Zeno effect and ergodicity in finite-time quantum measurements

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We demonstrate that an attempt to measure a nonlocal in time quantity, such as the time average $\langle A\rangle_T$ of a dynamical variable A, by separating Feynman paths into ever narrower exclusive classes traps the system in eigensubspaces of the corresponding operator $\hat{A}$. Conversely, in a long measurement of $\langle A\rangle_T$ to a finite accuracy, the system explores its Hilbert space and is driven to a universal steady state in which the von Neumann ensemble average of $\hat{A}$ coincides with $\langle A\rangle_T$. Both effects are conveniently analyzed in terms of singularities and critical points of the corresponding amplitude distribution and the Zeno-like behavior is shown to be a consequence of the conservation of probability.

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I. INTRODUCTION

Quantum Zeno effect (see, for instance, Refs. [1–3] and references therein) is often associated with the perturbation frequent projective von Neumann measurements [4] produce on the observed system. Suppose, for example, that one wishes to determine the duration τ_Ω a quantum system spends in a particular subspace Ω of its Hilbert space. Checking ever more frequently whether the system is indeed inside Ω one eventually destroys the transitions between Ω and the rest of the Hilbert space. Thus, because of the Zeno effect, a continuously observed system [5] prepared inside Ω would spend there all available time, while for a system initially outside Ω, τ_Ω would be exactly zero [6]. Alternatively, one can perform a *finite time* measurement of τ_Ω in which a meter monitors the system over a finite period of time, and one single observation is made at the end of the run. One example of such a meter is a Larmor clock [7] consisting of a spin which rotates only when the system resides in Ω. From the clock's final orientation one is able to determine the value of τ, but can learn nothing about the precise moments the system enters and leaves Ω. A conceptually similar measurement of the duration τ_q a qubit spends in its state $|q\rangle$ proposed in Ref. [8] employs a large number of bosons [e.g., a weakly interacting Bose-Einstein condensate (BEC)] trapped in one of the wells of a symmetric double-well potential. With the atomic current between the wells increased whenever the qubit occupies the state of interest, the number of bosons found in the other well contains, like a reading of a Larmor clock, information about $\tau(q)$. Depending on the accuracy, such a measurement was shown to yield two seemingly incompatible results: (a) the qubit spends all of the time in just one of the two states, (b) the qubit shares its time between the two states in equal proportion. To put it differently, the strong interaction with a BEC meter freezes the qubit in the state $|q\rangle$. For an interaction that is strong yet finite, and given enough time T, the qubit visits the rest of its Hilbert space and spends there approximately $T/2$.

The purpose of this paper is to present a general theory of this effect, applicable to an arbitrary system in a finite-dimensional Hilbert space, time average of an arbitrary physical quantity describing such a system, and an arbitrary (within reason) way to measure such a time average. In the following, with the minimal mathematical rigour required for such a presentation, we analyze the Zeno effect arising in the high accuracy limit of a general finite-time measurement. We also study the effects of a measurement of the long-time average of a quantum variable and search for any evidence of ergodic behavior. Various approaches to quantum ergodicity can be found in Refs. [9–12], with the importance of the measurement(s) performed on the system emphasised in Ref. [11]. A variant of the Zeno effect arising solely from the strong interaction between a system and its environment has been studied in Ref. [3].

II. TIME AVERAGES AND THE CONVOLUTION FORMULA

Consider a quantum system in an N-dimensional Hilbert space with a Hamiltonian $\hat{H}$. Choosing an orthogonal basis $|a_m\rangle$, $m = 1,2,\ldots,N$ in which $\hat{H}$ is not diagonal, we can write a transition amplitude between initial and final states $|i\rangle$ and $|f\rangle$ over a time T as a sum over Feynman paths

$$U^{f\leftarrow i}(T) \equiv \langle f|\exp(-i\hat{H}T|i\rangle = \lim_{K\to\infty}\sum_{m_1,\ldots,m_K}^{N}\langle f|a_{m_K}\rangle \times\prod_{j=1}^{K-1}\langle a_{m_{j+1}}|\exp(-i\hat{H}\epsilon|a_{m_j}\rangle\langle a_{m_1}|i\rangle \equiv \sum_{\text{paths}} U^{f\leftarrow i}[\text{path}], \quad (1)$$

