conflicting approaches where seemingly equivalent physical models lead to different quantitative statements. We show that the source of the problem lies in the use of quantum states of light, which are coherent superpositions of different total photon number terms without properly taking into account the role of an external phase reference beam.

Let a and b be the anihilation operators of, respectively, upper and lower input modes of the interferometer. For the purpose of this paper, we consider the input state of the form $|\psi_{\rm in}\rangle = |r,\alpha\rangle$, where $|\alpha\rangle$ is the coherent state, $a|\alpha\rangle = \alpha|a\rangle$, while $|r\rangle = \exp[\frac{1}{2}r^*a^2 - \frac{1}{2}r(a^\dagger)^2]|0\rangle$ is the squeezed vacuum state with squeezing parameter r (see Fig. 1). After it has evolved through the beam-splitter with power transmission τ, and experienced the relative phase shift inside the interferometer U_φ, the state becomes $|\psi_\varphi\rangle = U_\varphi B_\tau|\psi_{\rm in}\rangle$, where $B_\tau = \exp[-i\,\mathrm{asin}(\sqrt{\tau})(a^\dagger b + ab^\dagger)]$, $U_\varphi = \exp[-i\varphi a^\dagger a]$. In a standard Mach-Zehnder setup, one interferes the two modes on another balanced beam-splitter and detects number of photon clicks, n_a and n_b, in the two output modes. In an idealized setup with no losses, perfect interferometer, and 100% detection efficiency, this leads to a phase-dependent probability distribution of clicks:

$$p(n_a,n_b|\varphi) = |\langle n_a,n_b|B_{1/2}U_\varphi B_\tau|\psi_{\rm in}\rangle|^2. \quad (1)$$

Instead of looking for the best possible estimator of the phase, which in general is a hard task, one can invoke the Cramer-Rao bound [6], which states that for k repetitions of an experiment and any locally unbiased estimator $\varphi(n_a,n_b)$ the uncertainty of estimation is bounded from below by

$$\delta\varphi \geqslant \frac{1}{\sqrt{kF}},\ F = \sum_{n_a,n_b} \frac{1}{p(n_a,n_b|\varphi)}\left[\frac{dp(n_a,n_b|\varphi)}{d\varphi}\right]^2, \quad (2)$$

where F is the Fisher information. Moreover, the bound can be saturated in the limit $k \to \infty$, by making use of the maximum likelihood estimator.

A priori it is not obvious that this type of measurement is the optimal way to extract phase information from the state $|\psi_\varphi\rangle$. The *quantum* Cramer-Rao bound [4] provides an answer to this problem and states that whatever the measurement chosen, the following bound on the estimation uncertainty holds:

$$\delta\varphi \geqslant \frac{1}{\sqrt{kF_Q}},\ F_Q = 4\left(\langle\psi'_\varphi|\psi'_\varphi\rangle - |\langle\psi'_\varphi|\psi_\varphi\rangle|^2\right), \quad (3)$$

where $|\psi'_\varphi\rangle = \frac{d|\psi_\varphi\rangle}{d\varphi}$, and F_Q is called the quantum Fisher information. F_Q depends neither on the measurement nor on the estimator and it is solely a function of the probing state, which makes it an easy-to-calculate quantity. Moreover, one can always find a measurement (that may depend on the true value φ) for which $F = F_Q$. In what follows, we will drop k for simplicity and use notation where $\delta\varphi \equiv 1/\sqrt{F_Q}$.

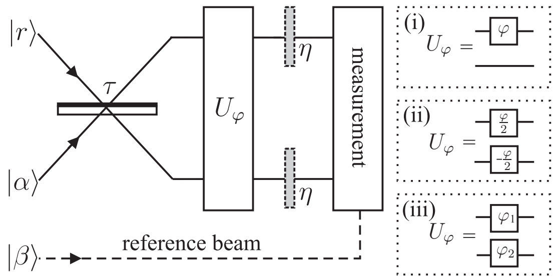

FIG. 1. An interferometric scheme with coherent and squeezed vacuum states interfered at a beam-splitter, with arbitrary quantum measurement potentially involving an additional reference beam. In general, the QFI bounds on the phase-estimation precision depend on the way the interferometer phase delay is modeled: (i) phase shift only in the upper arm, (ii) phase shift distributed symmetrically, (iii) phase shifts defined with respect to an additional reference beam.

A lot of work in quantum-enhanced interferometry has been based on utilizing the F_Q [7–9]. In a typical approach, one maximizes F_Q over a class of input states satisfying some constraint (e.g., total energy) and in this way finds the input states optimal for quantum interferometry.

