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Universal phase structure of dilute Bose gases with Rashba spin-orbit coupling

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A Bose gas subject to a light-induced Rashba spin-orbit coupling possesses a dispersion minimum on a circle in momentum space; we show that kinematic constraints due to this dispersion cause interactions to renormalize to universal, angle-dependent values that govern the phase structure in the dilute-gas limit. We find that, regardless of microscopic interactions, (a) the ground state involves condensation at two opposite momenta and is, in finite systems, a fragmented condensate and and (b) there is a nonzero-temperature instability toward the condensation of pairs of bosons. We discuss how our results can be reconciled with the qualitatively different mean-field phase diagram, which is appropriate for dense gases.

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The advent of ultracold gases has vastly increased the range of physically realizable many-body bosonic systems, enabling the exploration of quantum-degenerate Bose gases possessing tunable interactions and band structure as well as internal degrees of freedom. Among such systems, those of particular interest involve single-particle Hamiltonians having degenerate ground states related by symmetries. Bose-Einstein condensation (BEC), i.e., the macroscopic occupation of a *particular* single-particle state, typically entails breaking these symmetries; hence the order parameter space and defects of such BECs are richer than those of conventional BECs. For instance, spin-1 BECs [1] support fractionally quantized vortices, and in this sense resemble exotic *fermionic* condensates such as triplet superconductors.

Just as these exotic defects stem from broken *internal* symmetries, those of the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state, such as vortex-dislocation bound states [2], stem from its broken translational and rotational symmetries. The present work addresses purely bosonic analogs of the FFLO states, in which the degenerate single-particle ground states have distinct *spatial* wave functions. In particular, we consider the case in which the single-particle Hamiltonian possesses a dispersion minimum on a circle in momentum space, so that BEC occurs at one or more nonzero momenta. Our work is motivated by a recently proposed realization of such a Hamiltonian, viz., a spin-$\frac{1}{2}$ Bose gas subject to a light-induced Rashba spin-orbit coupling [3]. Simpler forms of spin-orbit coupling, having multiple discrete minima, have been experimentally demonstrated [4]. An alternative approach to realizing a circular dispersion minimum would be to load the atoms into the excited band of an optical lattice; in this case, too, multiple discrete minima have been realized [5], and under appropriate conditions (e.g., "sublattice-exchange-even faulted" stackings of bilayer honeycomb lattices [6]) continuous minima are realizable.

Spin-orbit-coupled BECs were originally addressed in Refs. [7,8] as examples of unconventional condensation; it was argued in Ref. [8] that, for a pure Rashba coupling and *isotropic* interactions, a fragmented condensate should form. More recently, the case of the Rashba-coupled BEC was treated using mean-field theory [9] and incorporating Gaussian fluctuations [10]; related systems have been studied in Refs. [11]. In general, two phases have been found, depending on the spin dependence of interactions: a time-reversal symmetry-breaking (TRSB) state and a density-wave state. In the present work, we describe how interaction-renormalization effects *qualitatively* change the phase diagram at low densities (see Fig. 1), destabilizing the TRSB state and giving rise to a number-squeezed (and, in finite systems, "fragmented") limit of the density-wave state. These changes are due to the strong, *emergent* angle dependence of renormalized interactions; such angle-dependent renormalizations are generic in systems whose low-energy modes occur around momentum-space surfaces, e.g., Fermi liquids [12]. Our results, while consistent with those of Ref. [8] in the special case of isotropic interactions, hold more generally for *any* interactions that are repulsive in all angular momentum channels.

Our primary results are as follows. At zero temperature, we find, exploiting the properties of a quantum critical point introduced in Ref. [13], that the renormalized interactions for a dilute gas *universally* favor a state in which the BEC forms at a pair of opposite momenta. For *finite*, weakly trapped systems, fragmented BEC is energetically favored over simple BEC at either a single momentum or a coherent momentum superposition such as a density wave. In the thermodynamic limit, the fragmented BEC, though favored over a *coherent* superposition, becomes degenerate with *squeezed* states that break translational symmetry. The resulting ground-state energy *per particle* scales unusually with the density n, i.e., as $n^{4/3}$; note that this scaling is the same as that of the "extremely anisotropic Wigner crystal" [14], which, in fact, approaches the fragmented state in the zero-density limit. At nonzero temperature, we argue using renormalization-group (RG) methods that the leading instability is toward condensation of boson *pairs* and estimate the condensation temperature.