where $\epsilon \equiv T/(K-1)$ and each path is defined by a sequence $\{m_1,\ldots,m_K\}$ numbering the states $|a_m\rangle$ through which the system passes until reaching the final state $|f\rangle$. Consider a quantity A represented by an operator $\hat{A}$ diagonal in the chosen representation,

$$\hat{A} = \sum_n A_n\hat{P}_n, \quad \hat{P}_n\hat{P}_m = \hat{P}_n\delta_{mn}, \quad \sum_n \hat{P}_n = 1, \quad (2)$$

where $\hat{P}_n$ are mutually orthogonal projectors on (one-dimensional or multidimensional) subspaces spanned by vectors $|a_m\rangle$ corresponding to the same eigenvalue A_n. We wish to measure the value of a Feynman functional

$$F[\text{path}] = \lim_{K\to\infty} \sum_{j=1}^{K-1} \beta(j\epsilon) A_{m_j} \epsilon \equiv \int_0^T \beta(t) A(t) dt, \quad (3)$$

where $\beta(t)$ is a known function [13] and $A(t)$ denotes the (highly irregular) function traced by the value of A along a given Feynman path. Equation (3) may, for example, represent the time average of a quantity A if one chooses $\beta(t) = 1/T =$ const and, in particular, the fraction of the time τ_Ω/T, the system has spent in Ω if $\hat{A}$ is also chosen to be the projector onto a subspace Ω spanned by $|a_n\rangle$, $\hat{A} = \hat{P}_\Omega \equiv \sum_{n\in\Omega} |a_n\rangle\langle a_n|$ [8]. The probability amplitude for the value of F to be y is given by the restricted path sum

$$\Phi^{f\leftarrow i}(y,T) = \sum_{\text{paths}} \delta(F[\text{path}] - y) U^{f\leftarrow i}[\text{path}], \quad (4)$$

where $\delta(z)$ is the Dirac delta. Without loss of generality we choose the measuring device to be a Neumann pointer with position y which interacts with the system over a time T, the full Hamiltonian being $\hat{\mathcal{H}}(t) = \hat{H} - i\partial_y \beta(t)\hat{A}$.

At $t = T$ the pointer states of the meter $|y\rangle$ are entangled with the system's states obtained by propagation along Feynman paths satisfying the condition $F[\text{path}] = y$ [14]. In particular, for the system and the meter prepared at $t = 0$ in an product state $|i\rangle|G\rangle$, the probability amplitude to find at $t = T$ the system in the state $|f\rangle$ and, simultaneously, the pointer reading y, $\Psi^{f\leftarrow i}(y)$, is given by

$$\Psi^{f\leftarrow i}(y,T) \equiv \langle y|\langle f|\exp\left[-i\int_0^T \hat{\mathcal{H}}(t)dt\right]|i\rangle|G\rangle = \int G(y-y')\Phi^{f\leftarrow i}(y',T)dy, \quad (5)$$

where $G(y) \equiv \langle y|G\rangle$.

Thus, the meter "probes" the amplitude distribution $\Phi^{f\leftarrow i}(y',T)$, obtained by rearranging the path amplitudes of an unobserved system, with an "apparatus function" $G(y)$ determined by the meter's initial state. With $G(y)$ narrowly peaked around the origin, Feynman paths with different values of $F[\text{path}]$ contribute to different final meter states, and $|\Psi^{f\leftarrow i}(y)|^2$ yields the probability to reach the final state and obtain the value y for the quantity in Eq. (3). The measured result contains, however, an intrinsic quantum uncertainty as the values of y' within the peak's width around y remain indistinguishable. With many different Feynman paths contributing to the transition (1) one might expect the values of F to have a broad distribution, which a more accurate measurement would resolve in ever greater detail.

III. A ZENO EFFECT: HIGH ACCURACY AND FIXED DURATION

First we consider what would happen if one tries to accurately determine the value of the functional (3) in a measurement of a fixed duration T. The accuracy of the measurement can be improved by making the initial pointer state narrower in the coordinate space, for example, by replacing $G(y)$ with

$$G_\alpha(y) = \alpha^{1/2}G(\alpha y), \qquad \lim_{\alpha\to\infty}|G_\alpha(y)|^2 = \delta(y), \quad (6)$$

where the factor $\alpha^{1/2}$ ensures the correct normalization of the new state $\int |G_\alpha(y)|^2 dy = \int |G(y)|^2 dy = 1$.

