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Perfect quantum state transfer in two- and three-dimensional structures

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We introduce a scheme for perfect state transfer in regular two- and three-dimensional structures. The interactions on the lattices are of the *XX* spin type with uniform couplings. In two dimensions, the structure is a hexagonal lattice, and in three dimensions, it consists of hexagonal planes joined to each other at arbitrary points. We will show that compared to other schemes, much less control is needed for routing, the algebra of global control is quite simple, and the same kind of control can upload and download qubit states to or from built-in read-write heads.

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Introduction. Since the inception of the fields of quantum information and computation, the task of coherently transferring quantum states through long and short distances has been of the utmost importance. While photons are the ideal carriers of quantum information over long distances, it has become evident that the best possible method for transferring quantum information over short distances, i.e., through regular arrays of qubits, is to exploit the natural dynamics of the many-body system. This idea was first introduced in the work of Bose [1] who showed that the natural dynamics of a Heisenberg ferromagnetic chain can achieve high-fidelity transfer of spin states over distances as long as 80 lattice units.

This has led to an active view in the study of the dynamics of many-body systems, traditionally studied in condensed-matter physics from a passive point of view. For example, scientists have tried to engineer the couplings in such a way that the natural dynamics of a spin chain achieve state transfer with perfect [2–5] or with arbitrary high fidelity [4,6–10]. In some protocols, the natural evolution is interrupted by some minimal control to achieve this task [12–15].

To overcome the necessity of engineered couplings, which usually restricts the experimental realization of such protocols, some kind of control was reintroduced in the scheme [16], where it was shown that quasi-one-dimensional chains with uniform $\pm$ couplings can achieve perfect transfer. The conversion of these linear structures to star configurations and arranging them in two-dimensional structures was shown to achieve perfect state transfer (PST) in two and higher dimensions. However, the nature of subsystems introduced in [16] required multiple control on external nodes of each subsystem, and different types of control for matching subsystems with each other. A different type of control was also necessary for loading and extracting the states to or from the lattice.

In this Rapid Communication, we introduce a very simple scheme for perfect transfer in two- and three-dimensional lattices. In addition to having all of the properties of the protocol of [16], like linear scaling of time with distance, and robustness to errors, this scheme has very desirable extra properties, namely,

(i) very simple global operations are needed for routing arbitrary states through arbitrary paths, that is, to each route a very simple sequence of operations corresponds,

(ii) the lattice has natural built-in local read-write (RW) heads for uploading and downloading qubit states, and

(iii) the same kind of global control which is used for routing is also used for uploading and downloading states to or from input and output heads.

(iv) As we will see, all of these properties are based on the geometry of hexagonal lattice and on a concept (or device) which we introduce here. The important point is that the effective Hamiltonian on these two- and three-dimensional hexagonal lattices, when written in the right basis, turns out to be the direct sum of one-dimensional PST chains, and the Hadamard switches allow us to route the particles by very simple controls through these chains in different directions.

Preliminaries. The prototype of many-body systems, which has been used in many protocols, is the *XX* spin chain,

$$H=\sum_{m,n}^{N}\frac{1}{2}K_{m,n}(X_mX_n+Y_mY_n). \tag{1}$$

This type of interaction preserves spin, $[H,\sum_m Z_m]=0$, and does not evolve the uniform background state of all spin-ups, i.e., $H^{\otimes N}|0\rangle^N=0$. This leads to the simple result that for transferring an arbitrary qubit state such as $\alpha|0\rangle+\beta|1\rangle$, it is enough to perfectly transfer only the state $|1\rangle$ through the lattice. Such a transfer occurs in the single excitation sector, which is spanned by N states of the form $|m\rangle$, where $|m\rangle$ means that the single spin in the mth place is down (or the local qubit is in the state $|1\rangle$).

