Lamb shift in the muonic deuterium atom

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(Received 26 July 2011; revised manuscript received 26 August 2011; published 21 November 2011)

We present an investigation of the Lamb shift ($2P_{1/2}$-$2S_{1/2}$) in the muonic deuterium (μD) atom using the three-dimensional quasipotential method in quantum electrodynamics. The vacuum polarization, nuclear-structure, and recoil effects are calculated with the account of contributions of orders α^3, α^4, α^5, and α^6. The results are compared with earlier performed calculations. The obtained numerical value of the Lamb shift at 202.4139 meV can be considered a reliable estimate for comparison with forthcoming experimental data.

DOI: 10.1103/PhysRevA.84.052514 PACS number(s): 31.30.jr, 12.20.Ds, 36.10.Ee, 31.30.jf

I. INTRODUCTION

The muonic deuterium (μD) is the bound state of a negative muon and deuteron. The lifetime of this simple atom is determined by the muon decay in time $\tau_\mu = 2.19703(4) \times 10^{-6}$ s. When passing from electronic hydrogen to muonic hydrogen we observe the variation of the relative value of the nuclear-structure and polarizability effects, the electron vacuum polarization corrections, and recoil contributions to the fine and hyperfine structure of the energy spectrum [1–6]. Muonic atoms represent a unique laboratory for the determination of the nuclear properties. The experimental investigation of the ($2P$-$2S$) Lamb shift in light muonic atoms (muonic hydrogen, muonic deuterium, and muonic helium ions) can give more precise values of the nuclear charge radii [7–11]. Since the early 1970s, measurement of the muonic hydrogen Lamb shift has been considered one of the fundamental experiments in atomic spectroscopy. Recent progress in muon beams and laser technology has made such an experiment feasible. The first successful measurement of the (μp) Lamb shift transition energy ($2P^{F=2}_{3/2}$-$2S^{F=1}_{1/2}$) at Paul Scherrer Institute (PSI) produced the result 49881.88 (76) GHz [206.2949 (32) meV] [12]. It leads to a new value of the proton charge radius, $r_p = 0.84184(36)(56)$ fm, where the first and second uncertainties originate, respectively, from the experimental uncertainty of 0.76 GHz and the uncertainty 0.0049 meV in the Lamb shift value which is dominated by the proton polarizability term. The new value of proton radius r_p improves the CODATA value [13] by an order of magnitude. Another important project which exists now at PSI in the charge radius experiment with muonic atoms (CREMA) collaboration proposes to measure several transition frequencies between the $2S$ and $2P$ states in muonic helium ions $(\mu^4_2\mathrm{He})^+$ and $(\mu^3_2\mathrm{He})^+$ with 50-ppm precision. As a result, new values of the charge radii of a helion and an α particle with the accuracy of 0.0005 fm will be determined. The program of the investigation of the energy levels in light muonic atoms suggests that the theoretical calculations of the fine and hyperfine structures of states with $n = 1,2$ will be performed with high accuracy. Note that the discrepancy in the new proton charge radius and CODATA values induced both a reanalysis of the earlier-obtained contributions to the observed transition frequency and a study of the hypothetical muon-proton interaction [14–17].