With the help of Eqs. (5) and (6) we can now prove a general result: an accurate finite time measurement $\alpha \to \infty$, $T < \infty$, would indicate that at all times $A(t)$ maintains a constant value equal to one of the eigenvalues A_n, with the value of the functional (3) equal exactly to $A_n \int_0^T \beta(t)dt$. The proof follows from observing first that $\Phi^{f\leftarrow i}(y)$ cannot be a smooth function for all final states $|f\rangle$. Indeed, increasing α while maintaining unit normalization will cause the integral $\int G_\alpha(y)dy = \alpha^{-1/2}\int G(y)dy$ to vanish. With it would also vanish $\Psi^{f\leftarrow i}(y)$, $\lim_{\alpha\to\infty} \Psi^{f\leftarrow i}(y) = \Phi^{f\leftarrow i}(y)\alpha^{-1/2}\int G(y-y')dy' = 0$, thus contradicting the conservation of probability for the pointer. Therefore $\Phi^{f\leftarrow i}(y)$ must have a singular part, which we evaluate by rewriting Eq. (4) as a Fourier integral

$$\Phi^{f\leftarrow i}(y) = (2\pi)^{-1}\int \exp(i\lambda y) \times \langle f|\exp\left\{-i\int_0^T [\hat{H} + \lambda\beta(t)\hat{A}]dt\right\}|i\rangle d\lambda. \quad (7)$$

Further, writing $\exp\{-i\int_0^T[\hat{H} + \lambda\beta(t)\hat{A}]dt\} = \lim_{K\to\infty} \prod_{j=1}^K \exp\{-i[\hat{H} + \lambda\beta(j\epsilon)\hat{A}]\epsilon\}$, $\epsilon \equiv T/K$, we note that [15] $\lim_{\lambda\to\infty}\exp\{-i[\hat{H} + \lambda\beta(j\epsilon)\hat{A}]\epsilon\} = \sum_n \exp\{-i\lambda\beta(j\epsilon)A_n\epsilon\}$ $\exp(-i\hat{P}_n\hat{H}\hat{P}_n\epsilon)\hat{P}_n + O(\lambda^{-1})$. Defining Zeno Hamiltonian as (cf. Ref. [3])

$$\hat{H}_Z = \sum_n \hat{P}_n\hat{H}\hat{P}_n, \quad (8)$$

we obtain

$$\left\langle f|\exp\left[-i\int_0^T [\hat{H} + \lambda\beta(t)\hat{A}]dt\right]|i\right\rangle = \sum_n \exp\left\{-i\lambda A_n\int_0^T \beta(t)dt\right\}\langle f|\exp(-i\hat{H}_Z T)\hat{P}_n|i\rangle + \langle f|\hat{u}(\lambda)|i\rangle, \quad (9)$$

where the last term vanishes as $\lambda \to \pm\infty$, $\lim_{\lambda\to\infty}$ $\langle f|\hat{u}(\lambda)|i\rangle = O(\lambda^{-1})$. Inserting Eq. (9) into Eq. (7) yields

$$\Phi^{f\leftarrow i}(y) = \sum_n \langle f|\exp(-i\hat{H}_Z T)\hat{P}_n|i\rangle \times \delta\left(y - A_n\int_0^T \beta(t)dt\right) + \Phi^{f\leftarrow i}_{\text{smooth}}(y), \quad (10)$$

where $\Phi^{f\leftarrow i}_{\text{smooth}}(y)$ is the smooth Fourier transform of the last term in Eq. (9). With the contributions from the smooth term vanishing in the limit $\alpha \to \infty$ [16] the monitored system is seen to undergo unitary evolution with a reduced Hamiltonian $\hat{P}_n\hat{H}\hat{P}_n$ in the subspaces corresponding to each of the distinct eigenvalues A_n, $n = 1,2,\ldots$. As in the case of the Zeno effect caused by frequent observations [1–3], an accurate finite-time measurement suppresses transitions between different subspaces.

