Scattering of intense laser radiation by a single-electron wave packet

John P. Corson,[1] Justin Peatross,[1] Carsten Müller,[2] and Karen Z. Hatsagortsyan[2]
[1]*Dept. of Physics and Astronomy, Brigham Young University, Provo, Utah 84602, USA*
[2]*Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, D-69117 Heidelberg, Germany*
(Received 26 July 2011; published 17 November 2011)

A quantum theoretical description of photoemission by a single laser-driven electron wave packet is presented. Energy-momentum conservation ensures that the partial emissions from individual momentum components of the electron wave packet do not interfere when the driving field is unidirectional. In other words, light scattering by an electron packet is independent of the phases of the pure momentum states comprising the packet; the size of the electron wave packet does not matter. This result holds also in the case of high-intensity multiphoton scattering. Our analysis is first presented in the QED framework. Since QED permits the second-quantized entangled electron-photon final state to be projected onto pure plane-wave states, the Born probability interpretation requires these projections to be first squared and then summed to find an overall probability of a scattering event. The QED treatment indicates how a semiclassical framework can be developed to recover the key features of the correct result.

DOI: 10.1103/PhysRevA.84.053831 PACS number(s): 42.50.Ct, 41.60.−m, 42.50.Xa

I. INTRODUCTION

The quantum theory of scattering processes relies on the concept of particle wave packets in order to describe the incoming and outgoing quantum states. A wave packet is localized both in coordinate and momentum space (within the constraints of the Heisenberg's uncertainty principle) and is, thus, the quantum mechanical counterpart of a classical moving point particle. In contrast, essentially all concrete calculations in textbooks and the literature are done for plane-wave states, which have definite momentum and are spread over all space. We show how this traditional simplified context has high relevance to the interaction between quantum mechanical packets—a connection that has been underappreciated.

Since the invention of the laser in 1960, many theoretical studies have considered scattering of electromagnetic radiation by particles (see [1] for a review). Due to its fundamental significance, the first process investigated was photoemission by an electron, which is kinematically allowed in the presence of a laser field and proceeds via Thomson (or Compton) scattering of laser photons; see [2] for early work and [3] for more recent treatments. All of these quantum mechanical treatments employ plane-wave electron states, following standard practice. The classical theory of photoemission by free electrons in laser fields has been treated in a seminal paper by Sarachik and Schappert [4] (see also [5]).

Early considerations of the dynamics of laser-driven single-electron wave packets date back to the 1960s [6]. More recently, interesting effects have been revealed, such as wave-packet spreading, deformation (tilting and Lorentz contraction), shearing, and the formation of multiple peaks when the wave packet spreads to the scale of the driving-field wavelength [7–11]. In fact, a free-electron wave packet with an initial spatial size on the scale of an atom undergoes rapid spreading in a realistic laser focus, the electron packet eventually reaching and even exceeding the scale of an optical wavelength [12,13]. This raises the question of how a single-electron wave packet radiates, especially when it undergoes such highly nondipole dynamics, where different parts of the same electron wave packet experience different phases of a stimulating laser field. A classical charge distribution oscillating in this way would exhibit pronounced suppression of the scattered radiation field, owing to interference. The question therefore arises as to whether, under these conditions, quantum (path) interference similarly influences field-induced photoemission by a single electron.

Theoretical efforts to answer this question have been undertaken recently [12–16]. Cheng *et al.* performed a one-dimensional (1D) quantum-field simulation showing that scalar bosons emitted by a spatially diffuse fermion packet exhibit no interference [16]. The problem is of growing interest since single-atom and single-electron experiments are becoming feasible where the behavior of the particle's wave function is relevant. Experiments on multiphoton Thomson and Compton scattering from *many*-electron samples in external laser fields have been performed in the past [17].

In this article, we extend our previous study [15] and provide a more comprehensive theoretical discussion of photoemission by a single-electron wave packet in a laser field. Although the outgoing light constitutes a photon wave packet, the probability interpretation of quantum mechanics constructs probability amplitudes by projecting the final state onto individual basis modes (such as plane waves). The probabilities for individual outgoing modes are then summed *incoherently* over the various possibilities [18]. We show then that, if the scattered photon is measured to be in a momentum eigenstate, the different momentum components of the initial electron state do not interfere if the stimulating field is unidirectional. This result is dictated by energy-momentum conservation. (See [19,20] for a discussion of conservation laws in the packet-packet context.) Only one momentum component of the initial electron wave packet can contribute to a given momentum component of the outgoing electron via photoemission into a given plane wave. Consequently, we find that the scattered light is insensitive to the spatial size of the electron wave packet.

This article is organized as follows: In Sec. II, we discuss the shortcomings of a first-quantized theory of radiation scattering by an electron. Section III computes the scattering amplitude in a second-quantized framework and highlights the requirement that the outgoing photon be projected onto an eigenstate before summing probabilities. Section IV develops a semiclassical theory where the first-quantized framework is salvaged via an *ad hoc* procedure prescribed by quantum electrodynamics (QED). Section V generalizes the analysis to the multiphoton regime. In Sec. VI, we comment on the scenario of multidirectional incident light and argue that unidirectional light is the appropriate context to explore the possibility of radiative interference in the emission from a large electron packet.

II. FAILURE OF A SELF-CONSISTENT FIRST-QUANTIZED THEORY

We consider a single electron interacting with an electromagnetic field. In the first-quantized picture, the electron is treated as a wave on equal footing with the classical electromagnetic field. Intuition suggests that the electron wave packet described by $\Psi(\mathbf{r},t)$ gives rise to the source term $\mathbf{J}(\mathbf{r},t)$ [and $\rho(\mathbf{r},t)$], which in turn is responsible for the scattered light represented by an electromagnetic potential $\mathbf{A}(\mathbf{r},t)$ [and $\Phi(\mathbf{r},t)$]. For simplicity, we neglect spin effects and model electron dynamics by the Klein-Gordon equation:

$$\left(i\hbar\frac{\partial}{\partial t}-e\Phi\right)^2\Psi=c^2\left(-i\hbar\nabla-\frac{e}{c}\mathbf{A}\right)^2\Psi+m^2c^4\Psi, \quad (1)$$

where the electron has charge $e=-|e|$ and mass m. Similarly, the vector potential satisfies the laws of electromagnetism, expressed as

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}=-\frac{4\pi}{c}\mathbf{J} \quad \text{and} \quad \nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi=4\pi\rho \quad (2)$$

in the Lorenz gauge. Throughout this article, we employ the Gaussian system of units.

To follow a self-consistent picture, one solves the Klein-Gordon equation (1) and Maxwell's equations (2) with coupling via the current density $\mathbf{J}=\frac{e}{2m}\Psi^*(-i\hbar\nabla-\frac{e}{c}\mathbf{A})\Psi+$ c.c. [and charge density $\rho=\frac{e}{2mc^2}\Psi^*(i\hbar\partial/\partial t-e\Phi)\Psi+\text{c.c.}$]. Under the initial condition of a light pulse directed toward the electron wave packet, one expects the interaction to create a classical scattered light field emerging from an altered electron wave packet. Solving these coupled equations either analytically or numerically is extremely difficult, which necessitates approximations. For long wavelengths, the equations can be approximately decoupled by including only the incident field in (1) and the expression for $\mathbf{J}$ (thereby forfeiting radiation reaction).

Regardless of the exact approach for generating a solution, this first-quantized approach leads to dramatic suppression of radiation for many directions when the electron wave packet becomes spatially large [12,13]. This is evident in the particular solution to (2) [21]:

$$\mathbf{A}_s(\mathbf{r},t)=\frac{1}{c}\int d^3\mathbf{r}'\frac{\mathbf{J}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}. \quad (3)$$

If the direction of $\mathbf{J}$ alternates across its distribution (i.e., owing to different phases of a driving field), the spatial sum resulting in $\mathbf{A}_s$ is severely suppressed for the majority of directions.

