Universal relations for the two-dimensional spin-1/2 Fermi gas with contact interactions

Manuel Valiente, Nikolaj T. Zinner, and Klaus Mølmer
Lundbeck Foundation Theoretical Center for Quantum System Research, Department of Physics and Astronomy, Aarhus University, DK-8000 Aarhus C, Denmark

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We present universal relations for a two-dimensional Fermi gas with pairwise contact interactions. The derivation of these relations is made possible by obtaining the explicit form of a generalized function—selector—in the momentum representation. The selector implements the short-distance boundary condition between two fermions in a straightforward manner and leads to simple derivations of the universal relations, in the spirit of Tan's original method for the three-dimensional gas.

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I. INTRODUCTION

The physics of strongly interacting quantum many-body systems has been one of the most active fields of research for a number of decades. Strongly correlated states of matter are ubiquitous in many areas of physics ranging from atomic, molecular, optical [1], condensed matter [2] and the study of quantum phase transitions [3], and quark-gluon plasma [4] to the physics of neutron stars [5]. A prominent example is the observed non-Fermi liquid behavior in high-temperature superconducting materials suggesting a strongly coupled state beyond the Landau paradigm [6]. Perturbative approaches to strongly coupled quantum systems are untenable, and even modern numerical techniques can be hard to apply. It is therefore of great interest to have analytical insights into systems where interactions are strong.

Research in strongly interacting three-dimensional (3D) Fermi gases has experienced an intense growth since the recent appreciation of Tan's universal relations [7]. These important results relate many of the many-body properties, such as the adiabatic change of the energy when varying the two-body scattering length, the asymptotic momentum distribution, the two-body loss rate [8], and the pressure to a single quantity called the contact, and were recently verified experimentally [9,10]. These universal relations and quantities involved therein have received much recent attention [11–21].

Interestingly, Tan relations have also been obtained for the one-dimensional (1D) Fermi gas [22], by means of the operator product expansion [23] that had been successfully applied in 3D [8]. However, less is known about these universal relations for the two-dimensional (2D) Fermi gas. Only the energy theorem [24–26], which relates the energy of the system to the momentum distribution, the large-momentum tail of the latter [25,26], and the adiabatic theorem [26], have been so far derived, and only in the homogeneous, or untrapped, case. The problem has been recently investigated numerically in Ref. [27].

In this paper, we derive the universal relations for the zero-range interacting spin-1/2 Fermi gas in 2D, with or without an external trapping potential. To do so, we use Tan's original method involving a generalized function, called Tan's selector. We find the explicit form of the selector in momentum space, in analogy with its 3D counterpart recently obtained by one of us [28]. Our approach greatly simplifies the derivation of the adiabatic theorem with respect to more conventional methods. Once the adiabatic theorem is shown, we use it to derive the generalized virial theorem, the pressure relation, and the two-body loss rate.

II. SYSTEM HAMILTONIAN

The zero-range interacting many-body Hamiltonian for $N = N_\uparrow + N_\downarrow$ spin-1/2 fermions has the form

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \sum_{i<j=1}^{N} V(\mathbf{r}_{ij}) + \sum_{i=1}^{N} W(\mathbf{r}_i), \tag{1}$$

where m is the particle's mass, $\mathbf{r}_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$, W is a single-particle external potential, and V is the 2D regularized pseudopotential [29]:

$$V(\mathbf{r}) = \frac{2\pi\hbar^2}{m}\delta(\mathbf{r}) r \frac{\partial}{\partial r}, \tag{2}$$

which is a particular member of the Olshanii-Pricoupenko Λ family of pseudopotentials [30]. Note that the pseudopotential (2) does not depend on the 2D scattering length a, which only enters the problem through the boundary condition at short-interparticle distances $\psi(\mathbf{r}_{ij}) \sim \ln(r_{ij}/a)$. The universal singlet binding energy in the homogeneous case ($W = 0$) can be seen to be [31]

$$E_B = \frac{4\hbar^2 e^{-2\gamma}}{ma^2}, \tag{3}$$

where $\gamma = 0.577\,215\,665\ldots$ is Euler's constant. The binding energy is the most appropriate fitting parameter of the theory in momentum space, as we see in the following paragraph.

III. TAN'S SELECTOR

To derive the universal relations for the Fermi gas in 2D, we will make use of Tan's original method [7], particularized to two dimensions. In Ref. [24], Tan defined two selectors, $\tilde{\eta}$ and l, the last of which is unnecessary in 2D, and involves the logarithm of a dimensioned quantity. The $\tilde{\eta}$ selector satisfies $\int d^2r\,\tilde{\eta}(r) = 1$, $\int d^2r\,\tilde{\eta}(r)\ln(r/a) = 0$, while $\tilde{\eta}(r) = 0$ for $r \neq 0$.