Let us investigate the consequences of this approach in our setup. The input state $|\psi_{\rm in}\rangle = |\alpha,r\rangle$ has the mean number of photons equal to $\bar{n} = |\alpha|^2 + \sinh^2 r$. Having fixed $\bar{n}$, we look for optimal α and r and the transmission coefficient τ that maximize F_Q. If one follows this procedure rigorously, then the solution depends strongly on the way the phase shift between the beams is modeled inside the interferometer. As a simple illustration of this counterintuitive behavior, the relative phase shift φ may be modeled in, e.g., two ways depicted in Fig. 1 as (i) and (ii). These two cases correspond with $U_\varphi^{(i)} = \exp[-i\varphi a^\dagger a]$ and $U_\varphi^{(ii)} = \exp[-i\frac{\varphi}{2}a^\dagger a + i\frac{\varphi}{2}b^\dagger b]$, respectively. When plugged into Eq. (3), they yield

$$F_Q^{(i)} = 4\tau^2|\alpha|^2 + 2(1-\tau)^2\sinh^2(2r) + \mathfrak{F}, \tag{4}$$

$$F_Q^{(ii)} = (1-2\tau)^2\left[|\alpha|^2 + \tfrac{1}{2}\sinh^2(2r)\right] + \mathfrak{F}, \tag{5}$$

where $\mathfrak{F} = 4\tau(1-\tau)(|\alpha|^2 e^{2r} + \sinh^2 r)$, and in order to simplify the formulas, we have put the relative phase between the input beams to be $\pi/2$ ($r = |r|, \alpha = i|\alpha|$), which is the optimal choice for this and all the examples presented in this paper. The formulas are clearly different, which becomes evident when we set $\tau = 1/2$, $r = 0$, in which case $F_Q^{(i)} = 2|\alpha|^2$, $F_Q^{(ii)} = |\alpha|^2$.

To understand what lies behind this discrepancy, consider an even more exotic case of $\tau = 1$, $r = 0$. The coherent state is simply transmitted to the upper arm so there is no interferometer at all, yet $F_Q^{(i)} = 4|\alpha|^2$, $F_Q^{(ii)} = |\alpha|^2$. To give a meaning to these "unphysical" results, notice that QFI simply depends on the change of the probe state under the variation of the parameter φ. Even if we send a coherent state $|\alpha\rangle$ to the upper arm alone, then under the phase shift φ it evolves to $|\alpha e^{i\varphi}\rangle$, which differs from $|\alpha\rangle$ and in principle may provide us with useful information on the value of phase φ. The physical content that is missing in this reasoning is that the phase information is only available once we have access to an additional reference beam with respect to which the phase shift φ is defined. In other words, there is no such thing as an absolute phase shift—a seemingly obvious fact that has nevertheless significant implications for the problem considered and has been treated in contradicting ways in the literature.

The whole problem revolves around quantum states that are coherent superpositions of different photon number states such as, e.g., coherent or squeezed states. Take a coherent state $||\alpha|e^{i\theta}\rangle$. Since all measurements in quantum optics rely ultimately on photon counts, no measurable consequences of these coherences may be observed unless these states are interfered (as, e.g., in a homodyne measurement) with a reference beam with respect to which the phase θ is defined. Otherwise, one is entitled to phase average the state without any observable consequences, i.e., replace $||\alpha|e^{i\theta}\rangle$ with $\rho = \int \frac{d\theta}{2\pi}||\alpha|e^{i\theta}\rangle\langle|\alpha|e^{i\theta}|$, which is an incoherent mixture of photon number states with Poissonian statistics [10].

Going back to our quantum interferometric setup, if we indeed consider just the two modes of the interferometer and do not allow any additional reference beam, then as an input we should rather consider a phase averaged state of the form

$$\rho(r,\alpha) = \int \frac{d\theta}{2\pi} V_\theta^a V_\theta^b |r,\alpha\rangle\langle r,\alpha| V_\theta^{a\dagger} V_\theta^{b\dagger}, \tag{6}$$

where $V_\theta^x = \exp(-i\theta x^\dagger x)$. Notice that squeezed and coherent states are averaged over a common phase θ, which reflects the fact there is a physical meaning in the *relative* phases between them. Calculation of QFI for $\rho(r,\alpha)$—$F_Q^{(\rho)}$—are more involved since the state is mixed and instead of Eq. (3) one needs to employ a general formula involving the concept of the symmetric logarithmic derivative [4]. The resulting $F_Q^{(\rho)}$ is different both from $F_Q^{(i)}$ and $F_Q^{(ii)}$ and does not depend on the choice of the phase shift generator—be it $U_\varphi^{(i)}$ or $U_\varphi^{(ii)}$. All that matters is the relative phase between the arms of the interferometer. $F_Q^{(\rho)}$ achieves maximum for $\tau = 1/2$ in which case it takes a simple form:

$$\max_\tau F_Q^{(\rho)} = F_{Q,\tau=1/2}^{(\rho)} = |\alpha|^2 e^{2r} + \sinh^2 r. \tag{7}$$