The scheme. We start with the hexagonal lattice shown in Fig. 1. Let v denote a vertex of the lattice. On the three links connected to this vertex, there are three qubits, which we denote by $v+e_1$, $v+e_2$, and $v+e_3$. The vectors e_1, e_2, and e_3 denote the three vectors directed along the links connected to a vertex. A fourth qubit $v+e_0$, called the read-write (RW) head, is also connected to this vertex, although we emphasize that the vector $v+e_0$ does not necessarily mean a vector in the plane. This RW head need only be near the vertex, and it can have any geometrical relation with respect to the main lattice. At each vertex v, there is a four-level quantum system, which will be specified later on. The Hamiltonian that governs the interaction on this system is of the form

$$H=\sum_v H_v, \tag{2}$$

where H_v is the local Hamiltonian connecting each vertex to its neighboring links and through these links to the other vertices.

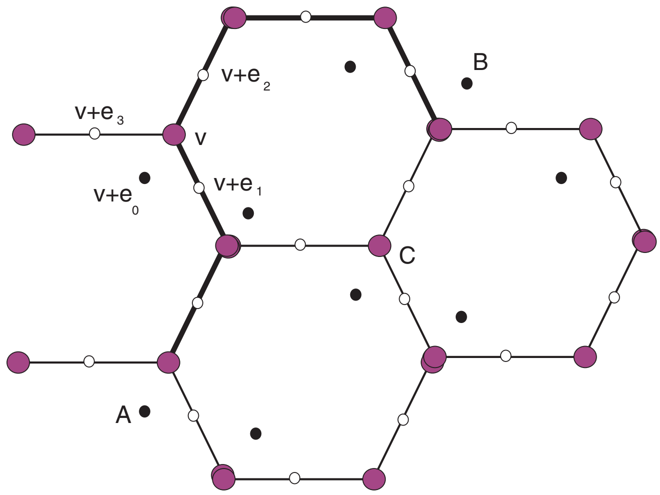

FIG. 1. (Color online) The hexagonal plane. Each edge is a PST chain of length 3, but it can also be replaced with PST chains of arbitrary length. The RW heads are the small black circles. The links accommodate qubit states (small white circles). The Hadamard switches (the big circles at the vertices) are used to switch the qubit in different directions and the RW head (if necessary). For transferring a state from A to B, we can take a longer (bold) path to route around an imperfection at C.

Now we describe the quantum system at each vertex. At each vertex v, there are four qubits which we denote by v_α, i.e., v_0, v_1, v_2, and v_3. These qubits are arranged on four different layers or four different global registers. In particular, the qubits v_0 for different v's lie in the hexagonal plane and the other qubits lie in different layers which we call control layers, or the control registers, to distinguish them from the main hexagonal plane.

Actually the qubits in the control layers need not have a fixed geometrical relation with the main layer, as Fig. 2 only shows the pattern of connections between different qubits. The geometry depends on the actual implementation (i.e., superconducting qubits, etc.).

As we will see later, the only control that we need is the possibility of applying a uniform magnetic field on each control layer or global register. No control on any individual qubits is necessary. The local Hamiltonian H_v is of the simple *XX* type,

$$H_v = \sum_{\alpha,\beta=0}^{3} J^{\alpha\beta}(X_{v_\alpha}X_{v+e_\beta} + Y_{v_\alpha}Y_{v+e_\beta}), \tag{3}$$

where $J^{\alpha\beta}$ are the entries of the Hadamard matrix in four dimensions, namely,

$$J = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{4}$$

Therefore, this is an *XX* spin lattice with uniform ± couplings between the four spins v_α in each vertex and the neighboring spins $v + e_\beta$. Note that while α indexes the four qubits on the same vertex, β indexes the four qubits on the links and the RW head.

In the above matrix, the rows and columns are numbered from 0 to 3, from left to right and from top to bottom, respectively. Note that the vertex v_0 is connected with each of the three links and also the RW head with equal couplings. We call this structure, described by the Hamiltonian H_v, a Hadamard switch. As we will see later, it can be used in a very effective way for routing states through two- and three-dimensional structures. Figure 2 shows this switch. In Fig. 1, these switches have been depicted as big colored circles at vertices of the hexagonal lattice.

A crucial step in our scheme is to show that this spin lattice, when written in an appropriate basis, is in fact the direct sum of uniformly coupled, perfect transfer chains, each pointing in a different direction. Figure 3 shows this effective structure where the Hadamard switch (the big bulb) puts the state in the initial node of a PST chain depending on the global control pulse. States are transferred between these blue bulbs, and in each bulb only global control pulses applied to the whole system determine whether the state should go to a given direction or else be downloaded to the RW head near the bulb. We also emphasize that the PST chains need not be of length 3; any type of PST chain [2–5] can be joined by these switches.