We here redefine the momentum representation $\eta(k)$ of $\tilde{\eta}(r)$ as follows:

$$\eta(k) = 1 \ (k < \infty), \tag{4}$$

$$\int d^2k\eta(k)\frac{1}{\hbar^2k^2/m + E_B} = 0, \tag{5}$$

where E_B is the two-body universal binding energy, Eq. (3). It is easy to see that Eq. (5) is equivalent to Eq. (16b) of Ref. [24], but our condition is more natural and represents a clear way of writing the two-body bound state integral equation. Proceeding in analogy to the 3D case [28], we see that

$$\eta(k) = 1 + \ln\left(\frac{2e^{-\gamma}}{ka}\right)\frac{\delta(1/k)}{k}. \tag{6}$$

The above explicit form for $\eta(k)$ will prove crucial for the derivation of the two-dimensional adiabatic theorem.

IV. ENERGY THEOREM

In Ref. [24] Tan stated, without derivation, that the energy of the 2D Fermi gas can be written as

$$E = \sum_{\mathbf{k}\sigma}\eta(k)\frac{\hbar^2k^2}{2m}n_{\mathbf{k}\sigma} + \langle\mathcal{W}\rangle, \tag{7}$$

where $n_{\mathbf{k}\sigma} = \langle c^\dagger_{\mathbf{k},\sigma}c_{\mathbf{k},\sigma}\rangle$, with $c_{\mathbf{k},\sigma}$ the spin-σ annihilation operator at momentum $\mathbf{k}$, is the momentum distribution, and $\mathcal{W} = \sum W(\mathbf{r}_i)$ is the total external potential.

To prove Eq. (7) it is sufficient to consider a pure, normalized state $|\phi\rangle$ with N_σ fermions of spin $\sigma = \uparrow, \downarrow$ and a total number of fermions $N = N_\uparrow + N_\downarrow$:

$$|\phi\rangle = \frac{1}{N_\uparrow!N_\downarrow!}\int d\mathbf{R}d\mathbf{S}\phi(\mathbf{R},\mathbf{S})\prod_{i=1}^{N_\uparrow}\psi^\dagger_\uparrow(\mathbf{r}_i)\prod_{j=1}^{N_\downarrow}\psi^\dagger_\downarrow(\mathbf{s}_j)\,|0\rangle, \tag{8}$$

where $\mathbf{R}\,(\mathbf{S})$ is shorthand for $\mathbf{r}_1,\ldots,\mathbf{r}_{N_\uparrow}$ $(\mathbf{s}_1,\ldots,\mathbf{s}_{N_\downarrow})$, and the integrals are done over the whole $2N$-dimensional space. Above, $\psi^\dagger_\sigma(\mathbf{r})$ is the spin-σ creation operator at position $\mathbf{r}$, and $|0\rangle$ is the vacuum.

Tan's derivation [7] for the 3D gas can be followed in parallel for the 2D case until the following expression is encountered:

$$\int d\mathbf{R}'d^2r_0\int d^2t\tilde{\eta}(t)\nabla^2_{\mathbf{t}}K(\mathbf{R}',\mathbf{r}_0,\mathbf{t}), \tag{9}$$

where $\mathbf{R}' = (\mathbf{r}_2,\ldots,\mathbf{r}_{N_\uparrow},\mathbf{s}_2,\ldots,\mathbf{s}_{N_\downarrow})$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{s}_1$, $\mathbf{r}_0 = (\mathbf{r}_1 + \mathbf{s}_1)/2$, and K is defined as

$$K(\mathbf{R}',\mathbf{r}_0,\mathbf{t}) = \int_{r<\epsilon} d^2r\phi^*(\mathbf{R}',\mathbf{r}_0,\mathbf{r})\phi(\mathbf{R}',\mathbf{r}_0 + \mathbf{t}/2,\mathbf{r} + \mathbf{t}), \tag{10}$$

where $\phi(\mathbf{R}',\mathbf{r}_0,\mathbf{r})$ stands for $\phi(\mathbf{R},\mathbf{S})$, with $\mathbf{r}_1$ and $\mathbf{s}_1$ replaced by $\mathbf{r}_0 + \mathbf{r}/2$ and $\mathbf{r}_0 - \mathbf{r}/2$, respectively. In Eq. (10), $\epsilon > 0$ is a small, positive quantity with dimensions of length; however it is much larger than $t = |\mathbf{t}|$. The goal is to show that the integral in Eq. (9) vanishes since, if it does, relation (7) holds [7]. First, we expand ϕ as

$$\phi(\mathbf{R}',\mathbf{r}_0,\mathbf{r}) = A(\mathbf{R}',\mathbf{r}_0)[\ln(r/a) + O(r)], \tag{11}$$

and

$$A(\mathbf{R}',\mathbf{r}_0 + \mathbf{t}/2) = A(\mathbf{R}',\mathbf{r}_0) + \nabla_{\mathbf{r}_0}A(\mathbf{R}',\mathbf{r}_0)\cdot\mathbf{t}/2 + O(t^2). \tag{12}$$