Taking trace over the system's variables we find the probability distribution for the functional (3)

$$\lim_{\alpha\to\infty} W^i(y,T) \equiv \sum_f |\Psi^{f\leftarrow i}(y,T)|^2$$
$$\approx \sum_n \delta\left(y - A_n \int_0^T \beta(t)dt\right) \langle i|\hat{P}_n|i\rangle. \tag{11}$$

In particular, a strongly observed system starting in a a state corresponding to a nondegenerate eigenvalue $|i\rangle = |a_M\rangle$, $\hat{P}_M = |a_M\rangle\langle a_M|$ would follow a constant Feynman path $A(t) = A_M$, thus having the time average of $\hat{A}$ exactly equal to A_M, and spending all available time in its initial state, $\tau_M = T$. This failure to find real evidence of the irregular virtual motion suggested in Eq. (1) by separating Feynman paths into ever narrower exclusive classes according to the value of a functional (3) constitutes the finite-time Zeno effect and is the first result of this paper.

IV. AN ERGODIC PROPERTY: FINITE ACCURACY AND LARGE DURATION

Next we show that the evidence of the virtual motion is recovered if one performs a long measurement of an arbitrarily high but fixed accuracy, $\alpha < \infty$ and $T \to \infty$. For simplicity we consider the time average of A, thus choosing $\beta(t) = 1/T = \text{const}$. Changing the variables in Eq. (7), $z = \lambda/T$, and using spectral representation for the operator $\hat{H} + z\hat{A}$ yields

$$\Phi^{f\leftarrow i}(y) = (2\pi)^{-1} T \sum_{n=1}^{N} \int \langle f|\psi_n(z)\rangle \exp\{i[zy - \mathcal{E}_n(z)]T\}$$
$$\times \langle \psi_n(z)|i\rangle dz, \tag{12}$$

where $(\hat{H} + z\hat{A})|\psi_n(z)\rangle = \mathcal{E}_n(z)|\psi_n(z)\rangle$. The long time behavior of $\Phi^{f\leftarrow i}(y)$ is now determined by the critical points $z_n^s(y)$ of the exponent in Eq. (12),

$$\partial_z \mathcal{E}_n(z)_{z=z_n^s} = y. \tag{13}$$

Evaluating the integrals in Eq. (12) by the stationary phase method yields N rapidly oscillating contributions containing factors $\exp[iS_n(y)T]$ with the phases given by the Legendre transforms of $\mathcal{E}_n(z)$, $S_n(y) = z_n^s y - \mathcal{E}_n(z_n^s)$. The critical points y_n^s of $S_n(y)$ are determined by the condition $z_n^s(y_n^s) = 0$ so that from Eq. (13) we have $y_n^s = \partial_z \mathcal{E}_n(z)|_{z=0}$. Calculating the derivatives with the help of the perturbation theory and evaluating the integrals in Eq. (5) by the stationary phase method, we find

$$\lim_{T\to\infty} \Psi^{f\leftarrow i}(y)$$
$$= \sum_n G_\alpha(y - \langle\phi_n|A|\phi_n\rangle)\langle f|\phi_n\rangle \exp(-iE_nT)\langle\phi_n|i\rangle, \tag{14}$$

where $\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$. Extension to mixed states is straightforward. For an initial state $\hat{\rho}_i$, all E_n nondegenerate and all $\langle\phi_n|A|\phi_n\rangle$ distinct, the probability distribution of the meter's readings is given by