The straightforward sourcing of (2) by (1) gives an entirely wrong result. While interference in the radiation field may seem plausible, consistency requires that one also include ρ and Φ, which leads to the absurd consequence of electron-wave self-repulsion. Classical charge distributions exhibit this effect when different regions of a single charge density repel each other via Coulomb's law. Interestingly, (1) and (2), as written, are the launching point for QED. The process of second quantization removes both single-particle self-repulsion and radiative interference, as will be highlighted in Sec. III. Moreover, it correctly describes radiation reaction.

Treating the electron as a classical point particle interacting with the Maxwell field gives the correct scattering cross section in the Thomson limit (which neglects radiation reaction). In contrast, the self-consistent first-quantized picture described above fails in this respect (unless the electron wave packet happens to be small compared to the stimulating wavelength). However, after a number of *ad hoc* procedures are imposed, the results of the first-quantized picture can be brought into close agreement with QED. We reserve the word "semiclassical" to refer to this QED-informed first-quantized theory. The first and obvious procedure is to drop single-electron self-repulsion from Φ, as is routinely done when solving the hydrogen atom. We describe further necessary modifications in Sec. IV.

III. SECOND-QUANTIZED SCATTERING AMPLITUDE

We begin by reminding the reader of a basic tenet of quantum mechanics. When calculating probabilities for observable measurements, one projects the normalized state onto an eigenstate of the measurement. These projection amplitudes are *first* squared and *then* summed over a subset of the basis eigenvalues. This principle does not change when the state includes more than one species of particles (i.e., an electron and photon).

When considering light scattered from electrons, we must project onto a complete basis that includes both the electron and the photon. In the subspace of states that include only a single electron and a single photon, we can resolve the identity as follows:

$$\mathbb{1}=\sum_{\mathbf{p}'}\sum_{\mathbf{k}'\lambda'}|\mathbf{p}';\mathbf{k}'\lambda'\rangle\langle\mathbf{p}';\mathbf{k}'\lambda'|, \quad (4)$$

where we have chosen a momentum plane-wave basis (for the sake of kinematic transparency in upcoming calculations). If we insert this expression inside the normalization condition of a single-electron–single-photon state $|\psi(t)\rangle$, we find that

$$1=\langle\psi(t)|\mathbb{1}|\psi(t)\rangle=\sum_{\mathbf{p}'}\sum_{\mathbf{k}'\lambda'}|\langle\mathbf{p}';\mathbf{k}'\lambda'|\psi(t)\rangle|^2. \quad (5)$$

This merely says that, for this state, the probability of measuring a single electron (with *any* momentum) and a single photon (also with *any* momentum and polarization) is equal to 1. Born's probabilistic interpretation [18] states that summing over subregions of $\{\mathbf{p}',\mathbf{k}'\lambda'\}$ yields a corresponding probability of finding the particles in that subregion.

We now proceed with a QED calculation of light scattered from a single electron. If the incident beam of light contains many photons, as is the case for this paper, the probabilistic statement in (5) must be augmented so as to contain those occupied states. Suppose that only modes parallel to $\hat{\mathbf{z}}$ (denoted by $V_{k_z\lambda_z}$) are occupied by the initial pulse, and that a single photon $\mathbf{k}'\lambda'$ is radiated into a different mode. Then (5) becomes

$$1 = \sum_{\mathbf{p}'} \sum_{\mathbf{k}'\lambda' \notin V_{k_z\lambda_z}} \sum_{\{n_{k_z\lambda_z}\}} |\langle \mathbf{p}'; \mathbf{k}'\lambda'; \{n_{k_z\lambda_z}\}|\psi(t)\rangle|^2, \tag{6}$$

where $\{n_{k_z\lambda_z}\}$ is the set of occupation numbers for modes belonging to $V_{k_z\lambda_z}$. To simulate an incident laser pulse, we choose the initial photon state to be a multimode coherent state $|\{\alpha_{k_z\lambda_z}\}\rangle$, which is an eigenstate of the annihilation portion of the quantized photon field operator:

$$\begin{aligned}
&\hat{\mathbf{A}}^{(+)}(x)|\{\alpha_{k_z\lambda_z}\}\rangle \\
&= \left[\sum_{\mathbf{k}\lambda} \left(\frac{2\pi\hbar c}{Vk}\right)^{(1/2)} \hat{a}_{\mathbf{k}\lambda}\boldsymbol{\epsilon}_{\mathbf{k}\lambda} e^{i(\mathbf{k}\cdot\mathbf{x}-ckt)}\right]|\{\alpha_{k_z\lambda_z}\}\rangle \\
&= \left[\sum_{V_{k_z\lambda_z}} \left(\frac{2\pi\hbar c}{Vk_z}\right)^{(1/2)} \alpha_{k_z\lambda_z}\boldsymbol{\epsilon}_{k_z\lambda_z} e^{ik_z(z-ct)}\right]|\{\alpha_{k_z\lambda_z}\}\rangle \\
&\equiv \mathbf{A}^{(+)}_{\{\alpha_{k_z\lambda_z}\}}(x)|\{\alpha_{k_z\lambda_z}\}\rangle.
\end{aligned} \tag{7}$$

The expectation value of the photon field operator $\hat{\mathbf{A}}(x) = \hat{\mathbf{A}}^{(+)}(x) + \hat{\mathbf{A}}^{(-)}(x)$ in the state $|\{\alpha_{k_z\lambda_z}\}\rangle$ is equal to the c number $\mathbf{A}^{(+)}_{\{\alpha_{k_z\lambda_z}\}}(x)$ + c.c., which could be a pulse. (Note the absence of the hat for the classical-field function.)

For the packet-packet problem that we wish to address in this paper, we take the initial state of the system to be

$$\begin{aligned}
|\psi(-\infty)\rangle &= \left(\int d^3p\beta_{\mathbf{p}}|\mathbf{p}\rangle\right) \otimes |\{\alpha_{k_z\lambda_z}\}\rangle \\
&= \int d^3p\beta_{\mathbf{p}}|\mathbf{p}; \{\alpha_{k_z\lambda_z}\}\rangle.
\end{aligned} \tag{8}$$

The electron wave packet might, for example, be a Gaussian with $\beta_{\mathbf{p}} = (p_0\sqrt{\pi})^{-3/2}\exp[-p^2/(2p_0^2)]$, where the wave packet is normalized to ensure $\langle\Psi|\Psi\rangle = 1$. To compute scattering probabilities based on (6), we are interested in objects of the form

$$|\langle \mathbf{p}'; \mathbf{k}'\lambda'; \{n_{k_z\lambda_z}\}|\hat{S}|\psi(-\infty)\rangle|^2. \tag{9}$$

The scattering operator $\hat{S}$ maps the initial state to the final state. Figure 1 depicts the bra and ket of (9). We approximate $\hat{S}$ by its lowest-order nonvanishing term in the Dyson expansion, given in scalar QED [22,23] by the normally ordered operator

$$\hat{S}^{(1)} = -\frac{ie^2}{\hbar c}\int d^4x : \hat{\mathbf{A}}(x)\cdot\hat{\mathbf{A}}(x)\hat{\phi}^\dagger(x)\hat{\phi}(x) :, \tag{10}$$

where the scalar field operator (representing the spinless Klein-Gordon electron) is given by

$$\begin{aligned}
\hat{\phi}(x) = &\int d^3p \frac{1}{\sqrt{2(2\pi\hbar)^3E_{\mathbf{p}}}} \\
&\times \left[\hat{b}(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-E_{\mathbf{p}}t)/\hbar} + \hat{d}^\dagger(\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-E_{\mathbf{p}}t)/\hbar}\right].
\end{aligned} \tag{11}$$

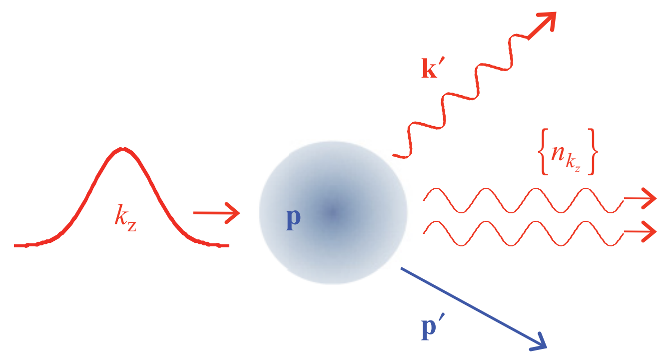

FIG. 1. (Color online) Light pulse (comprised of a range of k_z) incident on an electron wave packet (comprised of a range of $\mathbf{p}$). We consider scattering into a state $|\mathbf{p}'; \mathbf{k}'\lambda'; \{n_{k_z\lambda_z}\}\rangle$.

