Rotating superfluids in anharmonic traps: From vortex lattices to giant vortices

Michele Correggi

Dipartimento di Matematica, Università degli Studi Roma Tre, Largo San Leonardo Murialdo 1, 00146 Roma, Italy

Florian Pinsker

*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilbertforce Road, Cambridge CB3 0WA,
England, United Kingdom*

Nicolas Rougerie

Université Grenoble 1 and CNRS, LPMMC, UMR 5493, Maison des Magistères, 25 avenue des Martyrs, BP 166, 38042 Grenoble, France

Jakob Yngvason

*Faculty of Physics, University of Vienna, Boltzmanngasse 5 and Erwin Schrödinger Institute for Mathematical Physics,
Boltzmanngasse 9, A-1090 Vienna, Austria*

We study a superfluid in a rotating anharmonic trap and explicate a rigorous proof of a transition from a vortex lattice to a giant vortex state as the rotation is increased beyond a limiting speed determined by the interaction strength. The transition is characterized by the disappearance of the vortices from the annulus where the bulk of the superfluid is concentrated due to centrifugal forces while a macroscopic phase circulation remains. The analysis is carried out within two-dimensional Gross-Pitaevskii theory at large coupling constant and reveals significant differences between "soft" anharmonic traps (like a quartic plus quadratic trapping potential) and traps with a fixed boundary: in the latter case the transition takes place in a parameter regime where the size of vortices is very small relative to the width of the annulus, whereas in soft traps the vortex lattice persists until the width of the annulus becomes comparable to the vortex cores. Moreover, the density profile in the annulus where the bulk is concentrated is, in the soft case, approximately Gaussian with long tails and not of the Thomas-Fermi type like in a trap with a fixed boundary.

I. INTRODUCTION

A superfluid confined in a rotating trap undergoes several phase transitions as the rotational speed is increased. In anharmonic traps, where the speed can in principle be arbitrarily large, these transitions can essentially be associated with *three critical speeds*. At slow rotation the fluid is vortex free [1,2] but when the speed exceeds a first critical value a quantized vortex is created. As the speed goes further up the number of vortices increases [3–8] and a vortex lattice emerges.[1] This picture holds in harmonic and anharmonic traps alike, but in the latter case a new feature comes into play when a second critical velocity is exceeded: the centrifugal forces create a "hole" with strongly depleted density around the center of the trap [9–11], while the vortex lattice still prevails in the bulk. At a third critical speed a remarkable transition takes place: the vortex lattice disappears and the fluid becomes again vortex free in the bulk. Thus the process described above is in a sense reversed. All vorticity is now concentrated in a *giant vortex* situated in the hole and creating a macroscopic phase circulation in the bulk. In the past, several authors have studied this phenomenon theoretically by variational and numerical methods [9,12–18] but mathematically rigorous proofs of the

giant vortex transition have been obtained only very recently [19–23]. An experimental realization of this transition appears to be still out of reach, although anharmonic traps have been available already for some time [24–27].

In this paper we present rigorous results on the giant vortex transition in a two-dimensional trapping potential that is the sum of a quadratic and a homogeneous potential of the form

$$V_{\text{trap}}(r) = kr^s + \tfrac{1}{2}\Omega_{\text{osc}}^2 r^2, \tag{1}$$

with r the radial variable, $k > 0$, $s > 2$, and $0 \leqslant \Omega_{\text{osc}} < \Omega_{\text{rot}}$, where Ω_{rot} denotes the rotational speed. The case $s = 4$ was studied in [19] in an asymptotic regime that corresponds to a *fixed* value of the interaction strength while the rotational speed tends to infinity. In contrast, the papers [20–23] focus on the interplay between rotation speed and interaction strength and provide precise information about the third critical speed as a function of the interaction parameter when the latter is large. The model studied in these papers is that of a "flat" trap with the unit circle as the boundary, as in [10,11], that can formally be regarded as the limiting case $s = \infty$. A mathematical advantage of this model is that the extension of the system is fixed, while for finite s the system expands as Ω and/or ε^{-1} tend to ∞. There are, however, both physical and mathematical reasons for treating the latter case separately. One reason is that $s = 4$ corresponds to the lowest correction beyond quadratic in the Taylor expansion of a symmetric potential around its minimum and such a deviation from a quadratic potential has a better chance to be realized in experiments than the limiting

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[1]The emergence of a vortex lattice implies, among other things, that the solutions considered in [40–42] cannot be ground states of the interacting system at rapid rotation. We thank the referee for pointing out these references to us.

case $s \to \infty$. A further reason is that the limit $s \to \infty$ *cannot* be interchanged with the limit of strong coupling, which is the basis of almost all rigorous analysis of quantized vortices including the present one. This will be explained further below. In particular, the formulas for the limiting velocities obtained in this paper do not simply pass in the $s \to \infty$ limit over to those for a flat trap with Dirichlet boundary conditions, as considered in [21], contrary to what might be expected.

In fact, our analysis reveals significant qualitative differences between the two cases. In the flat trap the giant vortex transition takes place at a rotational velocity where the vortex cores are still vanishingly small relative to the width of the annulus containing the bulk of the density. In a trap with finite s, on the other hand, the vortex lattice persists until the width of the annulus becomes comparable to the size of the vortex cores. Moreover, the density profile in the flat trap is well approximated by a Thomas-Fermi (TF) functional without a kinetic-energy term, while in the case of finite s the radial kinetic energy cannot be neglected and the profile in the radial variable is approximately Gaussian centered at the middle of the annulus. The long tails of such a function impose the use of a larger domain than in former situations [20,21,28]. All together these differences necessitate new ideas for the proofs of the giant vortex transition compared to the earlier papers. Common to the present setting and [20,21] is a macroscopic phase circulation around the annulus as well as a breaking of rotational symmetry of the density in the ground state, even in the giant vortex regime.

The mathematical proofs of some of the statements in the sequel are rather lengthy and will not be detailed in the present paper, which is concerned with essential ideas and the main results. A full account of the proofs can be found in [22].