There is a great deal of confusion in the literature since formulas $F_Q^{(i)}$, $F_Q^{(ii)}$ are often used instead of $F_Q^{(\rho)}$ without discussing the need of an additional reference beam [9,11,12]. Despite this, one sometimes arrives at the correct result, since, e.g., for $\tau = 1/2$, $F_Q^{(ii)} = F_Q^{(\rho)}$, and that is why the results in Ref. [9] are indeed correct. However, had one used a phase shift generator (i) instead of (ii), one would arrive at a different solution. Similar objections can be raised in the context of Ref. [11], where $F_Q^{(i)}$ is used, and Ref. [12], where one defines standard quantum limit as $\delta\varphi^2 = (2|\alpha|^2)^{-1}$ instead of $(|\alpha|^2)^{-1}$, which is again due to the use of $F_Q^{(i)}$ instead of $F_Q^{(\rho)}$. Making use of $F_Q^{(i)}$, $F_Q^{(ii)}$ without mentioning the need of a reference beam is misleading since it is not clear what experimental setup these quantities really refer to.

Let us now consider a situation in which we indeed have an access to an additional reference beam—represented by the state $|\beta\rangle$ in Fig. 1—and want to properly analyze the quantum interferometric setup. If the reference beam is strong, we can treat it as a phase reference for the other two modes. Therefore, we introduce two phase shifts, φ_1 and φ_2, as in (iii) in Fig. 1, which are defined with respect to the reference beam. In a sense, we now face a two-parameter estimation problem. The proper way to proceed is to employ a two-parameter Cramer-Rao bound [4]:

$$\Sigma \geqslant \mathcal{F}^{-1},\ \mathcal{F}_{ij} = 4\mathrm{Re}(\langle\partial_i\psi|\partial_j\psi\rangle - \langle\partial_i\psi|\psi\rangle\langle\psi|\partial_j\psi\rangle), \quad (8)$$

where Σ_{ij}, $i = 1,2$ is the covariance matrix for parameters φ_1, φ_2, $\mathcal{F}$ is the quantum Fisher information matrix (QFIM), $|\psi\rangle = V^a_{\varphi_1} V^b_{\varphi_2} B_\tau |\psi_{\text{in}}\rangle$ is the probe state after sensing the phase shifts φ_1, φ_2 and $|\partial_i\psi\rangle = \frac{\partial|\psi\rangle}{\partial\varphi_i}$. If one is now interested in the bound on the uncertainty of φ_i, the proper formula reads

$$\delta\varphi_i \geqslant \sqrt{(\mathcal{F}^{-1})_{\text{ii}}}. \quad (9)$$

Note that in general $(\mathcal{F}^{-1})_{ii} \neq (\mathcal{F}_{ii})^{-1}$.

In quantum interferometry we are interested in the phase shift difference between the interferometer arms, i.e., $\varphi_- = \varphi_1 - \varphi_2$, so it is more convenient to write QFIM in basis $\varphi_\pm = \varphi_1 \pm \varphi_2$. Calculating QFIM in $\pm$ basis yields

$$\mathcal{F} = \begin{bmatrix} \mathfrak{G} & (1-2\tau)\mathfrak{H} \\ (1-2\tau)\mathfrak{H} & (1-2\tau)^2\mathfrak{G} + \mathfrak{F} \end{bmatrix}, \quad (10)$$

where $\mathfrak{G} = |\alpha|^2 + \sinh^2(2r)/2$, $\mathfrak{H} = \sinh^2(2r)/2 - |\alpha|^2$, and finally the bound on estimation precision of φ_- can obtained easily via Eq. (9). The minimal uncertainty is obtained for $\tau = \frac{1}{2}$ in which case $\delta\varphi_- \geqslant \sqrt{(|\alpha|^2 e^{2r} + \sinh^2 r)^{-1}}$.