The operators $\hat{b}$, $\hat{d}$, and their adjoints satisfy the usual bosonic commutation relations: $[\hat{b}(\mathbf{p}),\hat{b}^\dagger(\mathbf{p}')] = \delta^3(\mathbf{p}-\mathbf{p}')$, $[\hat{d}(\mathbf{p}),\hat{d}^\dagger(\mathbf{p}')] = \delta^3(\mathbf{p}-\mathbf{p}')$, with all other commutators vanishing.

A straightforward calculation shows that

$$\begin{aligned}
&\langle \mathbf{p}'; \mathbf{k}'\lambda'; \{n_{k_z\lambda_z}\}|\hat{S}^{(1)}|\psi(-\infty)\rangle \\
&= -\frac{i}{2mc^2\hbar}\langle\{n_{k_z\lambda_z}\}|\{\alpha_{k_z\lambda_z}\}\rangle\int d^3p\beta_{\mathbf{p}}\int_{-\infty}^{\infty}dt\int d^3r\Psi^*_{\mathbf{p}'}V_{\text{int}}\Psi_{\mathbf{p}},
\end{aligned} \tag{12}$$

where

$$\Psi_{\mathbf{p}}(\mathbf{r},t) = \sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\mathbf{p}}}}e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-E_{\mathbf{p}}t)} \tag{13}$$

and

$$\begin{aligned}
V_{\text{int}} &= \frac{\langle \mathbf{k}'; \{n_{k_z\lambda_z}\}|e^2 : \hat{\mathbf{A}}\cdot\hat{\mathbf{A}} : |\{\alpha_{k_z\lambda_z}\}\rangle}{\langle\{n_{k_z\lambda_z}\}|\{\alpha_{k_z\lambda_z}\}\rangle} \\
&= 2e^2\mathbf{A}^{(+)}_{\{\alpha_{k_z\lambda_z}\}}(x)\cdot\left(\sqrt{\frac{2\pi\hbar c}{Vk'}}\boldsymbol{\epsilon}^*_{\mathbf{k}'\lambda'}e^{-i(\mathbf{k}'\cdot\mathbf{x}-ck't)}\right).
\end{aligned} \tag{14}$$

Although the final state $|\psi(\infty)\rangle \cong \hat{S}^{(1)}|\psi(-\infty)\rangle$ represents an electron-photon packet, it is first projected onto our basis plane-wave states before squaring and then summing in (6). This is key to the fact that the outgoing scattered light does not interfere.

When computing probabilities in the state space of $\{\mathbf{p}',\mathbf{k}'\lambda'\}$, we should sum over the unobserved, forward-scattered photons. In this case, the factor $\langle\{n_{k_z\lambda_z}\}|\{\alpha_{k_z\lambda_z}\}\rangle$ in (12) disappears because

$$\sum_{\{n_{k_z\lambda_z}\}}|\langle\{n_{k_z\lambda_z}\}|\{\alpha_{k_z\lambda_z}\}\rangle|^2 = 1, \tag{15}$$

owing to completeness. Henceforth in this analysis, we ignore this factor with the understanding that the sum over $\{n_{k_z}\}$ has already been performed.

We express the positive-frequency component of the incident light pulse as

$$\mathbf{A}^{(+)}_{\{\alpha_{k_z\lambda_z}\}}(z-ct)=\int_0^\infty dk_z \boldsymbol{\epsilon}_{k_z} A_{k_z} e^{ik_z(z-ct)}, \tag{16}$$

which allows for an arbitrary electromagnetic pulse traveling in the z direction. For example, the (positive-frequency) Fourier components for the Gaussian waveform $\mathbf{A}_i(z-ct)=A_0\boldsymbol{\epsilon}e^{-(z-ct)^2/(2\Delta z^2)+ik_{zo}(z-ct)}+\text{c.c.}$ are given by $\boldsymbol{\epsilon}_{k_z}A_{k_z}=A_0\Delta z/\sqrt{2\pi}[\boldsymbol{\epsilon}e^{-\Delta z^2(k_z-k_{zo})^2/2}+\boldsymbol{\epsilon}^*e^{-\Delta z^2(k_z+k_{zo})^2/2}]$.

After plugging (16) into (12), we arrive at

$$\begin{aligned}&\langle \mathbf{p}';\mathbf{k}'\lambda';\{n_{k_z\lambda_z}\}|\hat{S}^{(1)}|\psi(-\infty)\rangle\\&=-i\sqrt{\frac{(2\pi)^3\hbar c e^4}{V}}\int_0^\infty dk_z\int d^3p\frac{A_{k_z}\beta_{\mathbf{p}}\boldsymbol{\epsilon}_{k_z}\cdot\boldsymbol{\epsilon}^*_{\mathbf{k}'\lambda'}}{\sqrt{E_{\mathbf{p}'}E_{\mathbf{p}}k'}}\\&\quad\times\delta^3(\mathbf{p}+\hbar k_z\hat{\mathbf{z}}-\mathbf{p}'-\hbar\mathbf{k}')\delta(E_{\mathbf{p}}+\hbar ck_z-E_{\mathbf{p}'}-\hbar ck').\end{aligned} \tag{17}$$

The arguments of the delta functions above enforce momentum and energy conservation between the initial and measured states.

The delta functions in (17) allow us to perform all integration (a benefit of having restricted the analysis to unidirectional incident light). After performing the momentum integration, the expression reduces to

$$\begin{aligned}&\langle \mathbf{p}';\mathbf{k}'\lambda';\{n_{k_z\lambda_z}\}|\hat{S}^{(1)}|\psi(-\infty)\rangle\\&=-i\sqrt{\frac{(2\pi)^3\hbar c e^4}{V}}\int_0^\infty dk_z\frac{A_{k_z}\beta_{\tilde{\mathbf{p}}}\boldsymbol{\epsilon}_{k_z}\cdot\boldsymbol{\epsilon}^*_{\mathbf{k}'\lambda'}}{\sqrt{E_{\tilde{\mathbf{p}}}E_{\mathbf{p}'}k'}}\\&\quad\times\delta(E_{\tilde{\mathbf{p}}}+\hbar ck_z-E_{\mathbf{p}'}-\hbar ck'),\end{aligned} \tag{18}$$

where $\tilde{\mathbf{p}}=\mathbf{p}'+\hbar\mathbf{k}'-\hbar k_z\hat{\mathbf{z}}$ so that $E_{\tilde{\mathbf{p}}}$ must now be considered to depend on k_z. The remaining delta function may be rewritten as

$$\begin{aligned}&\delta(E_{\tilde{\mathbf{p}}}+\hbar ck_z-E_{\mathbf{p}'}-\hbar ck')\\&=\frac{E_{\tilde{\mathbf{p}}}\delta(k_z-\tilde{k}_z)}{\hbar c[E_{\mathbf{p}'}+\hbar ck'-c(\mathbf{p}'+\hbar\mathbf{k}')\cdot\hat{\mathbf{z}}]},\end{aligned} \tag{19}$$

where $\tilde{k}_z=(k'E_{\mathbf{p}'}-c\mathbf{k}'\cdot\mathbf{p}')/[E_{\mathbf{p}'}+\hbar ck'-c(\mathbf{p}'+\hbar\mathbf{k}')\cdot\hat{\mathbf{z}}]$. Then (18) collapses to

$$\begin{aligned}&\langle \mathbf{p}';\mathbf{k}'\lambda';\{n_{k_z\lambda_z}\}|\hat{S}^{(1)}|\psi(-\infty)\rangle\\&=-i\sqrt{\frac{(2\pi)^3e^4E_{\tilde{\mathbf{p}}}}{\hbar cVE_{\mathbf{p}'}k'}}\frac{A_{\tilde{k}_z}\beta_{\tilde{\mathbf{p}}}\boldsymbol{\epsilon}_{\tilde{k}_z}\cdot\boldsymbol{\epsilon}^*_{\mathbf{k}'\lambda'}}{[E_{\mathbf{p}'}+\hbar ck'-c(\mathbf{p}'+\hbar\mathbf{k}')\cdot\hat{\mathbf{z}}]}.\end{aligned} \tag{20}$$