II. MATHEMATICAL SETTING

We now define precisely the mathematical setting which is that of two-dimensional Gross-Pitaevskii (GP) theory (cf. [3]). The general form of the energy functional for the superfluid order parameter Ψ (the wave function of the condensate) in a rotating trap is

$$\mathcal{E}^{\mathrm{GP}}_{\mathrm{phys}}[\Psi] = \int_{\mathbb{R}^2} d\mathbf{r} \left[\frac{1}{2} |(i\nabla + \mathbf{A})\Psi|^2 \right.$$
$$\left. + \left(V_{\mathrm{trap}} - \frac{1}{2}\Omega^2_{\mathrm{rot}}r^2 \right)|\Psi|^2 + \frac{|\Psi|^4}{\varepsilon^2} \right]. \quad (2)$$

Here $\mathbf{A} = \Omega_{\mathrm{rot}}\mathbf{e}_3 \wedge \mathbf{r}$ with $\Omega_{\mathrm{rot}} > 0$ being the rotation velocity, $\mathbf{e}_3$ is the unit vector in the x_3 direction, $\mathbf{r} = (x_1, x_2)$, $r = |\mathbf{r}|$, V_{trap} is the trap potential, and $1/\varepsilon^2$ with $\varepsilon > 0$ is the GP coupling constant. The latter is, for a dilute Bose gas, given by $2\pi Na/L$, with N being the particle number, a being the scattering length of the interaction potential between the particles, and L being a characteristic length in the x_3 direction [29]. Units have been chosen such that $\hbar$ and the particle mass are both 1. (This differs from [20,21], where the particle mass is taken to be $\frac{1}{2}$.) The normalization of the wave function is $\int |\Psi|^2 = 1$. The subscript "phys" indicates that the functional (2) is written in the terms of the original physical variables, in contrast to the scaled functional defined in Eq. (8) below. We denote by $E^{\mathrm{GP}}_{\mathrm{phys}}$ the GP energy, i.e., the minimum of Eq. (2)

under the normalization condition, and by Ψ^{GP} any of the (in general nonunique) minimizers. In this paper we always assume *strong coupling*, which means $\varepsilon \ll 1$.

We now specialize to external potentials of the form of Eq. (1) and $\Omega_{\mathrm{rot}} > \Omega_{\mathrm{osc}} \geqslant 0$. With the definition $\Omega_{\mathrm{eff}} = (\Omega^2_{\mathrm{rot}} - \Omega^2_{\mathrm{osc}})^{1/2}$ the sum of the external and centrifugal potentials in Eq. (2) becomes

$$V_{\mathrm{trap}}(r) - \frac{1}{2}\Omega^2_{\mathrm{rot}}r^2 = kr^s - \frac{1}{2}\Omega^2_{\mathrm{eff}}r^2. \quad (3)$$

The limiting case $s \to \infty$ and $\Omega_{\mathrm{osc}} = 0$ corresponds to the flat trap considered in [10,11,20,21]; the potential (3) is then simply $-\frac{1}{2}\Omega^2_{\mathrm{rot}}r^2$ and the integration is limited to the unit disk in $\mathbb{R}^2$.

In order that the effect of the quadratic term in the potential is visible also when $\Omega_{\mathrm{rot}} \to \infty$ it is natural to keep the ratio $\Omega_{\mathrm{eff}}/\Omega_{\mathrm{rot}}$ fixed, and we write accordingly

$$\Omega^2_{\mathrm{eff}} = \gamma\Omega^2_{\mathrm{rot}}, \quad (4)$$

with $0 < \gamma \leqslant 1$ fixed. The potential $(kr^s - \frac{1}{2}\gamma\Omega^2_{\mathrm{rot}}r^2)$ has a unique minimum at $r = R_{\mathrm{m}}$ with

$$R_{\mathrm{m}} = \left(\frac{\gamma\Omega^2_{\mathrm{rot}}}{sk} \right)^{1/(s-2)}. \quad (5)$$

We now write

$$\mathbf{r} = R_{\mathrm{m}}\mathbf{x}, \quad r = R_{\mathrm{m}}x, \quad \Psi(\mathbf{r}) = R^{-1}_{\mathrm{m}}\psi(\mathbf{x}), \quad \Omega_{\mathrm{rot}} = R^{-2}_{\mathrm{m}}\Omega, \quad (6)$$

and obtain

$$\mathcal{E}^{\mathrm{GP}}_{\mathrm{phys}}[\Psi] = R^{-2}_{\mathrm{m}}\mathcal{E}^{\mathrm{GP}}[\psi], \quad (7)$$

with the scaled energy functional

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathbb{R}^2} \left[\frac{1}{2}|(i\nabla + \Omega x\mathbf{e}_\vartheta)\psi|^2 \right.$$
$$\left. + \gamma\Omega^2 V(x)|\psi|^2 + \varepsilon^{-2}|\psi|^4 \right]d^2\mathbf{x}, \quad (8)$$

where we have written

$$V(x) = \left(\frac{1}{s}x^s - \frac{1}{2}x^2 \right). \quad (9)$$

Note that the scaled potential $\gamma\Omega^2 V(x)$ has a unique minimum at $x = 1$, independently of Ω, while the minimum of Eq. (3) wanders to infinity as $\Omega_{\mathrm{rot}} \to \infty$. Note also that we can take $\Omega \to \infty$ either by letting $\Omega_{\mathrm{rot}} \to \infty$ keeping k fixed, or by taking $k \to 0$ at fixed $\Omega_{\mathrm{rot}} > \Omega_{\mathrm{osc}}$. We note further that by Eqs. (5) and (6) the original rotational velocity Ω_{rot} is related to Ω by

$$\Omega_{\mathrm{rot}} = (sk/\gamma)^{2/(s+2)}\Omega^{(s-2)/(s+2)}. \quad (10)$$

In particular, for the important special case $s = 4$,

$$\Omega_{\mathrm{rot}} \sim \Omega^{1/3}. \quad (11)$$

The potential term and the interaction term in Eq. (8) become comparable when $\Omega \sim \varepsilon^{-1}$. As discussed below, this is the order of the second critical speed Ω_{c2}, above which the centrifugal force creates a hole. We are primarily interested in the case of fast rotation well above the second critical speed, which means that $\Omega \gg \varepsilon^{-1}$.