Theoretical investigations of the Lamb shift ($2P$-$2S$), the fine and hyperfine structures of light muonic atoms, was performed many years ago in Refs. [1,18–23] on the basis of the Dirac equation and the nonrelativistic three-dimensional method (see other references in review articles [1,6]). Their calculation took into account different QED corrections with the accuracy of 0.01 meV. Recently,Ref. [1] extended the case of muonic deuterium in Refs. [2,3] where the fine and hyperfine structures were analyzed with high accuracy. Different corrections to the fine and hyperfine structures of muonic hydrogen are calculated on the basis of the three-dimensional method in quantum electrodynamics in Refs. [4,24–28]. The vacuum polarization effects of order α^5 were considered in Refs. [29–31]. In this work we aim to present an independent calculation of the Lamb shift ($2P$-$2S$) in muonic deuterium (μD) with the account of contributions of orders α^3, α^4, α^5, and α^6 on the basis of the quasipotential method in quantum electrodynamics [26–28,32]. We consider the effects of the electron vacuum polarization, recoil, and nuclear-structure corrections which are crucial to attain high accuracy. With the exception of the nuclear-structure and polarizability contribution, we calculate all corrections in the intervals ($2P_{1/2}$-$2S_{1/2}$) and ($2P_{3/2}$-$2P_{1/2}$) with precision of 0.0001 and 0.00001 meV, respectively. We recalculate and improve the earlier obtained results [1,2] and derive a reliable independent estimate for the ($2P_{1/2}$-$2S_{1/2}$) Lamb shift and ($2P_{3/2}$-$2S_{1/2}$) Lamb shift, which can be used for comparison with forthcoming experimental data. Modern numerical values of fundamental physical constants are taken from Ref. [13] as follows: the electron mass $m_e = 0.510998910(13) \times 10^{-3}$ GeV, the muon mass $m_\mu = 0.1056583668(38)$ GeV, the fine-structure constant $\alpha^{-1} = 137.035999084(51)$ [33], and the deuteron mass $m_d = 1.875612793(47)$ GeV. Numerical values of the proton-structure corrections are obtained with the 2010 CODATA value for the deuteron charge radius $r_d = 2.1424(21)$ fm and $r_d = 2.130 \pm 0.003 \pm 0.009$ fm from Ref. [34].

II. EFFECTS OF VACUUM POLARIZATION IN THE ONE-PHOTON INTERACTION

Our approach to the investigation of the Lamb shift ($2P$-$2S$) in muonic deuterium is based on the use of the quasipotential method in quantum electrodynamics [27,28,35], where the two-particle bound state is described by the Schrödinger equation. In perturbation theory the basic contribution to the muon-deuteron interaction operator is determined by the Breit Hamiltonian [5,36] as follows:

$$H_B = \frac{\mathbf{p}^2}{2\mu} - \frac{Z\alpha}{r} - \frac{\mathbf{p}^4}{8m_1^3} - \frac{\mathbf{p}^4}{8m_2^3} + \frac{\pi Z\alpha}{2}\left(\frac{1}{m_1^2}+\frac{\delta_I}{m_2^2}\right)\delta(\mathbf{r}) - \frac{Z\alpha}{2m_1m_2r}\left[\mathbf{p}^2+\frac{\mathbf{r}(\mathbf{rp})\mathbf{p}}{r^2}\right] + \frac{Z\alpha}{r^3}\left(\frac{1}{4m_1^2}+\frac{1}{2m_1m_2}\right)(\mathbf{L}\boldsymbol{\sigma}_1) = H_0 + \Delta V^B, \quad (1)$$

where $H_0 = \mathbf{p}^2/2\mu - Z\alpha/r$, m_1 and m_2 are the muon and deuteron masses, and $\mu = m_1m_2/(m_1+m_2)$. The deuteron factor $\delta_I = 0$ because we used the common definition of the deuteron charge radius $r_d^2 = -6\frac{dF_C}{dQ^2}|_{Q^2=0}$ [37,38].

The wave functions of the $2S$ and $2P$ states are equal to

$$\psi_{200}(r) = \frac{W^{3/2}}{2\sqrt{2\pi}}e^{-\frac{Wr}{2}}\left(1-\frac{Wr}{2}\right),$$
$$\psi_{21m}(r) = \frac{W^{3/2}}{2\sqrt{6}}e^{-\frac{Wr}{2}}WrY_{1m}(\theta,\phi), W = \mu Z\alpha. \quad (2)$$