To show the decoupling, in Fig. (3) we note that the local *XX*Hamiltonian on spins m and n, $H_{m,n} := \frac{1}{2}(X_mX_n + Y_mY_n)$, has the following simple action: $H_{m,n}|m\rangle = |n\rangle$, $H_{m,n}|n\rangle = |m\rangle$, where $H_{m,n}|p\rangle = 0$ for $p \neq m, n$. This means that $H_{m,n}$, when restricted to a single-particle subspace, has the following expression:

$$H_{m,n} = |m\rangle\langle n| + |n\rangle\langle m|. \tag{5}$$

This allows us to rewrite the total Hamiltonian in the form

$$H = \sum_{v,\alpha,\beta} J^{\alpha,\beta}(|v_\alpha\rangle\langle v+e_\beta| + |v+e_\beta\rangle\langle v_\alpha|). \tag{6}$$

We now consider the four orthogonal states $|\xi_v^\alpha\rangle := \sum_{\beta=0}^{3} J^{\alpha,\beta}|v_\beta\rangle$, that is,

$$\begin{aligned} |\xi_v^0\rangle &:= \tfrac{1}{2}(|v_0\rangle + |v_1\rangle + |v_2\rangle + |v_3\rangle), \\ |\xi_v^1\rangle &:= \tfrac{1}{2}(|v_0\rangle + |v_1\rangle - |v_2\rangle - |v_3\rangle), \\ |\xi_v^2\rangle &:= \tfrac{1}{2}(|v_0\rangle - |v_1\rangle + |v_2\rangle - |v_3\rangle), \\ |\xi_v^3\rangle &:= \tfrac{1}{2}(|v_0\rangle - |v_1\rangle - |v_2\rangle + |v_3\rangle). \end{aligned} \tag{7}$$

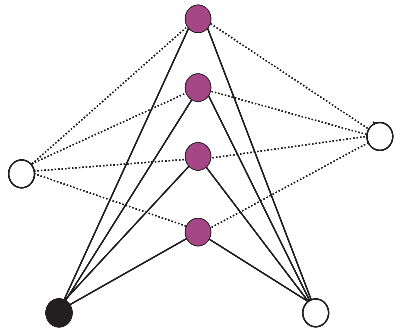

FIG. 2. (Color online) The Hadamard switch. The four spins in the middle (the colored circles) have an *XX* interaction with the four spins around, according to the pattern of ± signs in the Hadamard matrix J in (4). The white circles are the spins on the link and the black one is the RW head.

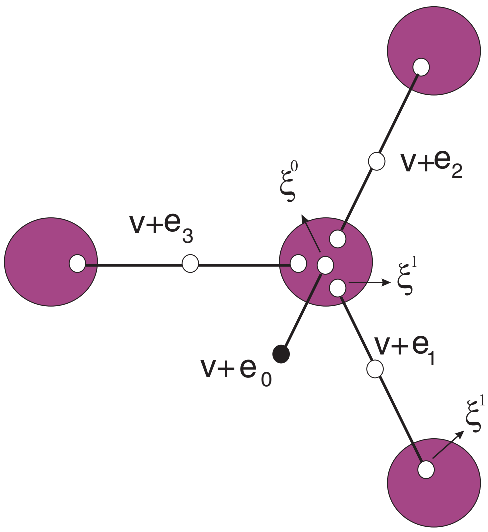

FIG. 3. (Color online) Decomposition of H_v into PST chains in different directions. The 2-chain transfers the particle between the RW head (the black circle) and the state $|\xi^0\rangle$. A state $|\xi^i\rangle$ perfectly goes in the direction i. Global control on the plane of spins v_1, v_2, and v_3 changes the states $|\xi^\alpha\rangle$ as desired.