For $\Omega \lesssim \varepsilon^{-1}$ it is more convenient to use a different scaling than Eq. (6), replacing $R_m \sim \Omega_{rot}^{2/(s-2)}$ by $R_\varepsilon \sim \varepsilon^{-2/(s+2)}$ [30, 31]. In terms of the scaled rotational velocity $\Omega' = R_\varepsilon^2 \Omega_{rot}$, the first critical velocity, where vortices start to appear, is $\Omega'_{c1} \sim |\ln \varepsilon|$. See [1,7,8] for the case of harmonic traps and [31] for an adaption to $s > 2$. Note that $\Omega' \sim \varepsilon^{-1}$ is equivalent to $\Omega \sim \varepsilon^{-1}$.

III. TF DENSITY PROFILE

In the parameter range $\Omega \ll \varepsilon^{-4}$ the bulk density profile of a minimizer ψ^{GP} of Eq. (8) can be approximately described by the TF density:

$$\rho^{TF}(x) = \frac{\varepsilon^2}{2}[\mu^{TF} - \gamma\Omega^2 V(x)]_+, \qquad (12)$$

where $[t]_+ = t$ if $t > 0$ and 0 otherwise. The chemical potential μ^{TF} is determined by the normalization $\int \rho^{TF} = 1$. The density ρ^{TF} is the minimizer of the TF functional:

$$\mathcal{E}^{TF}[\rho] = \int_{\mathbb{R}^2} [\gamma\Omega^2 V(x)\rho(x) + \varepsilon^{-2}\rho(x)^2]d^2\mathbf{x}, \qquad (13)$$

i.e, the GP functional (8) without the kinetic term. The corresponding energy will be denoted E^{TF}. For $\Omega \gtrsim \varepsilon^{-4}$ the radial kinetic energy significantly influences the bulk density profile and the TF approximation becomes inaccurate. This will be discussed further below.

From Eq. (12) it is clear that ρ^{TF} vanishes at the origin for $\mu^{TF} = 0$ and a hole of finite radius forms as soon as $\mu^{TF} < 0$. The normalization of Eq. (12) implies that the critical velocity for the appearance of the hole is given by

$$\Omega_{c2} = \varepsilon^{-1}\{(2/\gamma)\int[-V]_+\}^{-1/2}. \qquad (14)$$

As $(\varepsilon\Omega) \to \infty$ we have $\mu^{TF}/(\gamma\Omega^2) \to (s-2)/2s$ and the density ρ^{TF} becomes concentrated around $x = 1$. The inner and outer radii, $x_{in} < 1$ and $x_{out} > 1$, respectively, of the support, as well as the chemical potential μ^{TF}, are determined by the equations

$$\rho^{TF}(x_{in}) = \rho^{TF}(x_{out}) = 0, \quad 2\pi\int_{x_{in}}^{x_{out}} \rho^{TF}(x) x \, dx = 1. \qquad (15)$$

A Taylor expansion of V around its minimum [maximum for $\rho^{TF}(x)$] at $x = 1$ (see Sec. 2.3 in [30] for details) gives the thickness of the support:

$$x_{out} - x_{in} = (\varepsilon\Omega)^{-2/3}[12/(s-2)\gamma]^{1/3}[1 + O((\varepsilon\Omega)^{-2/3})]. \qquad (16)$$

By the normalization of ρ^{TF} it follows that the maximum $\|\rho^{TF}\|_\infty = \rho^{TF}(1)$ is $O((\varepsilon\Omega)^{2/3})$. In the flat trap, on the other hand, which corresponds formally to $s = \infty$, the thickness of the annulus where the TF density is concentrated is $O((\varepsilon\Omega)^{-1})$ and the density is of order $O(\varepsilon\Omega)$ [11].

The reason for the different powers of $\varepsilon\Omega$ can be understood by the following consideration. The Taylor expansion leading to Eq. (16) is justified as soon as the turning point x_{turn} where $V''(x_{turn}) = 0$ is much farther from 1 than the inner and outer radii x_{in} and x_{out}, which means that

$$1 - x_{turn} \gg (\varepsilon\Omega)^{-2/3}(s-1)^{-1/3}. \qquad (17)$$

Now $x_{turn} = [1/(s-1)]^{1/(s-2)}$, and since we are interested in large s we can write Eq. (17) as $1 - s^{-1/s} \gg (\varepsilon\Omega)^{-2/3}s^{-1/3}$. Since $1 - s^{-1/s} = 1 - \exp[-(\ln s)/s] = (\ln s)/s + O((\ln s/s)^2)$ we obtain

$$\varepsilon\Omega \gg s/(\ln s)^{3/2} \qquad (18)$$

as a condition for the validity of the Taylor expansion. While this condition is always fulfilled for each finite s if $\varepsilon\Omega$ is large enough it is clearly violated for every fixed value of $\varepsilon\Omega$ if $s \to \infty$. In fact, in the flat trap the Taylor expansion fails and the TF density has the form $\rho_{flat}^{TF}(x) \sim (\varepsilon\Omega)^2(x^2 - x_{in}^2)_+$ [cf. [11], Eq. (A.7)] with maximum value $\rho_{flat}^{TF}(1) \sim (\varepsilon\Omega)$.

In the following we consider a fixed, finite s and employ formula (16) above.