The ratio of the Bohr radius of muonic deuterium to the Compton wavelength of the electron is $m_e/W = 0.7$, so the basic contribution of the electron vacuum polarization (VP) to the Lamb shift is of order $\alpha(Z\alpha)^2$ [see Fig. 1(a)]. Accounting for the modification of the Coulomb potential due to the vacuum polarization in the coordinate representation

$$V^C_{\rm VP}(r) = \frac{\alpha}{3\pi}\int_1^\infty d\xi\rho(\xi)\left(-\frac{Z\alpha}{r}e^{-2m_e\xi r}\right),$$
$$\rho(\xi) = \frac{\sqrt{\xi^2-1}(2\xi^2+1)}{\xi^4}, \quad (3)$$

we present equations for the one-loop VP contributions to shifts of the $2S$ and $2P$ states and the Lamb shift ($2P$-$2S$) as follows:

$$\Delta E_{\rm VP}(2S) = -\frac{\mu(Z\alpha)^2\alpha}{6\pi}\int_1^\infty \rho(\xi)d\xi \times \int_0^\infty xdx\left(1-\frac{x}{2}\right)^2 e^{-x\left(1+\frac{2m_e\xi}{W}\right)} = -245.3194 \text{ meV}, \quad (4)$$

$$\Delta E_{\rm VP}(2P) = -\frac{\mu(Z\alpha)^2\alpha}{72\pi}\int_1^\infty \rho(\xi)d\xi\int_0^\infty x^3dxe^{-x\left(1+\frac{2m_e\xi}{W}\right)} = -17.6847 \text{ meV}, \quad (5)$$

$$\Delta E_{\rm VP}(2P\text{-}2S) = 227.6347 \text{ meV}, \quad (6)$$

where we round the number to four decimal digits for precision. The subscript VP designates the contribution of electron vacuum polarization. Experimental error in a determination of the particle masses and fine-structure constant does not influence on the digits given in Eq. (6). The muon one-loop vacuum polarization correction of order $\alpha(Z\alpha)^4$ is known in analytical form [6]. We included the corresponding value $\Delta E_{\rm MVP}(2P\text{-}2S) = \alpha^5\mu^3/30\pi m_1^2 = 0.01968$ meV to the total shift in Sec. V [Eqs. (71) and (72)]. This result agrees with that in Ref. [2]. Two-loop vacuum polarization effects in the one-photon interaction are shown in Figs. 1(b)–1(d). To obtain a contribution of the amplitude in Fig. 1(b) to the interaction operator, it is necessary to use the following replacement in the photon propagator:

$$\frac{1}{k^2} \to \frac{\alpha}{3\pi}\int_1^\infty \rho(\xi)d\xi\frac{1}{k^2+4m_e^2\xi^2}. \quad (7)$$

In the coordinate representation a diagram with two sequential loops gives the following particle interaction operator:

$$V^C_{\rm VP-VP}(r) = \frac{\alpha^2}{9\pi^2}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\left(-\frac{Z\alpha}{r}\right) \times \frac{1}{(\xi^2-\eta^2)}(\xi^2e^{-2m_e\xi r}-\eta^2e^{-2m_e\eta r}), \quad (8)$$

where the subscript (VP-VP) corresponds to two sequential loops in the Feynman amplitude [Fig. 1(b)]. Averaging Eq. (8) over the Coulomb wave functions (2), we find the following contribution to the Lamb shift of order $\alpha^2(Z\alpha)^2$:

$$\Delta E_{\rm VP-VP}(2P\text{-}2S) = -\frac{\mu\alpha^2(Z\alpha)^2}{18\pi^2}\int_1^\infty d\xi\int_1^\infty d\eta\frac{\rho(\xi)\rho(\eta)}{(\xi+\eta)} \times \{4m_e^2W^3[4m_e\xi\eta+W(\xi+\eta)] \times [8m_e^2\xi^2\eta^2+4m_eW\xi\eta(\xi+\eta)+W^2(\xi^2+\eta^2)]\} = 0.2956 \text{ meV}. \quad (9)$$

A higher-order $\alpha^2(Z\alpha)^4$ correction is determined by an amplitude with two sequential electron (VP) and muon (MVP) loops. The corresponding potential is given by

$$\Delta V_{\rm VP-MVP}(r) = -\frac{4(Z\alpha)\alpha^2}{45\pi^2m_1^2}\int_1^\infty\rho(\xi)d\xi\left[\pi\delta(\mathbf{r})-\frac{m_e^2\xi^2}{r}e^{-2m_e\xi r}\right]. \quad (10)$$

