# Propagator for the general time-dependent harmonic oscillator with application to an ion trap

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We present the simplest possible formula for the propagator of the general time-dependent quadratic Hamiltonian, including linear terms. The method is based on the use of a *linear* time-dependent invariant and requires only the solution of a *linear* homogeneous second-order ordinary differential equation corresponding to the classical quadratic Hamiltonian. We give an example for the case of the Paul trap.

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## I. INTRODUCTION

The harmonic oscillator is a fundamental model both in classical and in quantum mechanics. It is one of the few models that can be solved analytically and leads to many important insights. The solution of the quantum harmonic oscillator goes back to Schrödinger [1] where constant mass and frequency were used. Many papers addressed the issue of time-dependent mass and frequency [2,3]. The general time-dependent Hamiltonian is a key model in quantum optics, for it shows the general attributes of squeezing. Quadratic time-dependent Hamiltonians have many applications in many branches of physics (see, e.g., Ref. [4]). In particular, time-dependent frequency is encountered in ion traps; this is due to the fact that the electric potential can have no local minima; thus, in order to trap an ion one must push-pull the ion, resulting in time-dependent frequency (see, e.g., Refs. [5,6]). Using such traps, a number of coherent and squeezed states have been prepared in the vibrational motion of ions [7,8].

Many articles on this topic follow the quantum invariants method used by Lewis and Riesenfeld [9] to solve the time-dependent mass and frequency Hamiltonian. This invariant is quite complicated as it is nonlinear. Although a linear invariant was used for the time-dependent harmonic oscillator [4], Glauber's method [10], which uses complex initial conditions, simplifies the result significantly. Another approach has used canonical transformation (see, e.g., Leach [11], Yuen [12], and Brown [13]). Other works have analyzed the time-dependent harmonic oscillator by deriving a set of coupled differential equations for the evolution operator. In particular, Dodonov and Manko [2] have developed extensively the analysis of time-dependent harmonic oscillators.

In this article, a simple approach is presented to the general quantum quadratic Hamiltonian. It is a generalization of Glauber's method [10] and is based on the method of quantum invariants [2,9]. A simple and elegant *linear* invariant is constructed using Abel's identity [14] which allows us to solve the general quadratic time-dependent Hamiltonian. The method allows us to express the general wave functions in terms of Hermite polynomials and thus using Mehler's formula [15] to obtain the propagator of the general time-dependent Hamiltonian in a simple closed form. The method includes the case of linear terms in the Hamiltonian with no further complexity. In the present article we solve the general quadratic Hamiltonian and use it to solve a particular case relevant to ion traps. In Glauber's analysis for the Paul trap, a complete set of solutions is given; however, the general propagator is absent. This leads us to choose the example in the paper where an initial state which is not one of the complete set of solutions is propagated.

## II. GENERALIZATION OF GLAUBER'S INVARIANT AND THE GENERAL PROPAGATOR

We consider the Hamiltonian

$$\hat{H} = a(t)\hat{X}^2 + b(t)(\hat{X}\hat{P} + \hat{P}\hat{X}) + c(t)\hat{P}^2 + d(t)\hat{X} + e(t)\hat{P}, \tag{1}$$

with $a,b,d,e$ being arbitrary real functions of time and $c$ being a real positive function of time. A special case of this Hamiltonian ($b = e = d = 0,\ c = \text{const.}$) was addressed by Glauber [10]. In his treatment, a simple and elegant *linear* invariant which satisfies ladder commutation relations was introduced. We follow a similar path with the general Hamiltonian.

First, we examine the case with no linear terms ($d = e = 0$). The classical equation for the complex function $u(t)$ corresponding to the Heisenberg equation for $\hat{X}$, given the Hamiltonian (1) with no linear terms, is

$$\ddot{u} = \frac{\dot{c}}{c}\dot{u} + \left(4b^2 + 2\dot{b} - 2\frac{\dot{c}}{c}b - 4ca\right)u. \tag{2}$$

We look for the particular solution of this ordinary differential equation (ODE) satisfying

$$u(0) = 1, \quad \dot{u}(0) = i\nu, \quad \nu > 0. \tag{3}$$

Once this classical solution $u(t)$ of the second-order linear homogeneous differential equation is found, it is possible to write both the wave functions and the *propagator* in a closed form. Equation (2) is of the form

$$\ddot{u}(t) = \alpha(t)\dot{u}(t) + \gamma(t)u(t). \tag{4}$$

With the initial conditions (3), this ODE has, using Abel's identity [14], the following classical invariant:

$$W[u,u^*](t) = [u(t)\dot{u}^*(t) - \dot{u}(t)u^*(t)]e^{-\int_0^t \alpha(t)dx} = -2i\nu. \tag{5}$$