For a given mode of scattered light $\mathbf{k}'\lambda'$, the delta functions ensure that there is only one set of inputs (electron momentum component and incident photon energy) that gives rise to (20). The probability of a scattering event occurring is

$$P=\sum_{\lambda'=1,2}\frac{V}{(2\pi)^3}\int d^3k'\int d^3p'|\langle \mathbf{p}';\mathbf{k}'\lambda';\{n_{k_z\lambda_z}\}|\hat{S}^{(1)}|\psi(-\infty)\rangle|^2=\frac{e^4}{\hbar c}\sum_{\lambda'=1,2}\int d^3k'\int d^3p'\frac{E_{\tilde{\mathbf{p}}}|A_{\tilde{k}_z}|^2|\beta_{\tilde{\mathbf{p}}}|^2|\boldsymbol{\epsilon}_{\tilde{k}_z}\cdot\boldsymbol{\epsilon}^*_{\mathbf{k}'\lambda'}|^2}{E_{\mathbf{p}'}k'[E_{\mathbf{p}'}+\hbar ck'-c(\mathbf{p}'+\hbar\mathbf{k}')\cdot\hat{\mathbf{z}}]^2}. \tag{21}$$

Taking the limit of large V, we have replaced the summation over discrete modes by an integral: $\sum_{\mathbf{k}'}\rightarrow\frac{V}{(2\pi)^3}\int d^3k'$. In Appendix A, we recover the traditional single-mode cross section as a suitable limit of the above packet-packet formula.

The important thing to notice is that (20) contains only one term, which is squared in (21) before the integrations over d^3k' and d^3p' take place. This means that the probability is insensitive to the complex phases of $\beta_{\tilde{\mathbf{p}}}$ and $A_{\tilde{k}_z}$, as is immediately appreciated in (21). This feature is significant in that the initial wave packet may experience an arbitrary amount of free-particle spreading (described, say, by time T) before the stimulating field arrives, as this spreading is determined by relative phases of the form

$$\beta_{\mathbf{p}}\rightarrow\beta_{\mathbf{p}}e^{-iE_{\mathbf{p}}T/\hbar}. \tag{22}$$

Thus, the spatial extent of the packets does not impact the likelihood of scattering. We have developed this result in a packet-packet context, as opposed to standard pedagogy which delocalizes the incident photon and electron with single-mode initial states.

IV. SEMICLASSICAL RADIATION SCATTERING

We now return to the first-quantized picture and inject the attributes necessary to bring it into alignment with QED. It is standard practice simply to neglect $\mathbf{J}$ (and Φ) in (2) and to prescribe the form of both the incoming and scattered portions of the vector potential. This has the virtue of not only decoupling and thus greatly simplifying the equations, but, as it turns out, also brings the result into agreement with perturbative QED.

The total vector potential is decomposed as

$$\mathbf{A}=\mathbf{A}_i+\mathbf{A}_s, \tag{23}$$

where $\mathbf{A}_i$ and $\mathbf{A}_s$ are the incident and scattered vector potentials, respectively. We choose the real-valued incident field to be $\mathbf{A}^{(+)}_{\{\alpha_{k_z\lambda_z}\}}+\text{c.c.}$, where we have defined $\mathbf{A}^{(+)}_{\{\alpha_{k_z\lambda_z}\}}$ in (16). We assume that $\mathbf{A}_s$ is small. To the extent that $\mathbf{A}_s$ is ignored, the Klein-Gordon equation (1) can be solved exactly if the incident field has the form $\mathbf{A}_i(z-ct)$. In this case, a solution may be constructed from Volkov states $\Psi^V_{\mathbf{p}}(\mathbf{r},t)$, which satisfy (1) and are parametrized by asymptotic momentum $\mathbf{p}$. For the reader's convenience, a brief description of Volkov states is provided in Appendix B.

An arbitrary electron wave packet under the influence of only the incident field may be constructed as

$$\Psi(\mathbf{r},t) = \int d^3p\beta_{\mathbf{p}}\Psi^V_{\mathbf{p}}(\mathbf{r},t). \quad (24)$$

If we also include the scattered light $\mathbf{A}_s$, with its arbitrary direction, we may still use a superposition of Volkov states since they form a complete basis, but the coefficients now acquire time dependence. We can write this as

$$\Psi(\mathbf{r},t) \cong \int d^3p\left[\beta^{(0)}_{\mathbf{p}} + \beta^{(1)}_{\mathbf{p}}(t)\right]\Psi^V_{\mathbf{p}}(\mathbf{r},t). \quad (25)$$

We will allow the initial wave packet to be dictated by the time-independent coefficients $\beta^{(0)}_{\mathbf{p}}$, which might have a distribution of the example following (8). The time dependence is then carried by $\beta^{(1)}_{\mathbf{p}}(t)$, taken to be zero at $t = -\infty$, which can give rise to scattering phenomena.

The evolution of $\beta^{(1)}_{\mathbf{p}}(t)$ is governed by the Klein-Gordon equation (1). If $\beta^{(1)}_{\mathbf{p}}(t)$ is approximated by a first-order correction in a perturbative expansion, one arrives at (see Appendix B)

$$\beta^{(1)}_{\mathbf{p}'}(t=\infty) = -\frac{i}{2mc^2\hbar}\int d^3p\beta^{(0)}_{\mathbf{p}}\int_{-\infty}^{\infty}dt\int d^3r\Psi^{V*}_{\mathbf{p}'}V_{\text{int}}\Psi^V_{\mathbf{p}}. \quad (26)$$

Notice the resemblance between (26) and (12).

Since (26) involves the integration of Volkov states, we can greatly simplify the analysis if we limit the strength of the incident field such that $\frac{eA_i}{\hbar\omega} \ll 1$. This restricts the intensity to $I \ll 8\times10^{18}\ \text{W/cm}^2(\frac{\text{nm}}{\lambda})^4$. At low intensities, the Volkov wave functions (B1) reduce to the plane-wave states defined in (13). The high-intensity case is considered in Sec. V. For initial packets whose constituent momenta satisfy $p \ll mc$, the essential interaction term works out to be

$$V_{\text{int}} = 2e^2\mathbf{A}_i\cdot\mathbf{A}_s. \quad (27)$$

The first term in (B4) vanishes at this lowest order of perturbation theory when the integration in (26) is performed (although it would contribute in the next perturbative iteration if we had not assumed $p \ll mc$).