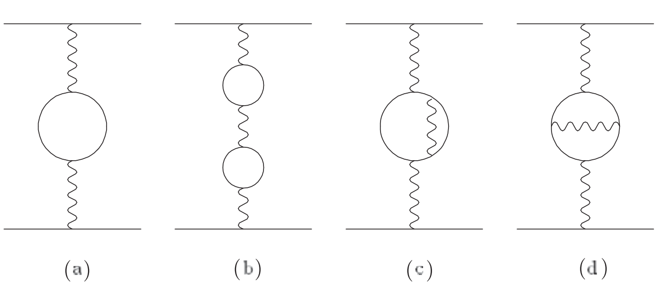


FIG. 1. Effects of one-loop and two-loop vacuum polarization in the one-photon interaction.

Its contribution to the shift ($2P$-$2S$) is equal to

$$\Delta E_{\mathrm{VP-MVP}}(2P\text{-}2S) = 0.0001\ \mathrm{meV}. \tag{11}$$

The two-loop vacuum polarization graphs [Figs. 1(b)–1(d)] were first calculated in Refs. [39,40]. The particle interaction potential, corresponding to two-loop amplitudes in Figs. 1(c) and 1(d) with the second-order polarization operator, takes the following form [39]:

$$\Delta V^{C}_{\mathrm{two-loopVP}} = -\frac{2}{3}\frac{Z\alpha}{r}\left(\frac{\alpha}{\pi}\right)^2\int_0^1 \frac{f(v)dv}{(1-v^2)}e^{-\frac{2m_e r}{\sqrt{1-v^2}}}, \tag{12}$$

where the subscript (two-loop VP) corresponds only to two-loop Feynman amplitudes shown in Figs. 1(c) and 1(d), and the spectral function is

$$f(v) = v\Big\{(3-v^2)(1+v^2)\Big[\mathrm{Li}_2\Big(-\frac{1-v}{1+v}\Big) + 2\mathrm{Li}_2\Big(\frac{1-v}{1+v}\Big) + \frac{3}{2}\ln\frac{1+v}{1-v}\ln\frac{1+v}{2} - \ln\frac{1+v}{1-v}\ln v\Big]$$
$$+\Big[\frac{11}{16}(3-v^2)(1+v^2) + \frac{v^4}{4}\Big]\ln\frac{1+v}{1-v} + \Big[\frac{3}{2}v(3-v^2)\ln\frac{1-v^2}{4} - 2v(3-v^2)\ln v\Big] + \frac{3}{8}v(5-3v^2)\Big\}, \tag{13}$$

where $\mathrm{Li}_2(z)$ is the Euler dilogarithm. The potential $\Delta V^{C}_{\mathrm{two-loopVP}}(r)$ gives a larger contribution, as compared with Eq. (8), to both the hyperfine structure and the Lamb shift ($2P$-$2S$). In the case of the Lamb shift we find the following contribution:

$$\Delta E_{\mathrm{two-loopVP}}(2P\text{-}2S) = 1.3704\ \mathrm{meV}. \tag{14}$$

In Eq. (12), by changing the electron mass to the muon mass one can obtain the two-loop muon vacuum polarization correction. It is known in analytical form from the Ref. [41] (we present their result with five decimal digits) as follows:

$$\Delta E_{\mathrm{two-loopMVP}}(2P\text{-}2S) = \frac{41}{324}\frac{\alpha^2(Z\alpha)^4\mu^3}{\pi^2 m_1^2} = 0.00017\ \mathrm{meV}. \tag{15}$$

The total two-loop electron vacuum polarization contribution given in Table I is the sum of Eqs. (9) and (14). The correction (15) is included further in Eq. (72). The numerical values of corrections (9) and (14) and the desired accuracy of the calculation show that it is important to consider three-loop contributions of the electron vacuum polarization (see Fig. 2). One part of the correction to the potential from the diagrams of the three-loop vacuum polarization in the one-photon interaction can be derived by means of Eqs. (8)–(12) [sequential loops in Figs. 2(a) and 2(b)] [28]. The corresponding contributions to the potential and the splitting ($2P$-$2S$) are given by