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Following Glauber [10] we thus define the invariant operator

$$\hat{C}(t) = i[4\hbar\nu c(0)]^{-\frac{1}{2}}\{u\dot{\hat{X}} - \dot{u}\hat{X}\}\frac{c(0)}{c(t)} = i\frac{(2bu-\dot{u})\hat{X} + 2cu\hat{P}}{2\sqrt{\hbar\nu c(0)}}\frac{c(0)}{c(t)}, \tag{6}$$

where the Heisenberg equation for $\dot{\hat{X}}$ was used. It satisfies

$$i\hbar\frac{d}{dt}\hat{C} = i\hbar\partial_t\hat{C} + [\hat{C},\hat{H}] = 0 \tag{7}$$

and the usual commutation relation

$$[\hat{C}(t),\hat{C}^\dagger(t)] = 1. \tag{8}$$

In the limit of a time-independent Hamiltonian, Eq. (6) reduces (with the choice $\nu = \omega$) to the standard lowering operator. One should notice that the parameter $a(t)$ does not appear explicitly in Eq. (6) but appears in the calculation of $u(t)$ according to Eq. (2). A complete set is then given by

$$|n,\nu,t\rangle = \frac{1}{\sqrt{n!}}[\hat{C}^\dagger(t)]^n|0,\nu,t\rangle, \tag{9}$$

where

$$\hat{C}(t)|0,\nu,t\rangle = 0. \tag{10}$$

Writing the ladder operators in the $x$ representation and solving for the wave functions in the usual manner we get

$$\psi_{n,\nu}(x,t) = \langle x|n,\nu,t\rangle = \frac{1}{\sqrt{n!}}\left[\frac{\nu}{2\pi\hbar c(0)u(t)^2}\right]^{\frac{1}{4}} \times \left[\frac{u^*(t)}{2u(t)}\right]^{\frac{n}{2}}\exp\left(\frac{2b(t)u(t)-\dot{u}(t)}{4i\hbar c(t)u(t)}x^2\right) \times H_n\left(\sqrt{\frac{\nu}{2\hbar c(0)|u(t)|^2}}x\right). \tag{11}$$

These wave functions form a ($\nu$-dependent) complete set satisfying the time-dependent Schrödinger equation

$$(i\hbar\partial_t - \hat{H})\psi_{n,\nu}(x,t) = 0. \tag{12}$$

The time evolution operator is given by

$$\hat{U}(t_2,t_1) = \Sigma|n,\nu,t_2\rangle\langle n,\nu,t_1|, \tag{13}$$

and therefore the propagator in the $x$ representation is

$$G(x_2,t_2;x_1,t_1) \equiv \langle x_2|\hat{U}(t_2,t_1)|x_1\rangle = \sum_n \psi_{n,\nu}(x_2,t_2)\psi^*_{n,\nu}(x_1,t_1). \tag{14}$$

Using Eq. (11) and Mehler's formula (see, e.g., Ref. [15]), we obtain for the quadratic Hamiltonian

$$G_0(x_2,t_2;x_1,t_1) = A(t_1,t_2)\exp\left(\sum_{i,j=1}^{2} x_i\alpha_{ij}(t_2,t_1)x_j\right), \tag{15}$$

with

$$A = \left(\frac{\nu}{4i\pi\hbar c(0)\mathrm{Im}(u(t_2)u^*(t_1))}\right)^{1/2}, \tag{16}$$

$$\alpha_{22}(t_2,t_1) = \alpha^*_{11}(t_1,t_2) = \frac{-\nu u(t_1)}{4i\hbar c(0)u(t_2)\mathrm{Im}\,(u(t_2)u^*(t_1))} + \frac{b(t_2)}{2i\hbar c(t_2)} - \frac{\dot{u}(t_2)}{4i\hbar c(t_2)u(t_2)}$$
$$\alpha_{21} = \alpha_{12} = \frac{\nu}{4i\hbar c(0)\mathrm{Im}\,(u(t_2)u^*(t_1))}. \tag{17}$$

Although $\nu$ appears in Eqs. (16) and (17) it should be noted that $G$ is independent of the choice of $\nu$ [although the $\psi_{n,\nu}$ of Eq. (11) are not]. In particular, $\nu$ may be chosen to simplify $u$ (see, e.g., Glauber's choice [10] $\nu = \omega$ for the time-independent harmonic oscillator).