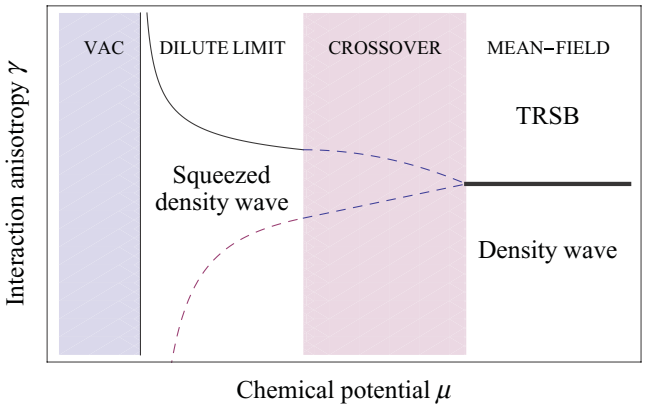

FIG. 1. (Color online) Zero-temperature phase diagram as a function of the interaction anisotropy $\gamma \equiv c_2/c_0$ and the chemical potential μ, showing the phases and transitions discussed in the main text. The (dashed) phase boundaries in the crossover region are schematic; the thick line indicates a first-order transition predicted by mean-field theory [9].

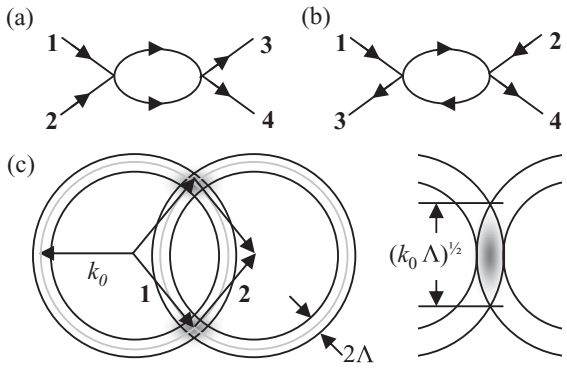

FIG. 2. (a) Loop correction in the particle-particle channel, which governs the hierarchy of couplings at the QCP. (b) Loop correction in the particle-hole channel. These corrections vanish at $T = 0$. (c) Kinematic constraints due to the dispersion structure: (left) case of $\theta \neq 0$, outgoing momenta are constrained to lie in the shaded region, of area $\sim\Lambda^2$; (right) case of $\theta = 0$, for which the shaded region's area scales as $\Lambda\sqrt{k_0\Lambda}$.

Model. We begin with the following effective model of a d-dimensional Bose gas having a circular dispersion minimum:

$$H = \int d^d k\, \Psi^\dagger(\mathbf{k})\left[-\mu + \frac{1}{2M}\{(|\mathbf{k}_{2D}| - k_0)^2 + k_\perp^2\}\right]\Psi(\mathbf{k}) + \int \prod_{i=1}^{4} d^d k_i\, U(\{\mathbf{k}_i\})\Psi^\dagger(\mathbf{k}_1)\Psi^\dagger(\mathbf{k}_2)\Psi(\mathbf{k}_3) \times \Psi(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3), \quad (1)$$

where $\Psi(\mathbf{k}_i)$ are Bose fields of momentum $\mathbf{k}_i$, $\mathbf{k}_{2D} \equiv (k_x, k_y)$, $k_\perp$ encodes all other momentum components, U is a possibly momentum-dependent interaction, and we have set $\hbar = 1$. For Rashba-coupled bosons, *spin*-dependent interactions in the microscopic model imply momentum-dependent interactions because, for modes near k_0, the spin is locked to the momentum. We shall first consider the universal properties of the general Hamiltonian H and then relate these to the phases of the specific microscopic model considered in Ref. [9]. We focus primarily on the two-dimensional (2D) case, in which $k_\perp = 0$, and then touch on the (similar) three-dimensional (3D) case.