IV. VORTEX LATTICE REGIME

An upper bound for the ground-state energy E^{GP} of Eq. (8) can be obtained by a variational ansatz that is analogous to Eq. (4.1) in [11]. It corresponds to a bulk profile determined by the TF density and a regular lattice of vortices localized at positions $\mathbf{x}_i$ in the disk with radius x_{out} centered at the origin. More precisely, the trial function is of the form

$$\psi(\mathbf{x}) = c\sqrt{\rho(x)}\xi(\mathbf{x})\phi(\mathbf{x}), \qquad (19)$$

where c is a normalization constant, ρ is a suitable regularization of the TF density ρ^{TF}, $\xi(\mathbf{x})$ is a function vanishing at the lattice points $\mathbf{x}_j$, and $\phi(\mathbf{x}) = \prod_j(\zeta - \zeta_j)/|\zeta - \zeta_j|$ is a phase factor generated by vortices of unit strength in each lattice point. We have here used the complex notation $\zeta = x_1 + ix_2$ for points in $\mathbb{R}^2$. The vortices are placed so that $\nabla\phi$ compensates as far as possible the vector potential term proportional to Ω in the kinetic energy, which means an arrangement in a triangular lattice with density Ω/π. Moreover, if t is the radius of a vortex core where the function ξ deviates significantly from 1 the kinetic energy of a vortex localized in $\mathbf{x}_j$ is, to lowest order in the small parameters, $\sim \rho(\mathbf{x}_j)|\ln(t^2\Omega)|$. Optimizing t to minimize the sum of kinetic and interaction energy gives $t \sim \varepsilon/\rho(\mathbf{x}_j)^{1/2}$, provided t is much smaller than the distance between vortices, which is $\sim \Omega^{-1/2}$. With $\rho(\mathbf{x}_j) \sim (\varepsilon\Omega)^{2/3}$ this leads to $t \sim \varepsilon^{2/3}\Omega^{-1/3}$ and this is $\ll \Omega^{-1/2}$ if $\Omega \ll \varepsilon^{-4}$. Following closely the computation in Sec. 4 in [11] one now obtains for $\varepsilon^{-1} \lesssim \Omega \ll \varepsilon^{-4}$ the upper bound

$$E^{GP} \leqslant E^{TF} + \tfrac{1}{6}\Omega|\ln(\varepsilon^4\Omega)|[1 + O((\varepsilon^4\Omega)^{1/3})]. \qquad (20)$$

The last term is the radial kinetic energy of the density profile ρ. It is smaller than the second term if $\Omega \ll \varepsilon^{-4}$.

A lower bound matching Eq. (20) is considerably more difficult to achieve, but it can be proved using techniques from Ginzburg-Landau (GL) theory in the same way as in Sec. 5 in [11]. The result is the following:

Theorem 1. (Energy between Ω_{c2} and Ω_{c3}). If $\varepsilon^{-1} \lesssim \Omega \ll \varepsilon^{-4}$ as $\varepsilon \to 0$, then

$$E^{GP} = E^{TF} + \tfrac{1}{6}\Omega|\ln(\varepsilon^4\Omega)|[1 + o(1)]. \qquad (21)$$

An important difference to the flat trap considered in [11] becomes apparent here: in [11] an upper bound corresponding to Eq. (20) [with $\ln(\varepsilon)$ in place of $\ln(\varepsilon^4\Omega)$] is shown to be valid under the condition $\Omega \ll \varepsilon^{-2}$, but the lower bound, derived

by using techniques from GL theory [35], holds only for $\Omega \ll \varepsilon^{-2} |\ln \varepsilon|^{-1}$.

For rotational speeds between the first and the second value the energy asymptotics can be proved in a similar way and is given by the following:

Theorem 2 (Energy between Ω_{c1} and Ω_{c2}). If $|\ln \varepsilon| \ll \Omega' \ll \varepsilon^{-1}$ as $\varepsilon \to 0$, then

$$E^{\text{GP}'} = E^{\text{TF}'} + \tfrac{1}{2}\Omega' |\ln(\varepsilon^2 \Omega')| [1 + o(1)]. \quad (22)$$

Here $\Omega' = R_\varepsilon^2 \Omega_{\text{rot}}$ as before while $E^{\text{GP}'}$ and $E^{\text{TF}'}$ denote the GP energy and the TF energy, respectively, multiplied by R_ε^{-2} rather than R_{m}^{-2}.

A further result that holds in the regimes of both Theorem 1 and Theorem 2 is that the vorticity is uniformly distributed in the bulk in the limit $\varepsilon \to 0$. A proof for the case of a flat trap with Dirichlet (or Neumann) boundary conditions is given in [21], Theorem 1.3, and can be generalized to the present situation. The precise formulation of the statement in the regime of Theorem 1 is delicate because of the concentration of the density in an annulus that gets thinner as $\varepsilon \to 0$, but the main point is that the phase circulation around a subset $\mathcal{S}$ with area $|\mathcal{S}|$ in the annulus is asymptotically equal to $2\Omega|\mathcal{S}|$, and this holds uniformly in $\mathcal{S}$ provided $|\mathcal{S}|$ is not too small.

V. GIANT VORTEX REGIME

The first step in a study of the giant vortex transition is to consider a variational ansatz for the wave function of the form

$$\psi(\mathbf{x}) = g(\mathbf{x}) \exp(i\Omega\vartheta), \quad (23)$$

with a real valued function g, normalized such that $\int g^2 = 1$. The ansatz (23) is well behaved as a function of the angular variable ϑ if Ω is an integer; otherwise it should be replaced by the integer part $[\Omega]$. In order to simplify the notation, however, we shall in the sequel always assume that Ω is an integer; since $\Omega \to \infty$ the inclusion of the difference $\Omega - [\Omega]$ for noninteger values leads only to negligible corrections. Inserting Eq. (23) into Eq. (8) gives

$$\mathcal{E}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} \left[\frac{1}{2}|\nabla g|^2 + \frac{1}{2}\Omega^2 (x - x^{-1})^2 g^2 \right.$$
$$\left. + \gamma \Omega^2 \left(\frac{1}{s}x^s - \frac{1}{2}x^2 \right) g^2 + \varepsilon^{-2} g^4 \right] d^2\mathbf{x} \equiv \mathcal{E}^{\text{gv}}[g]. \quad (24)$$

The unique positive minimizer g_{gv} of the functional $\mathcal{E}^{\text{gv}}[g]$ is rotationally symmetric, i.e., a function of the radial variable x alone. The corresponding energy will be denoted E^{gv}. A rough upper bound for it can be obtained by taking for g a regularization of $\sqrt{\rho^{\text{TF}}}$. Since g is concentrated in an annulus of width $\ell = O((\varepsilon\Omega)^{-2/3})$ and $g^2 = O((\varepsilon\Omega)^{2/3})$, the angular contribution to the kinetic energy, $\frac{1}{2}\Omega^2 \int (x - x^{-1})^2 g^2$, is $O((\ell\Omega)^2) = O(\varepsilon^{-4/3}\Omega^{2/3})$, while the radial kinetic term $\int |\nabla g|^2$ is $O((\varepsilon\Omega)^{4/3}|\ln(\varepsilon^4\Omega)|)$ as in Eq. (20). Hence,

$$E^{\text{GP}} \leqslant E^{\text{TF}} + O(\varepsilon^{-4/3}\Omega^{2/3}) + O((\varepsilon\Omega)^{4/3}|\ln(\varepsilon^4\Omega)|). \quad (25)$$