It is interesting to note that this is the same result as the one obtained for the phase averaged state using $F_Q^{(\rho)}$ from Eq. (7). This observation proves that in the setup considered (with $\tau = \frac{1}{2}$) there is no advantage in using the reference beam when estimating the phase difference between the two arms of the interferometer. More generally, it can be shown that this is a feature of all path-symmetric pure states, i.e., the states that are symmetric with respect to an exchange of the arms of the interferometer [13]. It is also worth mentioning that the optimal measurement in our setup when $\tau = 1/2$, and more generally whenever we deal with a pure path-symmetric state in the interferometer, is a standard photon count measurement after the two modes are interfered on a balanced beam splitter [8,14].

As a summary of the discussion, in Fig. 2 we plot in black the bounds on $\delta\varphi$ obtained using different QFIs. The bounds are plotted as a function of the total number of photons $\bar{n}$ used, and parameters (τ, α, r) are chosen to maximize the respective QFI. One can easily notice that the uncertainties calculated using $F_Q^{(i)}$ and $F_Q^{(ii)}$ are overly optimistic. The reason behind this is an implicit assumption that, e.g., in the case of $F_Q^{(i)}$, the lower arm of the interferometer (where there is no phase shift element) is perfectly aligned with the reference beam. Such an assumption can hardly be justified in practice.

Things become more complicated when one takes into account loss in the interferometer. Let η be the power transmission coefficient in both arms of the interferometer. All results presented in the paper may be rederived in this setup although calculations are more involved. Figure 2 depicts in gray the resulting uncertainties for exemplary loss coefficient $1 - \eta = 0.2$. Apart from a similar observation that $F_Q^{(i)}$ and $F_Q^{(ii)}$ yield overoptimistic results, we additionally observe that $(\mathcal{F}^{-1})_{--} < (F_Q^{(\rho)})^{-1}$, which is illustrated by a thick band and proves that having an additional reference beam helps in estimating the phase difference in a lossy interferometer.

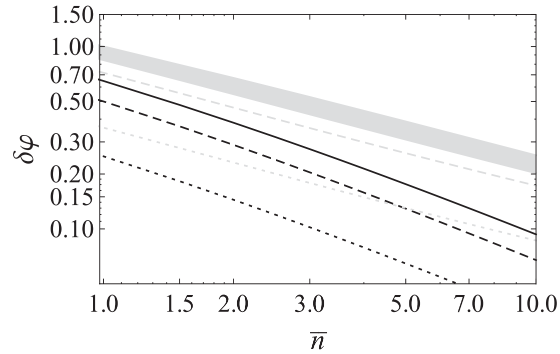

FIG. 2. Bounds on the phase estimation precision calculated using the QFI, in the cases of ideal (black) and lossy ($\eta = 0.8$, gray) interferometers. Different curves correspond to QFI calculated using different models: $F_Q^{(i)}$ (dotted), $F_Q^{(ii)}$ (dashed), $F_Q^{(\rho)}$ (solid). In the case of a lossy interferometer, the additional reference beam may improve the precision: $(\mathcal{F}^{-1})_{--}$(gray, solid, bottom) $<$ $(F_Q^{(\rho)})^{-1}$(gray, solid, top), while for the ideal interferometer these quantities coincide.

It is interesting to understand deeper what we really mean by *strong* reference beam. Clearly, if $|\beta|$ is not strong enough we can hardly treat it as a phase reference. To solve this problem, consider a phase-averaged three-mode state:

$$\rho(r,\alpha,\beta) = \int \frac{d\theta}{2\pi} V^a_\theta V^b_\theta V^c_\theta |r,\alpha,\beta\rangle\langle r,\alpha,\beta| V^{a\dagger}_\theta V^{b\dagger}_\theta V^{c\dagger}_\theta. \quad (11)$$

Calculating the QFIM in this case can be done only numerically. Finally, we can calculate the optimal estimation strategy (optimal τ, α, r) and the resulting bound on precision $\delta\varphi_- \geqslant (\mathcal{F}^{-1})_{--}$ as a function of $|\beta|$. With the increasing value of $|\beta|$ we will approach the regime discussed before, where we treated the reference beam as strong enough so it can serve as a perfect phase reference. In the case of the example depicted in gray in Fig. 2, this corresponds to improving the estimation precision by going from the upper to the lower boundary of the gray band with increasing $|\beta|$.

A deeper analysis [13] shows that a sufficient condition for treating the reference beam as a perfect phase reference is $|\beta|^2 \gg \bar{n}^2$. The reference beam needs to have much more than the square of the number of photons traveling in the proper modes of the interferometer, a fact observed also in Ref. [15].

In summary, we have pointed out some possible flaws in the interpretations of the results obtained using the QFI for states that are superpositions of different total photon number terms and showed that the full understanding of the problem is only possible if the role of an additional reference beam is properly taken into account.

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