The Hamiltonian can now be rewritten as

$$H = \sum_{v,\alpha} \left(|\xi_v^\alpha\rangle\langle v + e_\alpha| + |v + e_\alpha\rangle\langle\xi_v^\alpha|\right). \tag{8}$$

This means, for example, that the three qubit states $|\xi_v^1\rangle$, $|v + e_1\rangle$, and $|\xi_{v+e_1}^1\rangle$ form an XX chain of length 3 with equal couplings, hence a PST. Therefore, as shown in Fig. 3, in this new basis, the Hamiltonian has been decomposed into a direct sum of the *XX* spin chains with uniform couplings of length 2 (for the RW heads) and 3 (for the links). Such chains are capable of the perfect transfer of qubits in times $t_0 = \frac{\pi}{2}$ and $t_1 = \frac{\pi}{\sqrt{2}}$, respectively [2].

To switch the particle in different directions, we note the important property of the states $|\xi_v^\alpha\rangle$, which are turned into each other by global unitary operators. Let Z_1, Z_2, and Z_3 be the Pauli operators acting on spins 1, 2, and 3 on vertex v. Then, it is readily seen that

$$\begin{aligned} Z^1Z^2 &: |\xi^1\rangle \longleftrightarrow |\xi^2\rangle, \quad |\xi^0\rangle \longleftrightarrow |\xi^3\rangle, \\ Z^1Z^3 &: |\xi^1\rangle \longleftrightarrow |\xi^3\rangle, \quad |\xi^0\rangle \longleftrightarrow |\xi^2\rangle, \\ Z^2Z^3 &: |\xi^2\rangle \longleftrightarrow |\xi^3\rangle, \quad |\xi^0\rangle \longleftrightarrow |\xi^1\rangle. \end{aligned} \tag{9}$$

The crucial point is that the Z_i operations can be applied globally on all of the qubits in the ith control layer, since on the empty sites it has no effect, and on an occupied site it has the phase effect that we want. Therefore, there is no need to address single spins in each control layer, only the possibility of access to each layer is required. Such a control should be applied in a time much shorter than the time scales t_0 and t_1. We can also route many particles at the same time, in which case we have to control different regions of control layers differently, depending on the paths of these particles. In those time intervals when the paths become parallel, global (not regional) control pulses guide the particles through the lattice.

Now a clear and very simple method for perfect state transfer in the lattice emerges. A single particle $\alpha|0\rangle + \beta|1\rangle$ is uploaded to a given input head v_{in}. The part $\alpha|0\rangle$ does not evolve and indeed is ready for downloading at any output head. We only have to transfer the single-particle state $|1\rangle$, which in view of our notation has made the whole lattice be in the state $|v_{in} + e_0\rangle$. After a time t_0, this state evolves to $|\xi_{v_{in}}^0\rangle$, i.e., the particle has moved to the nearest vertex v_{in} in the form of the linear superposition $|\xi^0\rangle$. Once in a state $|\xi_v^0\rangle$, we can make a global control according to (9) to switch this vertex state to either of the states $|\xi_v^i\rangle$ $(i = 1,2,3)$, depending on the direction in which we want to route the state. For example, if we switch it to $|\xi_v^1\rangle$, then, according to Fig. 3, after a time t_1, the state will be transferred perfectly to the other end of the 3-chain in direction e_1. Continuing in this way, we can move the state via any path that we like to any other vertex, say v_{out}, where the final state will be one of the three states $|\xi_{v_{out}}^i\rangle$, $(i = 1,2,3)$. Switching this state to $|\xi_{v_{out}}^0\rangle$ will move this state to the nearest output head in the form $|v_{out} + e_0\rangle$, where it will be read off. The total time for routing is $2t_0 + Nt_1$, where N is the number of links which connect the input and output heads along the chosen path. The sequence of control operations is very simple. For uploading and downloading a qubit to or from a link e_i to its nearest head, the operation $\hat{Z}_i$ is applied when $\hat{Z}_i$ means that Z_i is removed from the triple $Z_1Z_2Z_3$, and at each vertex for routing a particle from direction e_i to direction e_j, the operation Z_iZ_j is applied. Except for uploading and downloading operations where a time lapse of t_0 is needed, all of the other control operations are applied at regular intervals of time t_1. We restate it as follows:

(i) for turning from direction i to j, apply Z_iZ_j; and

(ii) for uploading and downloading a qubit to or from a RW head to direction i, apply $\hat{Z}_i$.