From now on we shall always assume that

$$\Omega = \Omega_0 \varepsilon^{-4}, \quad (26)$$

with some fixed $\Omega_0 > 0$ while $\varepsilon \to 0$. [For the physical rotational velocity in Eq. (10) and $s = 4$ this means that $\Omega_{\text{rot}} \sim \varepsilon^{-4/3}$.] Then the second term in Eq. (25) is $O(\Omega_0^{2/3}/\varepsilon^4)$, while the second term in Eq. (21) (the vortex lattice kinetic energy) is $O(\Omega_0 |\ln \Omega_0|/\varepsilon^4)$ and thus larger if Ω_0 is sufficiently large. A radial kinetic-energy term $O(\Omega_0^{4/3}|\ln \Omega_0|/\varepsilon^4)$ is common to both Eqs. (21) and (25). The bottom line is that for large Ω_0 the giant vortex ansatz is energetically favorable to Eq. (21).

These simple considerations are, however, far from a proof that a true minimizer ψ^{GP} of Eq. (8) has no vortices in the bulk above some $\Omega_{c3} \sim \varepsilon^{-4}$. In [20,21] such a proof is carried out in full detail for the case of a flat trap ($s = \infty$), both with Neumann and Dirichlet boundary conditions, and it is shown that there $\Omega_{c3} \sim \varepsilon^{-2}|\ln \varepsilon|^{-1}$. The technique used in that proof depends on tools that were originally developed in the context of GL theory, in particular vortex ball constructions and Jacobian estimates [32–35]. A prerequisite for these techniques to apply is that potential vortices can be isolated in small discs with a radius much smaller than the thickness of the annulus where the bulk of the density is concentrated. As pointed out in the discussion preceding Eq. (20) above, the radius of vortices is expected to be of order $\varepsilon^{2/3}\Omega^{-1/3} \sim \varepsilon^2\Omega_0^{-1/3}$ while the thickness of the annulus defined by the TF profile is $\sim(\varepsilon\Omega)^{-2/3} \sim \varepsilon^2\Omega_0^{-2/3}$. It is thus clear that the methods used in [20,21] to prove the transition to a giant vortex in a flat trap do not apply in the present situation. Nevertheless the absence of vortices in the bulk can be proved for Ω as in Eq. (26), provided Ω_0 is sufficiently large. The rest of the paper is devoted to a precise statement of this result and an outline of its proof.

VI. GAUSSIAN DENSITY PROFILE

In contrast to the regime $\Omega \ll \varepsilon^{-4}$ and the situation discussed in [20,21] the TF profile is not a good approximation to the bulk density profile in the homogeneous trap beyond the vortex lattice regime, i.e., in the present situation for $\Omega \sim \varepsilon^{-4}$. In fact, the bulk of the density is contained in an annulus determined by a Gaussian density distribution that we consider first. We write the energy functional (24) as

$$\mathcal{E}^{\text{gv}}[g] = -\gamma \frac{(s - 2)}{2s}\Omega^2$$
$$+ \int_{\mathbb{R}^2} \left[\frac{1}{2}|\nabla g|^2 + \Omega^2 U(x)g^2 + \varepsilon^{-2}g^4 \right] d^2\mathbf{x}, \quad (27)$$

with

$$U(x) = \frac{1}{2}(x - x^{-1})^2 + \gamma\left(\frac{1}{s}x^s - \frac{1}{2}x^2 \right) + \gamma(s - 2)/(2s). \quad (28)$$

Taylor expansion of U around $x = 1$ (e.g., for $1/2 \leqslant x \leqslant 3/2$) yields

$$U(x) = \tfrac{1}{2}\alpha^2(x - 1)^2 + O((x - 1)^3), \quad (29)$$

with

$$\alpha^2 = 4 + \gamma(s - 2). \quad (30)$$

We consider now for Ω as in Eq. (26) the auxiliary one-dimensional functional:

$$\mathcal{E}^{\text{aux}}[f] = \int_{\mathbb{R}} \left[\frac{1}{2} |f'|^2 + \frac{1}{2} \Omega^2 \alpha^2 (x-1)^2 f^2 + \varepsilon^{-2} f^4 \right] dx$$

$$= \Omega \int_{\mathbb{R}} \left(\frac{1}{2} |\hat{f}'|^2 + \frac{1}{2} \alpha^2 y^2 \hat{f}^2 + \Omega_0^{-1/2} \hat{f}^4 \right) dy, \quad (31)$$

where the variable transformation $y = \Omega^{1/2}(x-1)$, $\hat{f}(y) = \Omega^{-1/4} f(x)$ has been employed. It is clear that all three terms in Eq. (31) are of the same order of magnitude, but the importance of the last term diminishes with increasing Ω_0. Without the last term the minimizer is the Gaussian

$$\hat{f}_{\text{osc}}(y) = \pi^{-1/4} \alpha^{1/4} \exp\left(-\frac{1}{2} \alpha y^2 \right). \quad (32)$$

In [22] it is proved that the unique positive minimizer g_{gv} of Eq. (27) is well approximated by the minimizer of Eq. (31) and that the latter is, indeed, approximately Gaussian for large Ω_0, so that

$$g_{\text{gv}}(x) \approx g_{\text{osc}}(x) = \Omega^{1/4} \hat{f}_{\text{osc}}(\Omega^{1/2}(x-1)). \quad (33)$$

In particular, the integral of g_{gv}^2 over an annulus

$$\mathcal{A}_\eta = \{\mathbf{x} : 1 - \Omega^{-1/2}\eta \leqslant x \leqslant 1 + \Omega^{-1/2}\eta\} \quad (34)$$

tends to 1 if and only if $\eta \to \infty$, even though $\Omega^{-1/2}\eta \to 0$. Furthermore, the same holds for the density $|\psi^{\text{GP}}|^2$ of the minimizer of the full GP functional (8), as shown in [22]. Thus any annulus of the form of Eq. (34) contains the bulk of the density if $\eta \to \infty$. For the proof of absence of vortices in $\mathcal{A}_\eta$ it is, however, necessary to restrict η. In fact, we prove that the annulus is vortex free if $\eta = O(|\ln \varepsilon|^{1/2})$. In the course of the proof, slightly larger annuli, with $\eta = O(|\ln \varepsilon|^{3/2})$ and $\eta = O(|\ln \varepsilon|^{3/4})$, respectively, have also to be considered for technical reasons.