Aside from needing to choose a specific initial electron packet via the coefficients $\beta^{(0)}_{\mathbf{p}}$, the scattered field $\mathbf{A}_s$ must be specified in (27). This is a key ingredient where QED is needed to guide the semiclassical approach. We want (26) and (27) to match the QED formulas (12)–(14). Within the semiclassical framework, we are tempted to use the real-valued field

$$\mathbf{A}_s(\mathbf{r},t) = \sqrt{\frac{2\pi\hbar c}{Vk'}}\boldsymbol{\epsilon}_{\mathbf{k}'\lambda'}e^{i(\mathbf{k}'\cdot\mathbf{r}-ck't)} + \text{c.c.}, \quad (28)$$

where $\boldsymbol{\epsilon}_{\mathbf{k}'\lambda'}$ is either of two orthogonal polarizations for $\mathbf{k}'$ ($\lambda' = 1,2$). This describes a plane wave with an amplitude chosen such that a large normalizing volume V contains the energy of one photon, $\hbar ck'$. However, it is only the second term in (28) (represented by c.c.) that gives rise to the correct QED result. We keep the extraneous term for now to better appreciate the problem that it causes.

Introducing the single-mode potential (28) as a perturbation in the electronic wave equation is typical [24], and it produces the effect of the *projection* discussed below (5). Keep in mind that by *choosing* the scattered field, we have overwritten what (2) sourced by $\mathbf{J}$ would dictate. In a technical sense, referring to (28) as the "emitted photon" is somewhat of a misnomer. Prior to the measurement, many k vectors may be present in the scattered field. Projecting onto a basis mode (in this case a monochromatic plane wave) allows one to connect measurements with calculable probabilities.

After plugging (27), (28), (16), and (13) into (26), we arrive at

$$\beta^{(1)}_{\mathbf{p}'}(\infty) = -i\sqrt{\frac{2\pi c}{\hbar V}}\frac{e^2}{(2\pi\hbar)^3}\int_0^{\infty}dk_z\int d^3p\frac{\beta^{(0)}_{\mathbf{p}}}{\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}k'}}\int_{-\infty}^{\infty}dt\int d^3r e^{-\frac{i}{\hbar}[(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}-(E_{\mathbf{p}'}-E_{\mathbf{p}})t]}$$
$$\times\left[\boldsymbol{\epsilon}_{k_z}\cdot\boldsymbol{\epsilon}_{\mathbf{k}'\lambda'}A_{k_z}e^{ik_z(z-ct)+i(\mathbf{k}'\cdot\mathbf{r}-ck't)} + \boldsymbol{\epsilon}^*_{k_z}\cdot\boldsymbol{\epsilon}_{\mathbf{k}'\lambda'}A^*_{k_z}e^{-ik_z(z-ct)+i(\mathbf{k}'\cdot\mathbf{r}-ck't)}\right.$$
$$\left.+\boldsymbol{\epsilon}_{k_z}\cdot\boldsymbol{\epsilon}^*_{\mathbf{k}'\lambda'}A_{k_z}e^{ik_z(z-ct)-i(\mathbf{k}'\cdot\mathbf{r}-ck't)} + \boldsymbol{\epsilon}^*_{k_z}\cdot\boldsymbol{\epsilon}^*_{\mathbf{k}'\lambda'}A^*_{k_z}e^{-ik_z(z-ct)-i(\mathbf{k}'\cdot\mathbf{r}-ck't)}\right]. \quad (29)$$

The integrations over time and space yield energy-momentum delta functions for each of the four terms in (29). One of the four terms produces the lowest-order QED result (17). Two of the four terms yield products of incompatible delta functions that vanish, as dictated by the constraints $E_{\mathbf{p}} = \sqrt{p^2c^2+m^2c^4}$ and $E_{\mathbf{p}'} = \sqrt{p'^2c^2+m^2c^4}$. Another term is proportional to $\delta^3(\mathbf{p}-\hbar k_z\hat{\mathbf{z}}-\mathbf{p}'+\hbar\mathbf{k}')\delta(E_{\mathbf{p}}-\hbar ck_z-E_{\mathbf{p}'}+\hbar ck')$, describing energy-momentum conservation for the wrong process. This problematic term does not arise if we keep only the complex-conjugate term in (28), as mentioned earlier.

The semiclassical result hinges crucially on the *ad hoc* treatment of the scattered light as a single mode, with summing over modes $\mathbf{k}'$ coming only after the probability is computed. This rightly seems at odds with the fact that the outgoing photon is undoubtedly some kind of packet. If the stimulating light has compact temporal support then, depending on distances involved, one would expect a photodetector monitoring scattered photons to click within a certain time interval (in the event that there is a click). On the other hand, a single-mode plane wave is unable to specify a time window. Nevertheless, if we tried to represent an outgoing photon with some sort of plausible packet (i.e., a superposition of modes) within the semiclassical framework, we would get a result inconsistent with QED.

V. SEMICLASSICAL MULTIPHOTON SCATTERING

We have shown in the previous sections that interference is kinematically forbidden in the low-intensity limit (single-photon absorption). One might suspect that this conclusion changes at high intensity of the driving laser field which allows for multiphoton Thomson scattering. However, we show below that this is not the case. The QED treatment in Sec. III treats the incident field perturbatively; second-quantizing in the Furry picture [25] upgrades the free-particle states in (12) to Volkov states, improving the agreement between (26) and (12).

We have to evaluate the amplitude (26) with the initial and final states given by Volkov functions (B1). To simplify the analysis, we consider the incident field to have the form

$$\mathbf{A}_i(\eta) = A_0\boldsymbol{\epsilon}\cos(k_z\eta), \tag{30}$$

where $\eta \equiv z - ct$. In this case, the Volkov states defined by (B1) read

$$\Psi_{\mathbf{p}}^V(\mathbf{r},t) = \sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}}}}\exp\left[\frac{i}{\hbar}(\mathbf{q}\cdot\mathbf{r} - E_{\mathbf{q}}t) - \frac{ieA_0\mathbf{p}\cdot\boldsymbol{\epsilon}}{\hbar k_z(E_{\mathbf{p}} - cp_z)}\times\sin(k_z\eta) + \frac{ie^2A_0^2}{8\hbar c k_z(E_{\mathbf{p}} - cp_z)}\sin(2k_z\eta)\right], \tag{31}$$

where we have introduced the dressed energy and momentum

$$\begin{aligned}\mathbf{q} &\equiv \mathbf{p} + \frac{e^2A_0^2}{4c(E_{\mathbf{p}} - cp_z)}\hat{\mathbf{z}},\\ E_{\mathbf{q}} &\equiv E_{\mathbf{p}} + \frac{e^2A_0^2}{4(E_{\mathbf{p}} - cp_z)}.\end{aligned} \tag{32}$$

By inserting these wave functions and the first two interaction terms of (B4) into (26), we obtain

$$\begin{aligned}\beta_{\mathbf{p}'}^{(1)}(\infty) = \frac{-i}{mc^2\hbar}\int d^3p\,\beta_{\mathbf{p}}^{(0)}\int dt\int d^3r\,\Psi_{\mathbf{p}'}^{V*}\Psi_{\mathbf{p}}^V \\ \times\mathbf{A}_s\cdot\left[-ec\mathbf{q} + e^2A_0\left(\boldsymbol{\epsilon} + \frac{c\mathbf{p}\cdot\boldsymbol{\epsilon}}{E_{\mathbf{p}} - cp_z}\hat{\mathbf{z}}\right)\cos(k_z\eta)\right. \\ \left. - \frac{e^3A_0^2}{4(E_{\mathbf{p}} - cp_z)}\cos(2k_z\eta)\hat{\mathbf{z}}\right]\end{aligned} \tag{33}$$

for the scattering matrix.