We assume that energies associated with temperature T, chemical potential μ, system size, etc., are smaller than the spin-orbit coupling scale $k_0^2/2M$. Typical values of k_0^{-1} are on the order of an optical wavelength [4], which is exceeded by the interparticle spacing in many experiments ($k_0^2/2M$ cannot be *smaller* than these scales if spin-orbit coupling is to play a significant role).

As we are concerned with the low-energy limit, it is convenient to study only the degrees of freedom in a momentum shell of thickness 2Λ centered on the dispersion minimum, giving rise to an energy scale $\Omega_\Lambda \equiv \Lambda^2/2M$ intermediate between $k_0/2M$ and the low-energy scales μ and T. Integrating out degrees of freedom with energies $\geqslant\Omega_\Lambda$ generates effective interactions for modes with energies $\leqslant\Omega_\Lambda$; as we discuss below, these interactions are further renormalized and, for energies $\ll\Omega_\Lambda$, take on universal values that are independent of Λ. A careful treatment of the high-energy renormalization, including the other Rashba bands, appears in a recent paper [15] and confirms this picture.

Quantum critical point. The model described by H has a quantum critical point (QCP) at $\mu = 0$, corresponding to the phase transition from the empty vacuum to a BEC. This QCP was analyzed in Ref. [13] *for fermions*, but the analysis extends straightforwardly to bosons. Given that $\Lambda \ll k_0$, kinematics constrains the resulting form of the effective interaction vertices within the momentum shell (i.e., those for which all four momenta satisfy $||\mathbf{k}_i| - k_0| \leqslant \Lambda$) to lie in the following channels: (i) forward-scattering processes, which involve momentum transfer $\lesssim\Lambda$ [denoted $F_{\Omega_\Lambda}(\theta)$, where θ is the angle between the incoming momenta], and (ii) "Cooper-channel" processes, in which incoming momenta are almost equal and opposite [denoted $V_{\Omega_\Lambda}(\theta)$, where θ is the angle between incoming and outgoing momentum pairs (see, e.g., Ref. [12])]. These channels renormalize differently: owing to the nonpolarizability of the vacuum [16], all renormalizations are due to the repeated scattering processes shown in Fig. 2(a), which have different amplitudes in the forward-scattering and Cooper channels. For forward scattering, intermediate momenta are constrained to lie in the regions shaded in Fig. 2(c), whereas in the Cooper channel intermediate momenta run over the entire circle of radius k_0.

The outcome of renormalization depends on the sign of the microscopic interactions. *Any* attractive interactions lead to an instability in the Cooper channel [13] and thereby to bound states; this case is not expected to yield universal behavior. If, on the other hand, the initial interactions are *all* repulsive, one arrives at the following expressions for the renormalized interactions for incoming frequencies of order $\Omega \leqslant \Omega_\Lambda$ (see Ref. [13]):

$$F_\Omega(\theta) = \frac{F_{\Omega_\Lambda}(\theta)}{1 + [M F_{\Omega_\Lambda}(\theta)/(2\pi \sin\theta)] \ln\left(\frac{\Omega_\Lambda}{\Omega}\right)}, \quad (2a)$$

$$F_\Omega(\theta = 0) = \frac{F_{\Omega_\Lambda}(0)}{1 + M F_{\Omega_\Lambda}(0)\sqrt{k_0/\sqrt{M\Omega}}\, f_1\left(\frac{\Omega_\Lambda}{\Omega}\right)}, \quad (2b)$$