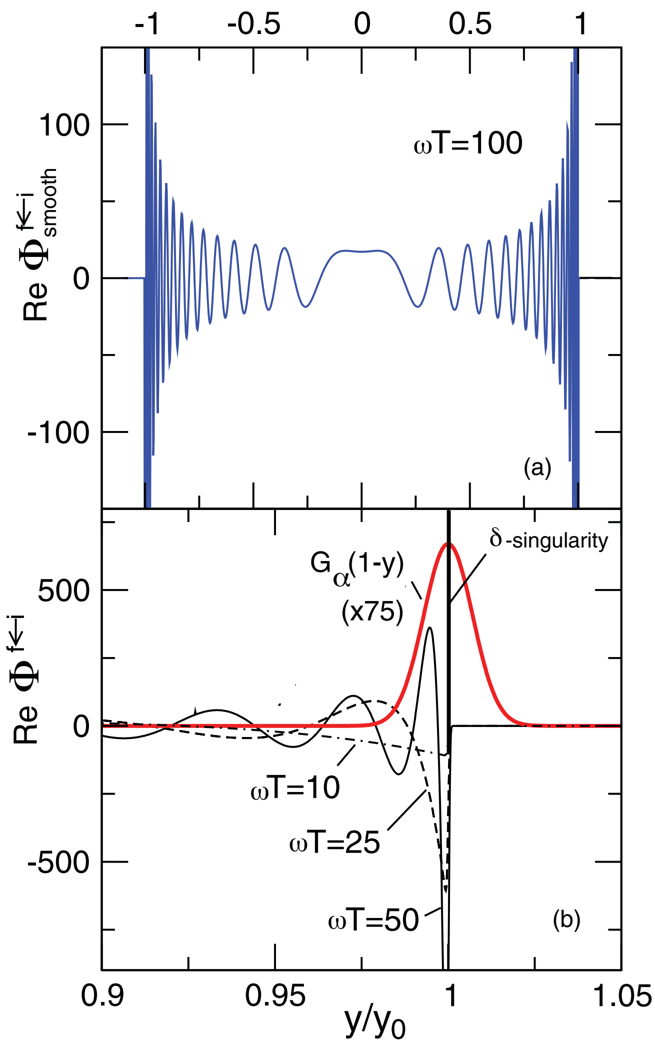

FIG. 1. (Color online) (a) Amplitude distribution $\Phi^{f\leftarrow i}_{\text{smooth}}(y,T)$ for $|i\rangle = |1\rangle$, $|f\rangle = |+\rangle$, $\hat{H} = \omega\hat{\sigma}_z$, $\hat{A} = \hat{\sigma}_y$, and $\omega T = 100$; (b) $\Phi^{f\leftarrow i}_{\text{smooth}}(y,T)$ for $\omega T = 50$ (solid line), $\omega T = 25$ (dashed line), and $\omega T = 10$ (dot-dashed line). Also shown is $G_\alpha(1-y)$ ($\times 75$) for $\alpha = 100$ (thick solid line).

$$\lim_{\alpha\to\infty}\lim_{T\to\infty} W^i(y,T) = \sum_n \delta(y - \langle\phi_n|\hat{A}|\phi_n\rangle)\langle\phi_n|\hat{\rho}_i|\phi_n\rangle, \tag{15}$$

where the order in which the limits are taken is essential. We note that a strongly observed system prepared with a known energy E_k, $\langle\phi_n|\hat{\rho}_i|\phi_n\rangle = \delta_{kn}$ explores its Hilbert space in such a way that the long-time time average of a dynamical variable A is sharply defined, with the value equal to the ensemble average in the projection von Neumann measurement of the operator $\hat{A}$, $\langle\phi_k|\hat{A}|\phi_k\rangle$. In particular, as $T \to \infty$, the fraction of time a system prepared in a pure state $|\phi_k\rangle$ spends in a subspace Ω spanned by the subset of eigenvectors $\{|a_m\rangle, m \in \Omega\}$ tends to the measure of the subset $\mu_\Omega = \sum_{m\in\Omega} |\langle a_m|\phi_k\rangle|^2$. This ergodic-like [17] property of bound quantum motion, to our knowledge not yet discussed in literature, is the second result of this paper. Further, as seen