The standard method to evaluate matrix elements involving Volkov functions exploits the fact that the periodic part of these functions can be expanded into a Fourier series. To this end, we write

$$\begin{aligned}\Psi_{\mathbf{p}'}^{V*}\Psi_{\mathbf{p}}^V = \frac{mc^2}{(2\pi\hbar)^3\sqrt{E_{\mathbf{p}'}E_{\mathbf{p}}}}e^{\frac{i}{\hbar}[(\mathbf{q}-\mathbf{q}')\cdot\mathbf{r} - (E_{\mathbf{q}} - E_{\mathbf{q}'})t]} \\ \times e^{i[\beta_1\sin(k_z\eta) + \beta_2\sin(2k_z\eta)]},\end{aligned} \tag{34}$$

where we define

$$\begin{aligned}\beta_1 &\equiv \frac{eA_0}{\hbar k_z}\boldsymbol{\epsilon}\cdot\left(\frac{\mathbf{p}'}{E_{\mathbf{p}'} - cp_z'} - \frac{\mathbf{p}}{E_{\mathbf{p}} - cp_z}\right),\\ \beta_2 &\equiv \frac{e^2A_0^2}{8\hbar c k_z}\left(\frac{1}{E_{\mathbf{p}} - cp_z} - \frac{1}{E_{\mathbf{p}'} - cp_z'}\right).\end{aligned} \tag{35}$$

The generating function for Bessel functions may be used to produce the following series expansions:

$$\begin{aligned}e^{i[\beta_1\sin(k_z\eta) + \beta_2\sin(2k_z\eta)]} &= \sum_{n=-\infty}^{\infty} B_n e^{ink_z\eta},\\ \cos(\eta)e^{i[\beta_1\sin(k_z\eta) + \beta_2\sin(2k_z\eta)]} &= \sum_{n=-\infty}^{\infty} C_n e^{ink_z\eta},\\ \cos(2\eta)e^{i[\beta_1\sin(k_z\eta) + \beta_2\sin(2k_z\eta)]} &= \sum_{n=-\infty}^{\infty} D_n e^{ink_z\eta},\end{aligned} \tag{36}$$

where the Fourier coefficients

$$\begin{aligned}B_n &= J_n(\beta_1,\beta_2),\\ C_n &= \tfrac{1}{2}\left[J_{n+1}(\beta_1,\beta_2) + J_{n-1}(\beta_1,\beta_2)\right],\\ D_n &= \tfrac{1}{2}[J_{n+2}(\beta_1,\beta_2) + J_{n-2}(\beta_1,\beta_2)],\end{aligned} \tag{37}$$

can be expressed in terms of ordinary Bessel functions via $J_n(\beta_1,\beta_2) = \sum_l J_{n-2l}(\beta_1)J_l(\beta_2)$. Combining (36), (33), and the complex conjugate term of (28) [in the spirit of (14)] yields:

$$\begin{aligned}\beta_{\mathbf{p}'}^{(1)}(\infty) = -i2\pi\sqrt{\frac{2\pi\hbar c}{Vk'}}\int d^3p\frac{\beta_{\mathbf{p}}^{(0)}}{\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}}\sum_n\delta(E_{\mathbf{q}'} + \hbar ck' - E_{\mathbf{q}} - n\hbar ck_z)\delta^3(\mathbf{q}' + \hbar\mathbf{k}' - \mathbf{q} - n\hbar k_z\hat{\mathbf{z}}) \\ \times\boldsymbol{\epsilon}_{\mathbf{k}'\lambda'}\cdot\left[-ec\mathbf{q}B_n + e^2A_0\left(\boldsymbol{\epsilon} + \frac{c\mathbf{p}\cdot\boldsymbol{\epsilon}}{E_{\mathbf{p}} - cp_z}\hat{\mathbf{z}}\right)C_n - \frac{e^3A_0^2}{4(E_{\mathbf{p}} - cp_z)}\hat{\mathbf{z}}D_n\right].\end{aligned} \tag{38}$$

This quantity must be squared and then summed in the sense of (5). The arguments of the energy-momentum delta functions indicate that a nonperturbative treatment of the incident field allows for the absorption and reemission of many photons.

A careful analysis of (38) indicates that the momentum integral indeed collapses. One may substitute from the $p_{(z)}$ delta function into the energy delta function, yielding

$$\begin{aligned}&\delta(E_{\mathbf{q}'} + \hbar ck' - E_{\mathbf{q}} - n\hbar ck_z)\\ &\quad\rightarrow \delta(E_{\mathbf{p}'} + \hbar ck' - E_{\mathbf{p}} - cp'_{(z)} - ck'_{(z)} + cp_{(z)}).\end{aligned} \tag{39}$$

This final constraint, along with the delta functions for $p_{(x)}$ and $p_{(y)}$, uniquely determines $\mathbf{p}$ in terms of $\mathbf{k}'$ and $\mathbf{p}'$—parameters that are *fixed* before (38) is squared. As before, we see that the relative phases of $\beta_{\mathbf{p}}^{(0)}$ have no influence on the emission of radiation. Even for high-intensity light beams, the size of the electron wave packet does not matter.

One should not confuse (the lack of) spatial interferences with the type of strong-field interference studied by Narozhny and Fofanov [26], where the quantum electron experiences a bichromatic laser field of commensurate frequencies. In this case, interferences occur between the different constituents of incident light pulse.

VI. MULTIDIRECTIONAL STIMULATION

In demonstrating that the probability of a scattering event (21) is independent of the phases of both $\beta_{\mathbf{p}}$ and A_{k_z}, we used an incident pulse (16) traveling strictly in one direction. Since the spatial size of the initial electron packet can be made arbitrarily large by simply adjusting the phases via (22), one concludes that the strength of photon scattering is independent of the packet size of the scattering electron. On the other hand, if the stimulating light is multidirectional, the scattering of the radiation does depend on the phases of both $\beta_{\mathbf{p}}$ and A_{k_z}. In this case, the size and shape of the electron wave packet and the electromagnetic pulse do matter. This, however, is expected and altogether ordinary. It does not negate the aforementioned conclusion.

Multidirectional light exhibits interference fringes, which means that different regions of space can host dramatically different amounts of fluence. For example, multiple-direction modes can be used to create a focused laser beam, where a small lateral translation in position can make the difference between being inside or outside of the beam. The phases on $\beta_{\mathbf{p}}^{(0)}$ determine not only the initial size of an electron packet, but also its location and, in particular, the amount of overlap with regions of high fluence. As illustrated in Fig. 2, the Fourier translation theorem can move the electron entirely out of the focus via phase adjustments.

In the same way, scattering by a classical point electron shows a similar sensitivity to position under multidirectional stimulation. It is therefore appropriate that we have addressed the question of whether scattering is sensitive to the size of the electron wave packet under a scenario of unidirectional stimulation. In this way, it is guaranteed that the entire electron wave packet experiences the same incident light pulse.

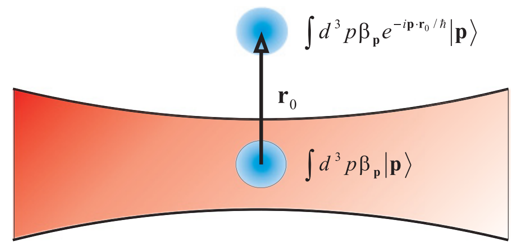

FIG. 2. (Color online) Electron may be inside or outside of a laser focus depending on the phases of $\beta_{\mathbf{p}}$.

VII. CONCLUSION

In this analysis, we have investigated the possibility of radiative interference from a laser-driven single-electron wave packet. Born's probability interpretation of quantum mechanics coupled with energy-momentum conservation predicts that radiative interference does not occur. We have outlined the various ingredients required to make the lowest-order semiclassical amplitude (for a single electron) exactly match the lowest-order QED amplitude (for a single-electron–single-photon system). The *ad hoc* prescription for this, as evidenced by (14), is to stimulate the first-quantized electron with the complex-conjugate piece of the single-mode scattered field (28). We then interpret the inherently *single*-particle scattering amplitude as a *two*-particle amplitude, intended to be first squared and then summed over the two-particle phase space in the sense of (6). Importantly, we find that sourcing Maxwell's equations with the single-particle probability current gives a result that disagrees with QED.

Measurements of Compton or Thomson scattering provide an indication that electrons do not radiate as extended charge distributions. For example, >10 keV photons scattered from electrons bound to helium corresponds to a scenario where the size of the electron wave packet is larger than the wavelengths involved. In this case, the scattered photons have energy well below the electron rest energy, and the forward-versus-back scatter is symmetric (i.e., Thomson limit) [21,27]. It is interesting to note that A. H. Compton initially proposed a "large electron" model to explain the decrease in cross section with angle for harder x-rays, which he later abandoned when the effect of momentum recoil was understood [28].