We obtain $K = K_0 + K_1$, with

$$K_0 \approx \pi|A(\mathbf{R}',\mathbf{r}_0)|^2\frac{t^2}{2}[\ln(t/a) - 1], \tag{13}$$

$$K_1 \approx \pi A^*(\mathbf{R}',\mathbf{r}_0)\frac{t^3}{4}[\ln(t/a) - 1]\nabla_{\mathbf{r}_0}A(\mathbf{R}',\mathbf{r}_0)\cdot\hat{\mathbf{t}}. \tag{14}$$

Inserting K into (9), we see that the integral vanishes, as we wanted to show.

We can express the energy relation in Eq. (7) more explicitly by using the form of the η selector in Eq. (6) as

$$E = \frac{\hbar^2\ln(2e^{-\gamma})}{2\pi m}\Omega C + \sum_{\mathbf{k},\sigma}\frac{\hbar^2k^2}{2m}\left[n_{\mathbf{k},\sigma} - \frac{C}{k^3(k + a^{-1})}\right] + \langle\mathcal{W}\rangle, \tag{15}$$

where Ω is the area and $C = \lim_{\mathbf{k}\to\infty}k^4n_{\mathbf{k},\sigma}$ is the 2D contact density. The above relation, Eq. (15), was expressed in a somewhat different manner by Combescot *et al.* in Ref. [25] and by Werner and Castin in Ref. [26]. The tail of the momentum distribution was found in Refs. [25,26] to be $\propto 1/k^4$, and therefore the contact C is a finite quantity.

V. ADIABATIC THEOREM

The adiabatic theorem states the relation between the contact and the change in energy as the scattering length is slightly varied. This is the central result among the universal relations for the Fermi gas, since it is needed to derive the virial theorem, the pressure relation, and the inelastic two-body loss rate.

The fact that the 2D pseudopotential, Eq. (2), does not depend explicitly on the scattering length adds a complication to the proof of the adiabatic theorem that was not present in 3D and 1D. Indeed, the Hellmann-Feynmann theorem, which was employed to derive this important result in 3D by Braaten and Platter [8] and in 1D by Punk and Zwerger [22], cannot be applied directly to Eq. (2) nor to any member of the so-called Λ family of pseudopotentials [29,30]. In addition, Tan's approach to the 2D problem [24] lacked a simple dependence of the η selector on the scattering length. These two facts have prevented a simple derivation of the adiabatic theorem using regularized pseudopotentials, which has been first shown by Werner and Castin [26] who obtained it using the Bethe-Peierls boundary condition. Below, we present a very simple proof of the adiabatic theorem by using the explicit form, Eq. (6), of the η selector.

We begin by defining the following operator:

$$\eta_aT \equiv \sum_{\mathbf{k},\sigma}\eta(k)(\hbar^2k^2/2m)c^\dagger_{\mathbf{k},\sigma}c_{\mathbf{k},\sigma}, \tag{16}$$

where $\eta(k)$ corresponds to the scattering length a. It is then easy to see that

$$E(a') - E(a) = \frac{\langle\phi(a)|\,\eta_aT - \eta_{a'}T\,|\phi(a')\rangle}{\langle\phi(a)|\phi(a')\rangle}, \tag{17}$$

where $E(a)$ and $\phi(a)$ are an energy eigenvalue and associated eigenstate of Hamiltonian (1), respectively, corresponding to the scattering length a. Using Eq. (6) we find

$$\frac{E(a')-E(a)}{\ln(a'/a)} = \sum_{\mathbf{k},\sigma} \frac{\hbar^2 k^2 \delta(1/k)}{2mk} \frac{\langle\phi(a)|\, c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma}\, |\phi(a')\rangle}{\langle\phi(a)|\phi(a')\rangle}. \tag{18}$$

If we now take the limit $a' \to a$ in the above equation, we obtain the desired adiabatic theorem

$$\frac{dE}{da}(a) = \frac{\hbar^2\Omega}{2\pi m a} C. \tag{19}$$

An immediate application of the adiabatic theorem is the calculation of the contact for a weakly coupled 2D Fermi gas. The energy density for the spin-balanced case is given by [32]

$$\frac{E}{N} \approx \frac{\hbar^2 k_F^2}{4m}\left(1 - \frac{1}{\ln(k_F a)}\right), \tag{20}$$

where N is the number of particles and k_F is the Fermi momentum. The contact in this limit is therefore given by

$$C = \frac{\pi \rho k_F^2}{2\ln^2(k_F a)}. \tag{21}$$

where $\rho = N/\Omega$ is the particle density.