VII. ENERGY ESTIMATES AND ABSENCE OF VORTICES

Our result on the giant vortex transition is as follows:

Theorem 3 (Absence of vortices in the bulk). There are constants $0 < \bar{\Omega}_0 < \infty$ and $c > 0$ such that for $\Omega = \Omega_0/\varepsilon^4$ with $\Omega_0 > \bar{\Omega}_0$ and ε sufficiently small the minimizer ψ^{GP} is free of zeros in the annulus $\mathcal{A}_{\text{bulk}} = \{\mathbf{x} : |1 - x| \leqslant c\Omega_0^{-1/2}\varepsilon^2 |\ln \varepsilon|^{1/2}\}$.

An essential part of the proof is the derivation of the precise energy asymptotics in the giant vortex regime:

Theorem 4. (Energy in the giant vortex regime). For $\Omega = \Omega_0/\varepsilon^4$ with $\Omega_0 > \bar{\Omega}_0$ the ground-state energy is

$$E^{\text{GP}} = E^{\text{gv}} + O(|\ln \varepsilon|^{9/2}) = -\gamma \frac{(s-2)}{2s} \Omega^2$$

$$+ \Omega \left[\frac{\alpha}{2} + \frac{1}{2\pi} \sqrt{\frac{\alpha}{2\pi \Omega_0}} + O(\Omega_0^{-3/4}) + O(\Omega^{-1/2}) \right]$$

$$+ O(|\ln \varepsilon|^{9/2}). \quad (35)$$

An upper bound to the energy is obtained by taking in Eq. (27) a trial function built from the Gaussian $g_{\text{osc}}(x)$. The lower bound is considerably more delicate and is discussed further below. As for the comparison with Eq. (21) we note that the negative first term in Eq. (35) is the same as the leading term in

E^{TF}, namely the potential energy in the minimum of $\gamma \Omega^2 V(x)$ at $x = 1$, while the term proportional to Ω is smaller than the term $\Omega |\ln \varepsilon^4 \Omega| = \Omega \ln \Omega_0$ in Eq. (21) for large Ω_0.

For technical reasons we consider besides the functional (27) also a functional $\mathcal{E}_\eta^{\text{gv}}$ defined by the same formula except that the integration is restricted to an annulus $\mathcal{A}_\eta$ with $\eta = O(|\ln \varepsilon|^{3/2})$. Its unique positive minimizer g_η can be shown to be close to g_{osc} on the smaller annulus $\mathcal{A}_{\sqrt{\eta}}$:

$$g_\eta(x) = \left[1 + O(\Omega_0^{-1/4}) \right] g_{\text{osc}}(x). \quad (36)$$

The corresponding energy $\mathcal{E}_\eta^{\text{gv}}(g_\eta)$ is denoted by E_η^{gv}. The choice of $\eta = O(|\ln \varepsilon|^{3/2})$ is to some extent arbitrary, but the method of sub- and supersolutions [36] used in the proof of Eq. (36) and of the exponential smallness of $|\psi^{\text{GP}}|$ outside of $\mathcal{A}_{\sqrt{\eta}}$ (which is needed for the energy estimates) requires that $|\ln \varepsilon| \ll \eta \ll \varepsilon^{-1}$.

The next step is a decoupling of the energy functional that has been used repeatedly in analogous contexts in GL and GP theory [37]. We define for $\mathbf{x} \in \mathcal{A}_\eta$ a function $u(\mathbf{x})$ by writing

$$\psi^{\text{GP}}(\mathbf{x}) = g_\eta(x) u(\mathbf{x}) \exp(i\Omega\vartheta). \quad (37)$$

Since g_η is without zeros, the function u contains all possible zeros of the minimizer ψ^{GP} in the annulus. The variational equation for g_η leads to the lower bound

$$E^{\text{GP}} \geqslant E_\eta^{\text{gv}} + \mathcal{E}_\eta[u], \quad (38)$$

with

$$\mathcal{E}_\eta[u] = \int_{\mathcal{A}_\eta} g_\eta^2 \left[\frac{1}{2} |\nabla u|^2 - \mathbf{B} \cdot \mathbf{J}(u) + \varepsilon^{-2} g_\eta^2 (1 - |u|^2)^2 \right],$$

$$(39)$$

where $\mathbf{B} = \Omega(x - x^{-1})\mathbf{e}_\vartheta$ and $\mathbf{J}(u) = \frac{i}{2}(u\nabla u^* - u^*\nabla u)$.

The main task is now to estimate the negative term involving $g_\eta^2 \mathbf{B} \cdot \mathbf{J}(u)$. As usual in the context of GP theory (see, e.g., [20, 21, 28]) an essential step is an integration by parts. Namely, one writes $g_\eta^2 \mathbf{B} = \nabla^\perp F$ with the dual gradient $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ and a potential function F. In order to employ Eq. (36) we also restrict the integration to $\mathcal{A}_{\sqrt{\eta}}$, which can be shown to create only negligible errors if $\eta \gg |\ln \varepsilon|$. If $g_\eta |\mathbf{B}|$ would be exactly symmetrical about $x = 1$ like g_{osc} we could choose F to vanish on both boundaries of the annulus $\mathcal{A}_{\sqrt{\eta}}$ and integration by parts would give

$$-\int_{\mathcal{A}_{\sqrt{\eta}}} g_\eta^2 \mathbf{B} \cdot \mathbf{J}(u) = \int_{\mathcal{A}_{\sqrt{\eta}}} F \nabla^\perp \cdot \mathbf{J}(u). \quad (40)$$

Moreover, a simple computation employing Eq. (36) gives $|F(x)| \leqslant \alpha^{-1}[1 + O(\Omega_0^{-1/4}) g_\eta^2(x)]$, while $|\nabla^\perp \cdot \mathbf{J}(u)| \leqslant |\nabla u|^2$. Thus, because $\alpha > 2$, the positive first term in Eq. (39) integrated over $\mathcal{A}_{\sqrt{\eta}}$ dominates Eq. (40) for Ω_0 large enough.