In conclusion, we have studied the amount of light that an electron scatters out the side of a laser focus. We have shown that individual electrons radiate with the strength of point emitters. Our results are soon to be tested in an experiment that combines the sensitive techniques of quantum optics (e.g., single-photon detectors) with the traditionally opposite and incompatible discipline of high-intensity laser physics.

ACKNOWLEDGMENTS

The authors acknowledge Mikhail Fedorov, Scott Glasgow, Christoph Keitel, Peter Milonni, Sergey V. Popruzhenko, Howard Reiss, and Michael Ware for helpful input. This work was supported in part by the National Science Foundation (Grant No. Phy-0970065).

APPENDIX A: SCATTERING CROSS SECTION

In this appendix, we derive the traditional scattering cross section, applicable to plane waves, starting from the packet-packet formulation (21). Our approach sidesteps the usual representation of the interaction time as $T = 2\pi\delta(0)$, which appears in the traditional derivation. We simply take a narrowband limit on the incident light pulse and the electron momentum spread. For example, we may let p_0 and Δz, defined in the example Gaussian distributions accompanying (8) and (16), be arbitrarily small positive numbers. In this case, we may approximate the momentum distributions by

$$|\beta_{\tilde{\mathbf{p}}}|^2 = \delta^3(\tilde{\mathbf{p}}) \text{ and } \left|A_{\tilde{k}_z}\right|^2 = \frac{\Phi}{k_{zo}^2}\delta(\tilde{k}_z - k_{zo}), \tag{A1}$$

where Φ is the fluence of the incident light pulse (i.e., Poynting vector integrated over time, and units of energy per area). Recall that $\tilde{\mathbf{p}} = \mathbf{p}' + \hbar\mathbf{k}' - \hbar\tilde{k}_z\hat{\mathbf{z}}$ and $\tilde{k}_z = (k'E_{\mathbf{p}'} - c\mathbf{k}'\cdot\mathbf{p}')/[E_{\mathbf{p}'} + \hbar ck' - c(\mathbf{p}' + \hbar\mathbf{k}')\cdot\hat{\mathbf{z}}]$.

Substituting the above distributions into (21) and integrating over momentum [aided by setting $\tilde{k}_z$ to k_{zo}, as enforced by $\delta(\tilde{k}_z - k_{zo})$], we arrive at

$$P = \frac{mce^4\Phi}{\hbar k_{zo}^2}\sum_{\lambda'=1,2}\int d\Omega_{k'}\int k'^2 dk' \times \frac{|\boldsymbol{\epsilon}_{k_z}\cdot\boldsymbol{\epsilon}^*_{\mathbf{k}'\lambda'}|^2\delta(\tilde{k}_z - k_{zo})}{k'E_{\hbar k_{zo}\hat{\mathbf{z}}-\hbar\mathbf{k}'}\left[E_{\hbar k_{zo}\hat{\mathbf{z}}-\hbar\mathbf{k}'} + \hbar ck' - c\hbar k_{zo}\right]^2}. \tag{A2}$$

The remaining delta function can be manipulated as follows:

$$\delta(\tilde{k}_z - k_{zo}) = \frac{k'E_{\hbar k_{zo}\hat{\mathbf{z}}-\hbar\mathbf{k}'}}{k_{zo}mc^2}\delta\left(k' - \left[\frac{1}{k_{zo}} + \frac{\hbar}{mc}(1-\cos\theta)\right]^{-1}\right). \tag{A3}$$

The argument of this delta function equivalently enforces $E_{\hbar k_{zo}\hat{\mathbf{z}}-\hbar\mathbf{k}'} + \hbar ck' - \hbar ck_{zo} = mc^2$, as evidenced by (19). The integration over k' in (A2) is then easily performed.

The well-known cross-section formula is then obtained by dividing the probability by the number of incident photons per area:

$$\sigma = \frac{P}{\Phi/(\hbar ck_{zo})} = \frac{e^4}{m^2c^4}\sum_{\lambda'=1,2}\int d\Omega_{k'}\left|\boldsymbol{\epsilon}_{k_z}\cdot\boldsymbol{\epsilon}^*_{\mathbf{k}'\lambda'}\right|^2\frac{k'^2}{k_{zo}^2}, \tag{A4}$$

where $\frac{1}{k'} = \frac{1}{k_{zo}} + \frac{\hbar}{mc}(1-\cos\theta)$.

APPENDIX B: *S* MATRIX

When the vector potential has (unidirectional) functional dependence $\mathbf{A}_i(z - ct)$ and in the absence of a scalar potential Φ, the Volkov states

$$\Psi^V_{\mathbf{p}}(\mathbf{r},t) = \sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\mathbf{p}}}}\exp\left\{i\frac{\mathbf{p}\cdot\mathbf{r} - E_{\mathbf{p}}t}{\hbar} + \frac{i}{\hbar(p_zc - E_{\mathbf{p}})} \times \int_{-\infty}^{z-ct}\left[e\mathbf{p}\cdot\mathbf{A}_i(\ell) - \frac{e^2}{2c}A_i^2(\ell)\right]d\ell\right\} \tag{B1}$$

satisfy the Klein-Gordon equation (1). These states, which form a complete solution basis, are parametrized by asymptotic momentum $\mathbf{p}$ and energy $E_{\mathbf{p}} = \sqrt{p^2c^2 + m^2c^4}$. The Volkov states (B1) are normalized according to

$$\int d^3r\left[\Psi^{V*}_{\mathbf{p}}(\mathbf{r},t)\dot{\Psi}^V_{\mathbf{p}'}(\mathbf{r},t) - \Psi^V_{\mathbf{p}'}(\mathbf{r},t)\dot{\Psi}^V_{\mathbf{p}}(\mathbf{r},t)\right] = \frac{2mc^2}{i\hbar}\delta(\mathbf{p} - \mathbf{p}'). \tag{B2}$$

When the complete vector potential $\mathbf{A}(\mathbf{r},t) = \mathbf{A}_i(z - ct) + \lambda\mathbf{A}_s(\mathbf{r},t)$ is inserted into the Klein-Gordon equation (1), we have

$$-\hbar^2\frac{\partial^2\Psi}{\partial t^2} = [(-i\hbar c\boldsymbol{\nabla} - e\mathbf{A}_i)^2 + m^2c^4]\Psi + \lambda V_{\text{int}}\Psi + \lambda^2V^{(2)}_{\text{int}}, \tag{B3}$$

where the interaction terms are given by (assuming $\boldsymbol{\nabla}\cdot\mathbf{A}_s = 0$)

$$V_{\text{int}} \equiv 2ie\hbar c\mathbf{A}_s\cdot\boldsymbol{\nabla} + 2e^2\mathbf{A}_i\cdot\mathbf{A}_s \quad \text{and} \quad V^{(2)}_{\text{int}} \equiv e^2\mathbf{A}_s^2. \tag{B4}$$

Here, λ is the usual expansion parameter of perturbation theory that will later be set to one.

We construct a wave packet comprised of Volkov states using the form (24). Since the full vector potential $\mathbf{A}(\mathbf{r},t)$ is no longer unidirectional, the coefficients must carry time dependence, which we write as

$$\beta_{\mathbf{p}}(t) = \beta^{(0)}_{\mathbf{p}} + \lambda\beta^{(1)}_{\mathbf{p}}(t) + \cdots. \tag{B5}$$

We take $\beta^{(0)}_{\mathbf{p}}$ to be time independent, specifying the initial state [implying $\beta^{(1)}_{\mathbf{p}}(t = -\infty) = 0$]. Installing (B5) into the Klein-Gordon equation (B3) and keeping terms up to order λ leads to

$$-\hbar^2\int d^3p\left(\ddot{\beta}^{(1)}_{\mathbf{p}}\Psi^V_{\mathbf{p}} + 2\dot{\beta}^{(1)}_{\mathbf{p}}\dot{\Psi}^V_{\mathbf{p}}\right) = \int d^3p\beta^{(0)}_{\mathbf{p}}V_{\text{int}}\Psi^V_{\mathbf{p}}. \tag{B6}$$

Terms that do not involve a power of λ cancel identically, since by definition $-\hbar^2\frac{\partial^2}{\partial t^2}\Psi^V_{\mathbf{p}} = [(-i\hbar c\boldsymbol{\nabla} - e\mathbf{A}_i)^2 + m^2c^4]\Psi^V_{\mathbf{p}}$.