$$V^{C}_{\mathrm{VP-VP-VP}}(r) = -\frac{Z\alpha}{r}\frac{\alpha^3}{(3\pi)^3}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta d\eta\int_1^\infty \rho(\zeta)d\zeta$$
$$\times\Big[e^{-2m_e\zeta r}\frac{\zeta^4}{(\xi^2-\zeta^2)(\eta^2-\zeta^2)} + e^{-2m_e\xi r}\frac{\xi^4}{(\zeta^2-\xi^2)(\eta^2-\xi^2)} + e^{-2m_e\eta r}\frac{\eta^4}{(\xi^2-\eta^2)(\zeta^2-\eta^2)}\Big], \tag{16}$$

$$V^{C}_{\mathrm{VP-two-loopVP}} = -\frac{4\mu\alpha^3(Z\alpha)}{9\pi^3}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \frac{f(\eta)d\eta}{\eta}\frac{1}{r(\eta^2-\xi^2)}(\eta^2 e^{-2m_e\eta r} - \xi^2 e^{-2m_e\xi r}), \tag{17}$$

$$\Delta E_{\mathrm{VP-VP-VP}}(2P\text{-}2S) = 0.0005\ \mathrm{meV}, \tag{18}$$
$$\Delta E_{\mathrm{VP-two-loopVP}}(2P\text{-}2S) = 0.0034\ \mathrm{meV}, \tag{19}$$

where the subscripts (VP-VP-VP) and (VP-two-loop VP) designate only the Feynman amplitudes shown in Figs. 2(a) and 2(b), respectively. Sixth-order vacuum polarization contributions, including Eqs. (18) and (19), were obtained in Ref. [2]. The contribution of other diagrams corresponding to the three-loop contribution in the 1γ approximation were calculated in Refs. [29,30] for muonic hydrogen. We estimated their contribution for the Lamb shift in μD from the results given in Eqs. (18) and (23) of Ref. [29]; the result is 0.0021 meV. This gives a total three-loop contribution of 0.0060 meV in the one-photon interaction, which is included in Table I. The two-loop and three-loop vacuum polarization corrections appearing in second-order perturbation theory are calculated in the next sections. Our sum of all three-loop VP contributions, 0.0086 meV, is very close to the total three-loop contribution, 0.00842 meV, given in Table I of Ref. [31], with rounding.

An additional one-loop vacuum polarization diagram is presented in Fig. 3. In the energy spectrum this diagram gives the correction of fifth order in α (the Wichmann-Kroll correction) [42,43]. The particle interaction potential can be written in this case in the integral form as follows:

$$\Delta V^{\mathrm{WK}}(r) = \frac{\alpha(Z\alpha)^3}{\pi r}\int_0^\infty \frac{d\zeta}{\zeta^4}e^{-2m_e\zeta r}\Big[-\frac{\pi^2}{12}\sqrt{\zeta^2-1}\theta(\zeta-1)$$
$$+\int_0^\zeta dx\sqrt{\zeta^2-x^2}f^{\mathrm{WK}}(x)\Big]. \tag{20}$$

TABLE I. Lamb shift ($2P_{1/2}$-$2S_{1/2}$) in muonic deuterium atom.