This reasoning is, however, not rigorous because $g_\eta |\mathbf{B}|$ is not perfectly symmetric about $x = 1$, and if F is chosen to vanish on one boundary, e.g., the inner one, it will not vanish exactly on the other. The integration by parts then creates a boundary term $F(R) = \oint_{x=R} \mathbf{J}(u) \cdot d\ell$ with R being the radius of that boundary. To control the circulation integral one would like to use a part of the kinetic energy $\int_{\mathcal{A}_{\sqrt{\eta}}} g_\eta^2 |\nabla u|^2$, so the first step is to transform the boundary integral into two-dimensional integrals. If $\bar{R} = R - c\Omega^{-1/2}$ with c small and χ is a smooth,

monotone radial function on $[\bar{R}, R]$ with $\chi(\bar{R}) = 0$, $\chi(R) = 1$, and $|\nabla^{\perp}\chi| \leqslant C\Omega^{1/2}$, we can write

$$\oint_{x=R} \mathbf{J}(u) \cdot d\ell = \int \nabla^{\perp}\chi \cdot \mathbf{J}(u) + \int \chi \, \nabla^{\perp} \cdot \mathbf{J}(u). \quad (41)$$

The analogous computation holds for $\bar{R} = R + c\Omega^{-1/2}$ and the interval $[\bar{R}, R]$. To bound the integrals in terms of $\int g_{\eta}^2 |\nabla u|^2$, however, we need g_{η} to be large in the interval $[\bar{R}, R]$ or $[R, \bar{R}]$. Because g_{η} is, in fact, very small on the boundary of the annulus $\mathcal{A}_{\sqrt{\eta}}$ this strategy runs into difficulties.

A way out is to introduce *two* potential functions: F_1, vanishing on the inner boundary of the annulus, and F_2, vanishing on the outer boundary. The former is applied below the radius $R_{\max}$ where g_{η} has its maximum, and the latter is applied above $R_{\max}$. The integration by parts now creates two boundary terms of opposite signs at $x = R_{\max}$, and one has to estimate

$$[F_1(R_{\max}) - F_2(R_{\max})] \oint_{x=R_{\max}} \mathbf{J}(u) \cdot d\ell. \quad (42)$$

Since g_{η} is large in a neighborhood of $R_{\max}$, Eq. (41) can be put to good use: Taking $R = R_{\max}$ we have $g_{\eta}^2 \geqslant C\Omega^{1/2}$ on $[\bar{R}, R]$ by Eq. (36), and from Eq. (41), using the normalization of $g^2|u|^2$, we obtain

$$\left| \oint_{x=R_{\max}} \mathbf{J}(u) \cdot d\ell \right|$$
$$\leqslant C \int_{\mathcal{A}_{\sqrt{\eta}}} g_{\eta}^2 |u| |\nabla u| + C\Omega^{-1/2} \int_{\mathcal{A}_{\sqrt{\eta}}} g_{\eta}^2 |\nabla u|^2$$
$$\leqslant C(\Omega^{-1/2} + \delta) \int_{\mathcal{A}_{\sqrt{\eta}}} g_{\eta}^2 |\nabla u|^2 + C\delta^{-1}, \quad (43)$$

for any $\delta > 0$. Here and in the following, C denotes a finite, positive constant that may differ from line to line. The difference $\left| F_1(R_{\max}) - F_2(R_{\max}) \right|$ is estimated separately. It is small because $g_{\eta}|\mathbf{B}|$ is approximately symmetric about $x = 1$, and, making use of the variational equation for g_{η}, it is shown to be at most $O(\Omega_0 \eta^{3/2})$. Choosing $\delta = C_{\delta}\Omega_0^{-1}\eta^{-3/2}$ with sufficiently small C_{δ} we now obtain for all $\Omega_0 > \bar{\Omega}_0$ sufficiently large the crucial bound

$$\int_{\mathcal{A}_{\sqrt{\eta}}} g_{\eta}^2 \left[\tfrac{1}{2} |\nabla u|^2 - \mathbf{B} \cdot \mathbf{J}(u) \right] \geqslant -C\Omega_0^2 \eta^3 + C' \int_{\mathcal{A}_{\sqrt{\eta}}} g_{\eta}^2 |\nabla u|^2, \quad (44)$$

with $C' > 0$. Replacing the integration domain by $\mathcal{A}_{\eta}$ only produces a negligible correction if $\eta \gg |\ln \varepsilon|$, and E_{η}^{gv} also differs from E^{gv} only by small terms. This completes the proof of the lower bound in Theorem 4.

A further consequence of Eq. (44), combined with the variational bound $E_{\eta}^{\mathrm{gv}} \leqslant 0$ and the exponential smallness of $|\psi^{\mathrm{GP}}|^2$ outside $\mathcal{A}_{\sqrt{\eta}}$, is the bound

$$\int_{\mathcal{A}_{\sqrt{\eta}}} \varepsilon^{-2} g_{\eta}^4 (1 - |u|^2)^2 \leqslant C\Omega_0^2 \eta^3, \quad (45)$$

on the interaction term for $\Omega_0 > \bar{\Omega}_0$. This leads to Theorem 3 by the following reasoning. Using the variational equation satisfied by g_{η} as well as the Gagliardo-Nirenberg inequality in

a similar way as in [20], Lemma 5.1, one obtains the gradient estimate:

$$|\nabla u(\mathbf{x})| \leqslant C\varepsilon^{-2+(c^2\alpha/2)}, \quad (46)$$

with α as in Eq. (30) for all $\mathbf{x}$ such that

$$|1 - x| \leqslant c\Omega_0^{-1/2}\varepsilon^2 |\ln \varepsilon|^{1/2}. \quad (47)$$

Here we have used that, because of Eq. (36),

$$g_{\eta}(x) \geqslant C\varepsilon^{-1+(c^2\alpha/2)}, \quad (48)$$

for x satisfying Eq. (47).