VI. GENERALIZED VIRIAL THEOREM

We assume now that the system is under an external potential of the form $W(\mathbf{r}) \propto r^\beta$, which has not been considered before for the 2D case. In this case, we will show that the virial theorem reads

$$E = \frac{\beta+2}{2}\langle\mathcal{W}\rangle - \frac{\hbar^2\Omega C}{4\pi m}. \tag{22}$$

To see this, we follow a technique used for the 3D gas [7]. We begin with a state ϕ corresponding to the scattering length a. We slightly change the scattering length to $a' = (1+\epsilon)a$, with ϵ small, associated with state ϕ' which has energy E'. Expanding E' in powers of ϵ, and using the adiabatic theorem, Eq. (19), we find

$$E' = E + \epsilon\frac{\hbar^2\Omega C}{2\pi m} + O(\epsilon^2). \tag{23}$$

We now define a rescaled wave function $\phi''(\mathbf{R},\mathbf{S}) \equiv (1+\epsilon)^N \phi'((1+\epsilon)\mathbf{R},(1+\epsilon)\mathbf{S})$. Expanding ϕ'' as in Eq. (11), we find that ϕ'' is a state at scattering length $a'' = a'/(1+\epsilon) = a$. Using the scaled wave function in the expectation value of the energy, we find its energy E'' to be

$$E'' = (1+\epsilon)^2 E'_{\text{in}} + (1+\epsilon)^{-\beta}\langle\phi'|\,\mathcal{W}\,|\phi'\rangle, \tag{24}$$

where $E'_{\text{in}} = E' - \langle\phi'|\,\mathcal{W}\,|\phi'\rangle$. Expanding E'' in powers of ϵ, we obtain $E'' - E' = 2\epsilon E'_{\text{in}} - \beta\epsilon\langle\phi'|\,\mathcal{W}\,|\phi'\rangle + O(\epsilon^2)$. From the quadratic convergence properties of variational energies [33], we have $E'' - E = O(\epsilon^2)$, and therefore, using Eq. (23) twice, we obtain

$$2\epsilon E_{\text{in}} - \beta\epsilon\langle\phi|\,\mathcal{W}\,|\phi\rangle + \epsilon\frac{\hbar^2\Omega C}{2\pi m} = O(\epsilon^2), \tag{25}$$

which, after taking the limit $\epsilon \to 0$, proves the virial theorem (22).

VII. PRESSURE RELATION

For a homogeneous system ($W = 0$), there is a relation between pressure, energy, and contact, which has been derived in 3D [7,8] and 1D [22]. In 2D, it is given by

$$P = \frac{E}{\Omega} + \frac{\hbar^2 C}{4\pi m}, \tag{26}$$

To show Eq. (26), we follow most of the steps taken for the proof of the virial theorem (22), and we arrive at the expression

$$\begin{aligned} P\Omega &= \lim_{\epsilon\to 0}\frac{E''-E}{1-(1+\epsilon)^{-2}} \\ &= \left(2E + \frac{\hbar^2\Omega C}{2\pi m}\right)\lim_{\epsilon\to 0}\frac{\epsilon}{1-(1+\epsilon)^{-2}}. \end{aligned} \tag{27}$$

Using L'Hôpital's rule in the above equation, we obtain the desired pressure relation (26).

VIII. INELASTIC TWO-BODY LOSS RATE

Two-particle loss rates can be calculated [8] by adding a small imaginary part a_I ($|a_I| \ll |a|$) to the scattering length, so that we replace $a \to a + ia_I$. The resulting complex energy is then expanded as $E(a+ia_I) = E(a) + ia_I dE/da(a) + O((a_I)^2)$, and the inelastic loss rate Γ is identified as $-\Gamma/2 = a_I dE/da$. Using the adiabatic theorem (19), we obtain

$$\Gamma = -a_I\frac{\hbar^2\Omega C}{\pi m a} + O((a_I)^2). \tag{28}$$

IX. CONCLUSIONS

We have obtained the universal Tan relations for the two-dimensional spin-1/2 Fermi gas with zero-range interparticle interactions. These results can be tested with current experimental techniques such as Bragg [10] or RF spectroscopy [9,34,35], already available for the two-dimensional Fermi gas [36]. Our methodology is close to the original approach for the three-dimensional problem [7]. The power of our approach relies on obtaining the explicit form, in momentum representation, of the so-called Tan's selector, which expresses the contact conditions between two particles with a given scattering length in closed form. This allowed us to overcome the fact that the Hamiltonian of the system does not depend on the two-body scattering length, and led us to a more straightforward derivation than with conventional methods. Our method is directly applicable also to universal three-body physics in two dimensions [37–39] where additional three-body contact parameters were recently proposed in the three-dimensional case [40,41]. This is an interesting direction for future study.

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