$$V_\Omega(m) = \frac{V_{\Omega_\Lambda}(m)}{1 + MV_{\Omega_\Lambda}(m)(k_0/\sqrt{M\Omega}) f_2\left(\frac{\Omega_\Lambda}{\Omega}\right)}, \qquad (2c)$$

where $f_1(x)$ and $f_2(x)$ are scaling functions that are of order unity as $x \to \infty$ and approach zero as $x \to 0$ and $V(m) \equiv \int_0^{2\pi} V(\theta) e^{im\theta} d\theta$. Subscripts Ω denote the incoming frequencies. Thus the low-energy (i.e., $\Omega/\Omega_\Lambda \to 0$) values of all couplings are "universal," i.e., independent of their microscopic values. Note that $F(\theta = \pi) = \sum_{m \text{ even}} V(m)$. Thus, given that $\Lambda/k_0 \ll 1$, the couplings assume the following hierarchy: $V_\Omega(m) \sim F_\Omega(\theta = \pi) \ll F_\Omega(\theta = 0) \ll F_\Omega(\theta \neq 0,\pi)$. Hence, interactions between particles at opposite momenta are negligible compared with other interactions [17].

Case of $T = 0$. We now turn from the QCP to phases in its vicinity. Suppose that the system is sufficiently dilute that when renormalization is cut off at a scale set by the chemical potential μ, the interactions are deep in the universal scaling regime. Then the interaction Hamiltonian is given by $H \sim \sum_{\theta,\theta'} F(\theta - \theta') n_\theta n_{\theta'}$, where n_θ denotes the boson density at a momentum of magnitude k_0 and direction θ. The hierarchy of universal coupling constants implies that H is minimized by a "fragmented" state, having precisely $N/2$ bosons at some θ and $N/2$ at $\theta + \pi$ [18]. Fragmentation is favored owing to a momentum-space analog of Coulomb blockade (cf. Sec. 2.6 of Ref. [19]): bosons with opposite momenta do not interact with one another to leading order in $\sqrt{\Lambda/k_0}$, whereas those at nonopposite momenta do interact.

In more quantitative terms we can deduce the ground-state energy from the relation [20] $\mu = (n/2)F_\mu(\theta = 0) \simeq (n/M)(\mu M/k_0^2)^{1/4}$, giving

$$E(N) \simeq \sum_{\sigma=\pm} \frac{\hbar^2 N_\sigma^{7/3}}{M\mathcal{A}^{4/3} k_0^{2/3}}, \qquad (3)$$

where $\mathcal{A}$ is the system area and $N_\pm$ denote the number of particles at θ and $\theta + \pi$, respectively. Note that this expression is *universal*, i.e., independent of the microscopic interaction strengths, and its unusual scaling is a consequence of the renormalization discussed above. As $E(N)$ is minimized when $N_+ = N_- = N/2$, the ground state for finite N is fragmented. Such a fragmented state can be understood as a density wave of wave vector k_0 along the direction θ with a randomly varying phase (analogous, e.g., to interfering independent condensates [21]).

Fragmented states are typically unstable relative to simple condensates (i.e., those having a fixed phase relation) because spatial inhomogeneities tend to phase lock the fragments [19]. In the present case, a phase-locked, coherent superposition would involve fluctuations of order $\sqrt{N}$ in $N_\pm$ and hence cost an energy of order unity relative to the fragmented state even in the thermodynamic limit. Thus, a few scattering sites cannot overcome the tendency toward fragmentation. Similarly, a *weak* harmonic potential (i.e., of characteristic length much larger than the interparticle spacing) would *not* stabilize a coherent superposition relative to a fragmented state, even in the thermodynamic limit, provided that, according to the standard prescription, the trap frequency $\omega \to 0$ so as to keep $N\omega^2$ constant. This is because the typical matrix element between $\pm k$ due to the trap is of order $\exp(-2k_0^2 N)$, which rapidly decreases as $N \to \infty$.