Contribution to the splitting	$\Delta E(2P\text{-}2S)$ (meV)	Equation, reference
1	2	3
VP contribution of order $\alpha(Z\alpha)^2$ in one-photon interaction	227.6347	(6), [2]
Two-loop VP contribution of order $\alpha^2(Z\alpha)^2$ in one-photon interaction	1.6660	(9), (14), [2]
VP and MVP contribution in one-photon interaction	0.0001	(11), [2]
Three-loop VP contribution in one-photon interaction	0.0060	(17), (18), [29,31]
The Wichmann-Kroll correction	−0.0011	(21), [2,31]
Light-by-light contribution	0.0001	[31]
Relativistic and VP corrections of order $\alpha(Z\alpha)^4$ in first-order PT	−0.0353	(27)–(30), [5]
Relativistic and two-loop VP corrections of order $\alpha^2(Z\alpha)^4$ in first-order PT	−0.0002	(32)
Two-loop VP contribution of order $\alpha^2(Z\alpha)^2$ in second-order PT	0.1720	(39)–(41), [31]
Relativistic and one-loop VP corrections of order $\alpha(Z\alpha)^4$ in second-order PT	0.0530	(43), [5]
Relativistic and two-loop VP corrections of order $\alpha^2(Z\alpha)^4$ in second-order PT	0.0004	(44) and (45)
Three-loop VP contribution in second-order PT of order $\alpha^3(Z\alpha)^2$	0.0025	(46)–(47), [31]
Three-loop VP contribution in third-order PT of order $\alpha^3(Z\alpha)^2$	0.0001	(48), [29,31],
Nuclear-structure contribution of order $(Z\alpha)^4$	−27.8749	(49), [2,6]
Nuclear-structure and polarizability contribution of order $(Z\alpha)^5$	1.6800	[51]
Nuclear-structure and VP contribution in 1γ interaction of order $\alpha(Z\alpha)^4$	−0.0620	(51)
Nuclear-structure and VP contribution in second-order PT of order $\alpha(Z\alpha)^4$	−0.0940	(52)
Nuclear-structure and two-loop VP contribution in 1γ interaction of order $\alpha^2(Z\alpha)^4$	−0.0005	(56)
Nuclear-structure and two-loop VP contribution in second-order PT of order $\alpha^2(Z\alpha)^4$	−0.0004	(57)
Nuclear-structure and polarizability contribution of order $\alpha(Z\alpha)^5$ with VP correction	−0.0001	(60)
Nuclear-structure contribution of order $\alpha(Z\alpha)^5$ with muon-line radiative correction	0.0044	(62), [61]
Nuclear-structure contribution of order $(Z\alpha)^6$	−0.0069	(63), [22,54]
Recoil correction of order $(Z\alpha)^4$	0.0672	(64), [5]
Recoil correction of order $(Z\alpha)^5$	−0.0266	(69), [2,6,55]
Recoil correction of order $(Z\alpha)^6$	0.0001	(70), [6]
Recoil correction to VP of order $\alpha(Z\alpha)^5$ (seagull term)	0.0002	[5]
Radiative-recoil corrections of orders $\alpha(Z\alpha)^5$, $(Z^2\alpha)(Z\alpha)^4$	−0.0026	(71), Table 9 [6]
Muon self-energy and MVP contribution	−0.7747	(72) and (73), [2,6]
VP contribution to muon form factors $F_1'(0)$, $F_2(0)$ of order $\alpha^2(Z\alpha)^4$	−0.0018	(78), [4,6,63]
VP correction to muon self-energy	−0.0047	(80), [4,6]
HVP contribution	0.0129	[64,65]
Total contribution	202.4139 ± 0.0573 [$r_d = 2.1424(21)$ fm]	
	202.7375 ± 0.2352 [$r_d = 2.130(9)$ fm]	

The exact form of the spectral function f^{WK} is presented in Refs. [6,42,43]. Numerical integration in Eq. (20) with the wave functions (2) gives the following contribution to the Lamb shift as follows:

$$\Delta E^{\mathrm{WK}}(2P\text{-}2S) = -0.0011 \text{ meV}. \tag{21}$$

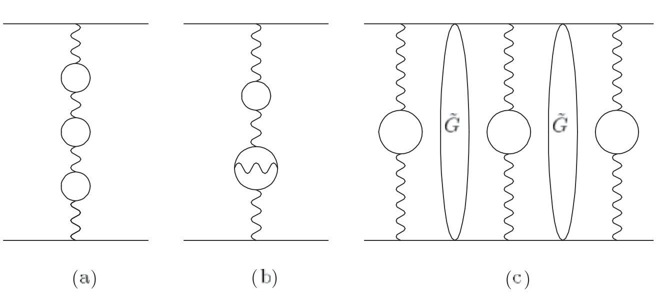


FIG. 2. Effects of three-loop vacuum polarization in the one-photon interaction [(a) and (b)] and in third order perturbation theory (c). $\tilde{G}$ is the reduced Coulomb Green's function (33).

This agrees well with other calculations [2,31]. The detailed calculation of all three light-by-light graphs is presented in Ref. [44]. We included their estimation using Eq. (21) and the results from Ref. [44] in Table I.