In order to include the linear terms as well, we define the *linear* quantum invariant (time dependence is left implicit)

$$\hat{C}_l(t) = \frac{i}{2}\sqrt{\frac{c(0)}{\hbar\nu}}\left(\frac{u\dot{\hat{X}} - \dot{u}\hat{X}}{c(t)} + \delta(t)\right) = \frac{i}{2}\sqrt{\frac{c(0)}{\hbar\nu}}\left(\frac{2bu-\dot{u}}{c}\hat{X} + 2u\hat{P} + \delta\right), \tag{18}$$

where

$$\delta(t) = \frac{ue}{c} - \zeta(t),$$
$$\zeta(t) = \int_0^t \frac{u}{c}\left[\dot{e} + e\left(2b - \frac{\dot{c}}{c}\right) - 2cd\right]dt', \tag{19}$$

and where $u$ is defined again in (2). $\hat{X}$ and $\hat{P}$ satisfy the Heisenberg equations with the linear terms. It is easy to verify that $\hat{C}_l$ is invariant and that the standard ladder commutation relation (8) holds. Writing the ladder operators in the $x$ representation and solving for the wave functions we get

$$\psi_{n,\nu}(x,t) = \langle x|n,\nu,t\rangle = \frac{1}{\sqrt{n!}}r(t)u^{-1/2}\left(\frac{u^*}{2u}\right)^{\frac{n}{2}} \times \exp\left(\frac{2bu-\dot{u}}{4i\hbar cu}\left(x + \frac{c\delta}{2bu-\dot{u}}\right)^2\right) \times H_n\left(\sqrt{\frac{\nu}{2\hbar c(0)|u|^2}}(x+\beta)\right), \tag{20}$$

where

$$r(t) = \left[\frac{\nu}{2\pi\hbar c(0)}\right]^{\frac{1}{4}}\exp\left(\frac{ic\delta^2}{4\hbar u(2bu-\dot{u})} + s(t)\right),$$
$$\beta(t) = \frac{c(0)}{\nu}\mathrm{Im}(u^*(t)\zeta(t)), \tag{21}$$
$$s(t) = \frac{i}{4\hbar}\int_0^t\left(\frac{e^2}{c} - \frac{c\zeta^2}{u^2}\right)dt'.$$

Using Mehler's formula again we thus get for the propagator $G(x_2,t_2;x_1,t_1)$

$$G = G_0(x_2,t_2;x_1,t_1)B(t_2,t_1)\exp\left(\sum_{i=1}^{2}\gamma_i(t_2,t_1)x_i\right), \tag{22}$$

$$B = \exp\left(\nu\frac{2\beta(t_2)\beta(t_1) - \beta^2(t_2)\frac{u(t_1)}{u(t_2)} - \beta^2(t_1)\frac{u^*(t_2)}{u^*(t_1)}}{4ic(0)\hbar\mathrm{Im}\,(u(t_2)u^*(t_1))}\right) \times \exp(s(t_2) + s^*(t_1)), \tag{23a}$$

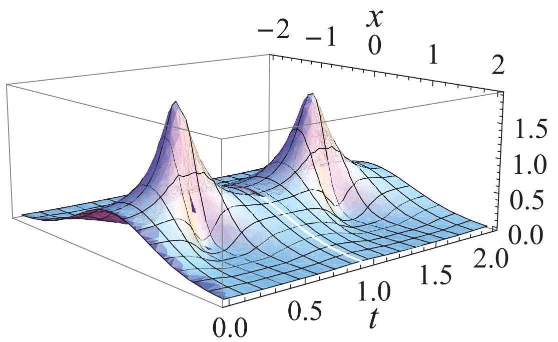


FIG. 1. (Color online) Evolution of the state given by Eqs. (25)–(27), where $|\psi(x,t)|^2$ is plotted in $(x,t)$ diagram. The "breathing" effect is shown.

$$\gamma_2(t_2,t_1) = \gamma_1^*(t_1,t_2) = \frac{\delta(t_2)}{2i\hbar u(t_2)} + \frac{\nu\left[\beta(t_1) - \beta(t_2)\frac{u(t_1)}{u(t_2)}\right]}{2i\hbar c(0)\mathrm{Im}(u(t_2)u^*(t_1))}. \tag{23b}$$