Although a coherent superposition is disfavored in the thermodynamic limit, the energy cost of number fluctuations of order *unity* vanishes as $1/N$. Thus, the thermodynamic ground state (e.g., in a trap) is likely to be a squeezed state with small but nonvanishing phase variance, as opposed to the fragmented state, in which the phase is entirely random. This observation extends to translation-invariant systems, which should therefore exhibit spontaneously broken translational invariance in the thermodynamic limit.

Implications for $T = 0$ phase diagram. The dilute-limit phase diagram is *simpler* than that obtained from mean-field theory: it predicts that BEC occurs at two momenta regardless of microscopic interactions, provided these are repulsive. By contrast, mean-field theory [9] predicts a TRSB state or a density-wave state, depending on microscopic interactions. We now give an account of the crossover between universal and mean-field regimes and estimate the minimum densities required for mean-field results to apply.

The dilute-gas results apply when, upon renormalization, the pertinent interactions have already achieved their universal forms at a length-scale shorter than the interparticle spacing; thus, a TRSB state is disfavored if $F_\mu(\pi) \leqslant F_\mu(0)$, regardless of whether the (larger) $F_\mu(\theta \neq 0,\pi)$ couplings have approached their universal values. Note that $F_\Lambda(\theta = 0,\pi)$ are related to the parameters c_0 and c_2 in Ref. [9] as follows: $F_\Lambda(\pi)/F_\Lambda(0) \approx 1 + c_2/c_0$. [These relations, and similar ones for other couplings, can be derived as outlined following Eq. (3) in Ref. [9]. Provided $c_2 \lesssim c_0$, all microscopic couplings are of comparable magnitude.] Therefore, in terms of c_0 and c_2, the TRSB state is favored only if

$$\frac{c_0}{1 + \frac{M}{2\pi} c_0 \sqrt{k_0/n^{1/2}}} < \frac{c_0 + c_2}{1 + qM(c_0 + c_2)(k_0/n^{1/2})}, \qquad (4)$$

where q is a constant of order unity.

Note that, in addition to the TRSB phase, the Hamiltonian of Ref. [9] also exhibits a regime in which a coherent superposition is *lower* in energy than the fragmented state, owing to terms of the form $\psi^\dagger_{2\mathbf{k}_0} \psi^\dagger_{-\mathbf{k}_0} \psi_{\mathbf{k}_0} \psi_0$, which involve momenta of order $2k_0$ and thus do not appear in H.

These considerations lead us to the phase diagram shown in Fig. 1, in which there is no *direct* transition from the vacuum to the TRSB state. The transition from the vacuum to the density-wave state is unusual in being a *continuous* transition (*known* to be continuous as the properties of the QCP are understood exactly [13]) at which both rotational and translational symmetry are broken. As a general rule (see, e.g., Refs. [2,22]), transitions that break rotational *and*translational symmetry are first order. For densities $\gtrsim \Lambda^2$, at which the renormalization effects discussed in the present work are not present, mean-field simulations show evidence of metastability [9]; this would suggest a first-order transition between the density-wave and TRSB states.

Case of $T > 0$. Following standard treatments of the dilute Bose gas [20], we assume that $T \gg \mu$. Beyond the momentum scale $\Lambda_T \equiv 1/\sqrt{2MT}$, the physics is captured by a classical free-energy functional of the form

$$S = \int d^d k [-\mu + (|\mathbf{k} - k_0)^2] |\psi_\mathbf{k}|^2 + S_4, \qquad (5)$$

where S_4 denotes the set of angle- and channel-dependent couplings and we have set $2M = 1$. S is a complex-field version of Brazovskii's model [22] (the relevance of Brazovskii's model was previously suggested in Ref. [7]). The initial values for the couplings in S_4 are the renormalized interactions at a scale $\Omega = T$. At scales $\lesssim \Lambda_T$, the vacuum is nontrivial, owing to the presence of thermal particles; hence, all couplings are renormalized by the particle-hole channel [Fig. 2(b)]. It is convenient to expand F as well as V in terms of angular momenta. One can then implement the momentum-shell RG procedure described in Ref. [23] by integrating out modes satisfying $\Lambda_T(1 - dl) < |k - k_0| < \Lambda_T$ and rescaling $k \to (1 + dl)k, \psi \to [1 - (3/2)dl]\psi$, and $\mu \to \mu/\Lambda_T^2$. The couplings transform as follows (ignoring the flow of μ):