Next we multiply both sides of (B6) by $\Psi^{V*}_{\mathbf{p}'}$ and integrate over $\mathbf{r}$ and t. The first term that results on the left-hand side can be rewritten by performing an integration by parts:

$$\int dt\Psi^{V*}_{\mathbf{p}'}\ddot{\beta}^{(1)}_{\mathbf{p}}\Psi^V_{\mathbf{p}} = \dot{\beta}^{(1)}_{\mathbf{p}}\Psi^{V*}_{\mathbf{p}'}\Psi^V_{\mathbf{p}}\Big|_{-\infty}^{+\infty} - \int dt\dot{\beta}^{(1)}_{\mathbf{p}}\left(\dot{\Psi}^{V*}_{\mathbf{p}'}\Psi^V_{\mathbf{p}} + \Psi^{V*}_{\mathbf{p}'}\dot{\Psi}^V_{\mathbf{p}}\right). \tag{B7}$$

The boundary term is zero because we may assume $\dot{\beta}^{(1)}_{\mathbf{p}}(t = \pm\infty) = 0$ if the electromagnetic disturbance has a beginning and end. This yields

$$-\hbar^2\int dt\int d^3p\dot{\beta}^{(1)}_{\mathbf{p}}\int d^3r\left(\Psi^{V*}_{\mathbf{p}'}\dot{\Psi}^V_{\mathbf{p}} - \dot{\Psi}^{V*}_{\mathbf{p}'}\Psi^V_{\mathbf{p}}\right) = \int d^3p\beta^{(0)}_{\mathbf{p}}\int d^3r\int dt\Psi^{V*}_{\mathbf{p}'}V_{\text{int}}\Psi^V_{\mathbf{p}}. \tag{B8}$$

The spatial integral on the left can be performed using (B2), and the resulting delta function collapses the momentum integral. The time integration becomes trivial, immediately yielding (26).

[1] D. B. Milošević and F. Ehlotzky, Adv. At. Mol. Opt. Phys. **49**, 373 (2003); F. Ehlotzky, K. Krajewska, and J. Z. Kaminski, Rep. Prog. Phys. **72**, 046401 (2009).
[2] N. D. Sen Gupta, Bull. Calcutta Math. Soc. **41**, 187 (1949); **44**, 157 (1952); I. I. Goldman, Phys. Lett. **8**, 103 (1964); Zh. Eksp. Teor. Fiz. **46**, 1412 (1964) [Sov. Phys. JETP **19**, 954 (1964)]; L. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964); T. W. B. Kibble, Phys. Rev. B **138**, B740 (1965); A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **46**, 776 (1964) [Sov. Phys. JETP **19**, 529 (1964)]; **46**, 1768 (1964); **19**, 1191 (1964); **47**, 1130 (1964); **20**, 757 (1965); N. B. Narozhnyi, A. I. Nikishov, and V. I. Ritus, Zh. Eksp. Teor. Fiz. **47**, 930 (1964) [Sov. Phys. JETP **20**, 622 (1965)].
[3] P. Panek, J. Z. Kaminski, and F. Ehlotzky, Phys. Rev. A **65**, 022712 (2002); Y. Y. Lau, Fei He, Donald P. Umstadter, and Richard Kowalczyk, Phys. Plas. **10**, 2155 (2003); Fei He, Y. Y. Lau, Donald P. Umstadter, and Richard Kowalczyk, Phys. Rev. Lett. **90**, 055002 (2003); J. Gao, *ibid.* **93**, 243001 (2004); F. V. Hartemann, W. J. Brown, D. J. Gibson, S. G. Anderson, A. M. Tremaine, P. T. Springer, A. J. Wootton, E. P. Hartouni, and C. P. J. Barty, Phys. Rev. ST Accel. Beams **8**, 100702 (2005); M. Boca and V. Florescu, Phys. Rev. A **80**, 053403 (2009); D. Seipt and B. Kämpfer, *ibid.* **83**, 022101 (2011).
[4] E. S. Sarachik and G. T. Schappert, Phys. Rev. D **1**, 2738 (1970).
[5] F. V. Hartemann and A. K. Kerman, Phys. Rev. Lett. **76**, 624 (1996).
[6] H. R. Reiss, Bull. Am. Phys. Soc. **11**, 96 (1966); **11**, 1054 (1967).
[7] J. San Román, L. Roso, and H. R. Reiss, J. Phys. B **33**, 1869 (2000); J. San Román, L. Plaja, and L. Roso, Phys. Rev. A **64**, 063402 (2001).
[8] G. R. Mocken and C. H. Keitel, Phys. Rev. Lett. **91**, 173202 (2003).
[9] M. Mahmoudi, Y. I. Salamin, and C. H. Keitel, Phys. Rev. A **72**, 033402 (2005).
[10] M. Verschl and C. H. Keitel, Laser Phys. **15**, 529 (2005).
[11] J. Peatross, C. Müller, and C. H. Keitel, Opt. Expr. **15**, 6053 (2007).
[12] P. Krekora R. E. Wagner, Q. Su, and R. Grobe, Laser Phys. **12**, 455 (2002).
[13] E. A. Chowdhury, I. Ghebregziabiher, and B. C. Walker, J. Phys. B **38**, 517 (2005).
[14] G. R. Mocken and C. H. Keitel, Comp. Phys. Comm. **166**, 171 (2005).
[15] J. Peatross, C. Müller, K. Z. Hatsagortsyan, and C. H. Keitel, Phys. Rev. Lett. **100**, 153601 (2008).
[16] T. Cheng, C. C. Gerry, Q. Su, and R. Grobe, Euro. Phys. Lett. **88**, 54001 (2009).
[17] T. J. Englert and E. A. Rinehart, Phys. Rev. A **28**, 1539 (1983); C. I. Moore, J. P. Knauer, and D. D. Meyerhofer, Phys. Rev. Lett. **74**, 2439 (1995); C. Bula *et al.*, *ibid.* **76**, 3116 (1996); C. Bamber *et al.*, Phys. Rev. D **60**, 092004 (1999); H. Schwoerer, B. Liesfeld, H. P. Schlenvoigt, K. U. Amthor, and R. Sauerbrey, Phys. Rev. Lett. **96**, 014802 (2006).
[18] John von Neumann, *Mathematical Foundations of Quantum Mechanics*, translated by Robert Beyer (Princeton University Press, Princeton, 1955), p. 198.
[19] J. Taylor, *Scattering Theory* (John Wiley & Sons, New York, 1972), pp. 48–51.
[20] M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, New York, 1995), pp. 102–106.
[21] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1998), Eq. (14.1), Eq. (14.70), and Eq. (14.124).
[22] K. Huang, *Quantum Field Theory: From Operators to Path Integrals* (Wiley, New York, 1998), p. 129.
[23] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), p. 285.
[24] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), p. 121.
[25] W. Furry, Phys. Rev. **81**, 115 (1951).
[26] N. B. Narozhny and M. S. Fofanov, Zh. Eksp. Teor. Fiz. **117**, 476 (2000); J. Exp. Theor. Phys. **90**, 415 (2000).
[27] E. O. Wollan, Phys. Rev. **37**, 862 (1931).
[28] R. H. Stuewer, *The Compton Effect* (Science History Publications, New York, 1975).