It can be checked that the general result for the propagator in the present article reduces to special known cases. For the case of the simple harmonic oscillator and for the forced harmonic oscillator [simple harmonic oscillator with time-dependent force term $d(t)$] one can verify that the general propagator of Eq. (22) reduces to the well-known propagators [15,16] using the same $u$ for both cases:

$$u(t) = \cos(\omega t) + i\frac{\nu}{\omega}\sin(\omega t). \tag{24}$$

For a particle in a linear potential ($a = b = e = 0,\ c,d =$ const.), $u(t) = 1 + i\nu t$ and we obtain the well-known propagators [16]. We note that our method yields the propagator also for problems in the continuum (see, e.g., the two last mentioned cases and the case $a < 0$, which is relevant for tunneling [17]).

## III. PROPAGATOR FOR THE PAUL TRAP

For the Paul trap, $\hat{H}$ is given by

$$a = \frac{\omega^2}{2m}[a_x + q_x\cos(\omega t)], \quad b = d = e = 0, \quad c = \frac{1}{2m}, \tag{25}$$

and the function $u$ is the solution of the Mathieu equation with the initial conditions (3) (see, e.g., Mathematica). This gives an analytic formula for the propagator, and the propagation of, for example, any Gaussian initial state may be carried out analytically. To see the behavior, we show in Fig. 1 the propagation of a Gaussian with

$$m = 1/2, \quad \omega = 1, \quad a_x = 7, \quad q_x = 1, \quad \nu = 2. \tag{26}$$

The chosen initial state is

$$\psi(x,0) = \frac{1}{\pi^{1/4}}e^{-\frac{x^2}{2}}, \tag{27}$$

which is not one of the $\psi_{n,\nu}$. The evolution of this state is shown in Fig. 1 where $|\psi(x,t)|^2$ is plotted. A significant "breathing" effect is obtained in the evolution of the wave function (i.e., the wave function becomes broader and narrower as function of time).

## IV. CONCLUSION

In Ref. [13], first-order transition amplitudes have been obtained by assuming a simple harmonic oscillator at times $t = -\infty$ and $t = +\infty$ and a harmonic oscillator with time-dependent parameters between these times. Our method enables us to get the *exact* transition amplitudes without the use of perturbation theory at any time. Such transition amplitudes can be calculated in our method as

$$T_{n,n'}(t,t') = \int dx \int dx'\Psi_n(x)G(x,t;x',t')\Psi^*_{n'}(x'). \tag{28}$$

Here, $\Psi_n(x)$ and $\Psi^*_{n'}(x')$ are the corresponding simple harmonic oscillator wave functions and $G(x,t;x',t')$ is the propagator given by Eq. (15) (without linear terms) or, more generally, by Eq. (22). One should notice the simplicity of our propagators relative to those obtained by other authors. It is especially useful for cases in which the initial state is not one of the $\psi_{n,\nu}(x,t)$.

Let us summarize our results as follows: Using the method developed in the present article we give in Eq. (11) a general $\nu$-dependent solution for the wave functions of the time-dependent quadratic Hamiltonian (without linear terms) using a *linear* invariant. Using Mehler's formula a general propagator for this Hamiltonian was obtained, independent of the choice of $\nu$, in Eq. (15). The calculation of the wave functions was then generalized and the propagator for time-dependent Hamiltonians including linear terms [Eqs. (20) and (22), respectively] by generalizing the invariant as given in Eq. (18). The importance of the linear terms is immediate. The linear term in $\hat{X}$ represents the external force applied, whereas the linear term in $\hat{P}$ appears most commonly in electromagnetic (EM) interactions. One of the beautiful aspects of Glauber's method is that it allows the calculation of the propagator of *any* quadratic (plus linear) Hamiltonian (even if one expects it to be in the continuum). This is because the method yields a *complete* (time-dependent) set of states which can be used to obtain the propagator, irrespective of the fact that these states may be very different from the eigenstates of $\hat{H}$ at any particular time. Finally, we note that our method yields the propagator in terms of the basic parameters of the Hamiltonian, without any need for transformations of variables or of canonical transformations or calculation of the classical action.

There is a lot of scientific activity in this field (see, e.g., Refs. [2,4,18–22]) but use of Glauber's method for the general time-dependent Hamiltonian (with linear terms) seems to have been exploited only in the present work. Our general results for the propagator of the general time-dependent Hamiltonian, including time-dependent linear terms, should be useful for various physical problems (see, e.g., Ref. [4]).

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