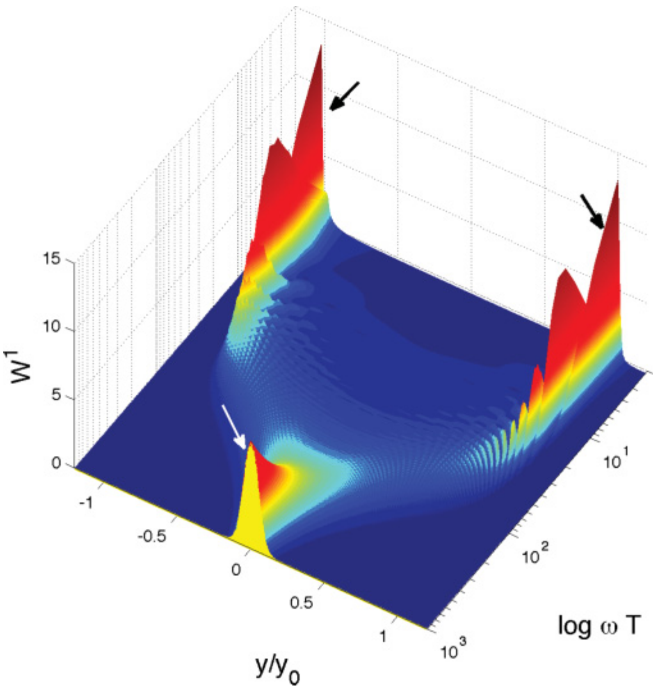

FIG. 2. (Color online) Probability distribution of the time average of the y-component of the spin $\hat{A} = \hat{\sigma}_y$, for a two-level system with the Hamiltonian $\hat{H} = \omega\hat{\sigma}_x$ vs. the averaging time T. The system is prepared in the eigenstate $|1\rangle$, $\hat{H}|1\rangle = \omega|1\rangle$, and the accuracy of the measurement is $\Delta y/y_0 = 0.05$. The ergodic peak and two Zeno peaks are indicated by white and black arrows, respectively. (Note the logarithmic scale used for the T-axis.).

from Eq. (14), a system starting in a mixed state $\hat{\rho}_i$, undergoes relaxation to the same steady-state diagonal in the energy representation $\hat{\rho}_f = \sum_n |\phi_n\rangle\langle\phi_n|\hat{\rho}_i|\phi_n\rangle\langle\phi_n|$, regardless of the choice of $\hat{A}$. Thus the finite-time average of a quantity A, $\langle A\rangle_T \equiv \int yW^i(y,T)dy$, tends, as $T \to \infty$, to the von Neumann ensemble average $Tr\{\hat{A}\hat{\rho}_f\}$. As in the case of a frequently observed system [12] the equivalence between the time and ensemble averages is established in the state produced at the end of measurement $\hat{\rho}_f$, rather that in the initial state $\hat{\rho}_i$.

V. AN EXAMPLE: SPIN-1/2 IN A MAGNETIC FIELD

We illustrate the above with a simple example, where one wishes to measure the time average of the y-component of a spin 1/2, $\hat{A} \equiv \hat{\sigma}_y$, for a two-level system with Hamiltonian $\hat{H} = \omega\hat{\sigma}_x$ prepared in its eigenstate $|1\rangle$, $\hat{H}|1,2\rangle = \pm\omega|1,2\rangle$. Choosing in Eq. (3) $\beta =$ const we then convert to dimensionless variables, $T \to \omega T$, $y \to y/y_0$, $y_0 \equiv \beta T$, $\beta \to 1/T$.

As in Ref. [8], $G(y)$ in Eq. (5) is chosen to be a Gaussian,

$$G_\alpha(y) = (2/\pi)^{1/4}(\alpha)^{1/2}\exp(-\alpha^2 y^2). \qquad (16)$$

Figure 1(a) shows $\Phi^{+\leftarrow 1}_{\text{smooth}}(y)$, $\hat{\sigma}_z|\pm\rangle = \pm|\pm\rangle$, for $\omega T = 100$, with the stationary region clearly seen around $y^s_2 = y^s_1 = \langle 1|\hat{\sigma}_y|1\rangle = 0$. Two Zeno peaks at $y/y_0 = \pm 1$ predicted by Eq. (11) and a single "ergodic" (for want of a better word) peak predicted by Eq. (15) are shown in Fig. 2. The transition between the regimes described by Eqs. (11) and (15) occurs as the Zeno peaks, however narrow G_α may be, are eventually canceled by the contributions from $\Phi^{f\leftarrow i}_{\text{smooth}}(y)$, whose oscillations become more rapid as the time T increases. This is illustrated in Fig. 1(b), for the spin-1/2 system described above, with $\Psi^{+\leftarrow 1}(y=1)$ given by Eq. (5). For $\omega T = 10$ the contribution from $\Phi^{+\leftarrow 1}_{\text{smooth}}(y)$ (dot-dashed line) is negligible, and the Zeno peak is formed by the δ singularity of $\Phi^{+\leftarrow 1}$ [cf. Eq. (10)] at $y = 1$. For $\omega T = 50$ the contribution of the singular term is largely canceled by the first negative oscillation of $\Phi^{+\leftarrow 1}_{\text{smooth}}(y)$ (solid line) that fits under the Gaussian $G_\alpha(1-y)$ (thick solid line), thus making $\Psi^{+\leftarrow 1}(y=1)$ negligible.