We now claim that, as $\varepsilon \to 0$, $|1 - |u(\mathbf{x})|| < |\ln \varepsilon|^{-a}$ holds for all $a > 0$ on the annulus defined by Eq. (47), provided $c < (2/\alpha)^{1/2}$. The proof is by contradiction: suppose that $|1 - |u(\mathbf{x})|| \geqslant |\ln \varepsilon|^{-a}$ at some $\mathbf{x}$ and $a > 0$. Then the gradient estimate (46) implies that $|1 - |u(\mathbf{x})|| \geqslant |\ln \varepsilon|^{-a}/2$ on a disk of radius $C\varepsilon^{2-(c^2\alpha/2)}$ around $\mathbf{x}$. We thus obtain

$$\int_{\mathcal{A}_{\sqrt{\eta}}} \varepsilon^{-2} g_{\eta}^4 (1 - |u|^2)^2 \geqslant C\varepsilon^{-2+c^2\alpha} |\ln \varepsilon|^{-2a}. \quad (49)$$

This is a contradiction to Eq. (45) for $c < (2/\alpha)^{1/2}$ and $\eta = |\ln \varepsilon|^{3/2}$. Thus $|1 - |u(\mathbf{x})|| < |\ln \varepsilon|^{-1}$ holds, implying that u, and hence also ψ^{GP}, is free of zeros in the bulk defined by Eq. (47).

VIII. CIRCULATION AND SYMMETRY BREAKING

The degree (winding number) of the giant vortex ansatz (23) is clearly Ω. This can also be shown to hold, to very good accuracy, for the true minimizer ψ^{GP} in the giant vortex regime, ensuring a macroscopic circulation around the central hole where the density is strongly depleted:

Theorem 5 (Asymptotics for the degree). If Ω is given by Eq. (26) with $\Omega_0 > \bar{\Omega}_0$ and R is any radius satisfying $R = 1 + O(\Omega^{-1/2})$, then as $\varepsilon \to 0$ the degree of ψ^{GP} around the circle with radius R is $\Omega + O(\Omega_0 |\ln \varepsilon|^{9/4})$.

Indeed, a simple computation, using Eq. (37), gives

$$\text{degree of } \psi^{\mathrm{GP}} = \Omega + i(2\pi)^{-1} \oint_{x=R} u^{-1} |u| \partial_{R\vartheta}(u|u|^{-1}), \quad (50)$$

and the second term is easily estimated exploiting Eq. (43) and the bound $\int g_{\eta}^2 |\nabla u|^2 \leqslant C\Omega_0^2 \eta^3$ that follows from Eq. (44) together with $E_{\eta}^{\mathrm{gv}} \leqslant 0$.

According to Theorems 3–5 the ansatz (23) gives an excellent approximation to the energy and the qualitative properties of a true minimizer ψ^{GP} if Ω_0 is large enough. Nevertheless, while Eq. (23) is an eigenfunction of angular momentum and its modulus is therefore rotationally symmetric, this is not the case for a true minimizer:

Theorem 6 (Symmetry breaking). No minimizer ψ^{GP} in the giant vortex regime $\Omega_0 > \bar{\Omega}_0$ is an eigenfunction of angular momentum.

The indirect proof is very similar to the proof of a corresponding result in [21], Theorem 1.6, which in turn is inspired by Theorem 2 in [38], and [39]. One assumes that $\psi^{\mathrm{GP}}(\mathbf{x}) = f(x) \exp(in\vartheta)$ with a real radial function f and $n \in \mathbb{Z}$. As a byproduct of the analysis in the previous section

it can be shown that $n = \Omega[1 + O(\varepsilon^4 |\ln \varepsilon|^{9/4})]$. Moreover, f is exponentially small with respect to ε outside the annulus $\mathcal{A}_{\sqrt{\eta}}$ with $\eta = |\ln \varepsilon|^{3/2}$. It has a unique maximum at x_{m}, close to 1. Define

$$A(x) = x^2 f'(x) \text{ for } \mathbf{x} \in \mathcal{A}_{\sqrt{\eta}}, x \leqslant x_{\mathrm{m}},$$

and $A(x) = 0$ for $x > x_{\mathrm{m}}$, as well as a smooth interpolation to zero inside the inner boundary of $\mathcal{A}_{\sqrt{\eta}}$. Likewise define

$$B(x) = nxf(x) \quad \text{for} \quad \mathbf{x} \in \mathcal{A}_{\sqrt{\eta}},$$

with a smooth interpolation to 0 outside $\mathcal{A}_{\sqrt{\eta}}$. Consider then

$$\begin{aligned}
w(\mathbf{x}) = &[A(x) + B(x)] \exp[i(n+1)\vartheta] \\
&+ [A(x) - B(x)] \exp[i(n-1)\vartheta].
\end{aligned} \quad (51)$$

A computation, employing the variational equation for f, then shows that the second variation of the GP functional, i.e., the quadratic form Q given by Eq. (2.3) in [38], is negative when evaluated on w, implying that $f(x) \exp(in\vartheta)$ cannot be a minimizer.

IX. CONCLUSIONS

We have analyzed the change in the density and vortex patterns of a superfluid in a rotating, anharmonic trap as the rotational velocity and the interaction parameter both tend to infinity. In particular we have shown rigorously that the fluid undergoes a transition into a giant vortex state where there are no vortices in the bulk if the rotational velocity exceeds a certain limit depending on the interaction strength while a macroscopic circulation remains. In this paper we have focused on soft trapping potentials, e.g., the sum of a quartic and a quadratic potential, where the problems turns out to differ markedly, both physically and mathematically, from the previously considered case of a flat trapping potential with a fixed boundary. The differences concern both the shape of the bulk density in the giant vortex state, which in soft traps turns out to be approximately Gaussian rather than of a Thomas-Fermi type, as well as the relative size of vortex cores and the annulus where the bulk is concentrated.

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