$$\frac{dF_l(m)}{dl} = 3F_l(m) - \frac{A\,F_l^2(m)}{(1-\mu_l)^2} - \frac{A}{2}\frac{\sum_m V_l^2(m)}{(1-\mu_l)^2}, \quad \text{(6a)}$$

$$\frac{dV_l(m)}{dl} = 3V_l(m) - \frac{A\,V_l^2(m)}{2(1-\mu_l)^2} - A\frac{\sum_m F_l^2(m)}{(1-\mu_l)^2}, \quad \text{(6b)}$$

where $A \equiv 2\pi k_0/\Lambda_T$. If the coupling constants at Λ_T are in the universal regime, one can use the fact that $V \ll F$ to drop terms of order V^2. The last term in the flow equations drives all *even* $V(m)$ (which are initially near zero) to negative values at some $\Lambda_2 = \Lambda_T[1 - o(\Lambda_T/k_0)]$, triggering a runaway growth of the even-parity $V(m)$ couplings. Such runaway growth is associated with a pairing instability, which should, in principle, occur simultaneously in all even-m channels. (However, as noted in Ref. [8], the confining trap acts as a kinetic-energy term of the form ∇_θ^2, which penalizes higher-m channels.)

The pair-condensation temperature can be estimated by observing that arbitrarily weak attractive interactions in the Cooper channel give rise to pairs [24] whose binding energy is $\Delta \sim MV^2k_0^2$. Pairing is favored for $\Delta \geqslant T$. As $T/E_0 \simeq (\Lambda_1/k_0)^2 \ll (\Lambda/k_0)^2 \ll V \simeq \ln(\Lambda/\Lambda_1)$, one expects pairs to be tightly bound at length scales comparable to $1/\Lambda_T$; at longer distances they can be treated as nonoverlapping. The system is thus a dilute gas of pairs, which condense at a temperature given implicitly [20] by $T_c \approx (\hbar^2 n/4m) \times 1/\ln\ln(na^2)$, where a is an effective dimer-dimer scattering range, which is of order Λ.

Pairing would be straightforward to detect experimentally via radio-frequency spectroscopy, which should reveal a peak corresponding to the pair binding energy; moreover, a pair condensate would support half-quantized vortices detectable via rotation.

3D case. For this case, $k_z \equiv k_\perp$ in Eq. (1); thus, the dispersion minimum is *circular* rather than spherical and imposes the same kinematic constraints as in two dimensions. The 2D analysis thus generalizes readily; the chief difference is that the forward-scattering couplings in 3D renormalize to nonuniversal T matrices rather than universal values, and the ground-state energy thus depends on microscopic couplings. However, Cooper-channel couplings approach the following universal expression as $\Omega/\Omega_\Lambda \to 0$:

$$V_\Omega \sim 1/\left[k_0 M \ln(\Omega_\Lambda/\Omega)\right]. \quad (7)$$

Hence, as in two dimensions, $\sum_m V(m) \sim F(\theta = \pi) \ll F(\theta \neq \pi)$ at low energies. It follows that the dilute-limit ground state universally preserves time-reversal symmetry. This qualitative resemblance to two dimensions extends to the $T > 0$ case, in which the free-energy functional, in this case, the variant of Brazovskii's model having two transverse dimensions discussed in Ref. [25], develops a pairing instability as in two dimensions. As the Cooper-channel couplings approach universal values more slowly, however, the conditions for the dilute limit to obtain are more stringent in three dimensions than in two dimensions.

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[18] Our conclusions thus agree with those of Ref. [8]; however, our argument does *not* use the assumption that $(\theta, \theta + \pi)$ are associated with orthogonal internal states and thus also applies to the higher-band lattice case.

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