VI. CONCLUSION AND DISCUSSION

In summary, we have considered a general finite-time measurement based on separating Feynman paths into exclusive classes according to the value of a functional such as time average of a dynamical variable A, $\langle A\rangle_T$. We have shown the following. (i) A highly accurate measurement of a fixed duration traps the measured system in the eigenstates (eigensubspaces) of the corresponding operator $\hat{A}$. (ii) For any quantity A, a prolonged measurement of an arbitrary but fixed accuracy destroys coherences in the energy representation, thus leaving the system in a steady state with $\langle A\rangle_T$ equal to the von Neumann projection average of $\hat{A}$. In the special case of a system prepared in a pure stationary state, $\langle A\rangle_T$ is sharply defined, and the proportion of time spent in a given subspace Ω is exactly equal to the von Neumann probability to find the system there. Both effects are readily explained in terms of singularities and critical points of the corresponding amplitude distribution. Finally, these results have applications beyond the von Neumann meter model we used for our derivation. Central to our analysis is the convolution formula (5) in which an amplitude distribution describing an uncoupled system is projected onto an "apparatus function," whose shape determines the accuracy of the measurement. Equation (5) arises in a much wider context of measurement situations, for example, for a BEC meter (cf. Eq. (10) of Ref. [8]), when the electrons or cold atoms experience spin-orbit coupling (cf. Eq. (5) of Ref. [18]) and in the case of "self-measurement," such as wave-packet tunneling where no external meter is employed (see Ref. [19] and references therein).

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[15] Consider, as $\gamma \to \infty$, $\exp[-i(\hat{B}+\gamma\hat{A})] = \{\exp[-i(\hat{A}+\hat{B}/\gamma)]\}^{\gamma}$. Let $|\phi_{nk}\rangle$ be the orthogonal basis spanning the nth degenerate subspace of $\hat{A}$, $\hat{A}|\phi_{nk}\rangle = A_n|\phi_{nk}\rangle$, in which $\hat{B}$ is diagonal with nondegenerate eigenvalues B_{nk}, $\hat{B}|\phi_{nk}\rangle = B_{nk}|\phi_{nk}\rangle$. By the perturbation theory, the eigenvalues of $\hat{A}+\hat{B}/\gamma$ are given by $A_n + B_{nk}/\gamma + O(1/\gamma^2)$ while the eigenstates, unchanged to the zeroth order in $1/\gamma$, are $|\phi_{nk}\rangle + O(1/\gamma)$. Inserting the above into the spectral decomposition of $\exp\{\hat{A}+\hat{B}/\gamma\}$ and raising it to the power of γ yields the required result $\lim_{\gamma\to\infty}\exp[-i(\hat{B}+\gamma\hat{A})] = \sum_n \exp(-i\gamma A_n)\sum_k |\phi_{nk}\rangle \exp(-iB_{nk})\langle\phi_{nk}| + O(1/\gamma)$, where the second sum is just $\exp(-i\hat{B})$ acting on the nth subspace, $\exp(-i\hat{P}_n\hat{B}\hat{P}_n)\hat{P}_n$.
[16] More precisely, by Parceval's theorem $\Phi^{f\leftarrow i}_{\text{smooth}}(y)$ is a square integrable function, $\int|\Phi^{f\leftarrow i}_{\text{smooth}}(y)^2|dy < \infty$. Therefore $|\Phi^{f\leftarrow i}_{\text{smooth}}(y)^2|$ can have, at most, integrable singularities which would be even weaker for $|\Phi^{f\leftarrow i}_{\text{smooth}}(y)|$. Thus the integrals $\int G_\alpha(y-y')\Phi^{f\leftarrow i}_{\text{smooth}}(y')dy'$ are finite and would vanish for $\alpha\to\infty$.
[17] We do not address the macroscopic aspect central to most studies of quantum ergodicity [9,11]. We do, however, address the nontrivial statistical behavior of an observed completely specified quantum system, something not considered if an unperturbed Schrödinger evolution is assumed [12].
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