Perfect transfer in three dimensions. The Hadamard switch can be used in a natural way for achieving PST in three-dimensional structures. Figure 4 shows a Hadamard switch connecting two hexagonal planes. Such planes can be joined by any number of switches. The number and positions of Hadamard switches are determined to optimize the accessibility of all of the heads in the two planes by the shortest

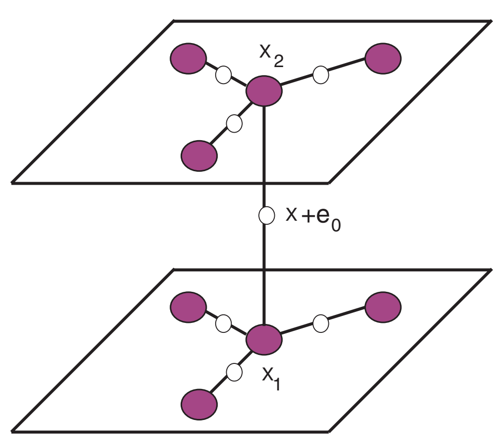

FIG. 4. (Color online) The Hadamard switch can transfer a state between planes. The leg which was previously connected to RW heads is replaced by a PST chain to connect two switches in two different planes.

possible paths. When used in this way, the RW head gives its role to the qubit on the link which joins the two planes. For example, when the two planes are joined to each other at points x_1 and x_2 on the two planes (Fig. 4), the effective Hamiltonian for the states $|x + e_0\rangle$ on the joining link, and the states $|\xi^0_{x_1}\rangle$ on plane 1 and $|\xi^0_{x_2}\rangle$ on the upper plane, is nothing but a perfect XX 3-chain. This effective Hamiltonian transforms the state $|\xi^0\rangle$ perfectly between the two planes. This time we should wait for time t_1 instead of t_0. Therefore, we route a particle within each plane as before and bring it to the position of the nearest switch, where by appropriate control we move it to another plane and continue there.

A note on implementation. Solid-state qubits are attractive due to their inherent scalability using well-established microfabrication techniques. The main challenge is obviously to implement the $\pm$ couplings. It has been shown that these kinds of tunable couplings can be implemented in charge-based [17] and flux-based [18] superconducting qubits. In the latter case, the coupling strength is adjusted by the current bias applied to the superconducting quantum interference device (SQUID) and can be varied continuously from positive to negative values.

Imperfections. It has been shown that a modest amount of static and dynamic disorder in couplings does not affect the optimal transfer time [19] and has little effect on transfer fidelity in the chains [20]. For a linear chain of size up to $N = 51$, a randomness of strength $\sigma_J = 0.1\ J$ decreases the fidelity of perfect state transfer by only 10% [20]. Moreover, there are recent proposals for achieving PST via certain classes of random, unpolarized (infinite-temperature) spin chains [4]. These kinds of chains can be joined to each other in a hexagonal lattice via Hadamard switches, in which case longer chains up to the point where localization will be important can be used. In such schemes, fewer switches will be used and when any of them fails, it can be routed around as in Fig. 1 at the cost of taking a slightly longer path.

In summary, we have developed a scheme on a hexagonal lattice with uniform ± 1 XX couplings for perfect state transfer in two and three dimensions. In this scheme, qubit states are uploaded from RW heads, routed via specific paths, and downloaded to RW heads by a very simple sequence of control pulses applied to global registers. No individual addressing is required. The transfer time scales linearly with the path length. The simple features of the scheme, namely, the hexagonal lattice, the RW head, single-step controls, and the uniform couplings, are all naturally linked with the beautiful mathematical fact of a unique Hadamard matrix (4) in four dimensions. (Such matrices exist only in certain dimensions.) We believe that once the challenge of implementing $\pm$ couplings is solved, for example, in arrays of Josephson junctions [21], which have been shown to allow tunable $\pm$ couplings [17,18], this will be the natural choice for state transfer in two and three dimensions, and it will be worthwhile to investigate other issues, such as the optimal control sequences for the simultaneous routing of many particles, independence of the initial state, the effect of static and dynamic random couplings, random magnetic fields, and temperature fluctuations, in this scenario.

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