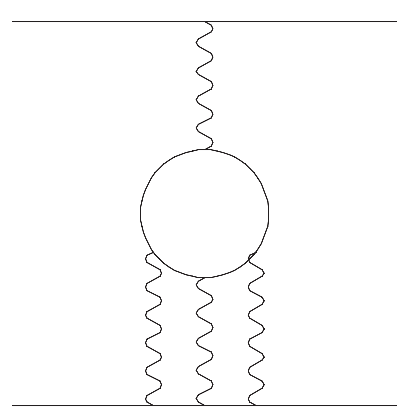


FIG. 3. The Wichmann-Kroll correction. The wave line shows the Coulomb photon.

III. RELATIVISTIC CORRECTIONS WITH THE VACUUM POLARIZATION EFFECTS

The electron vacuum polarization effects lead not only to corrections in the Coulomb potential (3) but also to the modification of other terms of the Breit Hamiltonian (1). The one-loop vacuum polarization corrections in the Breit interaction were obtained in Refs. [4,5,25]:

$$\Delta V^B_{\rm VP}(r)=\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)d\xi\sum_{i=1}^4\Delta V^B_{i,\rm VP}(r),\tag{22}$$

$$\Delta V^B_{1,\rm VP}=\frac{Z\alpha}{8}\left(\frac{1}{m_1^2}+\frac{\delta_I}{m_2^2}\right)\left[4\pi\delta(\mathbf{r})-\frac{4m_e^2\xi^2}{r}e^{-2m_e\xi r}\right],\tag{23}$$

$$\Delta V^B_{2,\rm VP}=-\frac{Z\alpha m_e^2\xi^2}{m_1m_2r}e^{-2m_e\xi r}(1-m_e\xi r),\tag{24}$$

$$\Delta V^B_{3,\rm VP}=-\frac{Z\alpha}{2m_1m_2}p_i\frac{e^{-2m_e\xi r}}{r}\left[\delta_{ij}+\frac{r_ir_j}{r^2}(1+2m_e\xi r)\right]p_j,\tag{25}$$

$$\Delta V^B_{4,\rm VP}=\frac{Z\alpha}{r^3}\left(\frac{1}{4m_1^2}+\frac{1}{2m_1m_2}\right)e^{-2m_e\xi r}(1+2m_e\xi r)(\mathbf{L}\boldsymbol{\sigma}_1),\tag{26}$$

where the superscript B designates the Breit interaction. In first-order perturbation theory (PT) the potentials $\Delta V^B_{i,\rm VP}(r)$ give necessary contributions of order $\alpha(Z\alpha)^4$ to the shift $(2P\text{-}2S)$:

$$\Delta E^B_{1,\rm VP}(2P\text{-}2S)=-0.0353\ \text{meV},\tag{27}$$

$$\Delta E^B_{2,\rm VP}(2P\text{-}2S)=0.0011\ \text{meV},\tag{28}$$

$$\Delta E^B_{3,\rm VP}(2P\text{-}2S)=0.0012\ \text{meV},\tag{29}$$

$$\Delta E^B_{4,\rm VP}(2P\text{-}2S)=-0.0023\ \text{meV}.\tag{30}$$

The potentials $\Delta V^B_{2,\rm VP}$, $\Delta V^B_{3,\rm VP}$, $\Delta V^B_{4,\rm VP}$ take into account the recoil effects over the ratio m_1/m_2. In Table I we have included the summary correction of order $\alpha(Z\alpha)^4$, which is determined by Eqs. (27)–(30). The next-to-leading-order correction of order $\alpha^2(Z\alpha)^4$ appears in the energy spectrum from the two-loop modification of the Breit Hamiltonian. We consider in the potential the term of the leading order in m_1/m_2 [the function $f(v)$ is determined by Eq. (13)]:

$$\Delta V^B_{\rm two-loopVP}(r)=\frac{\alpha^2(Z\alpha)}{12\pi^2}\left(\frac{1}{m_1^2}+\frac{\delta_I}{m_2^2}\right)\times\int_0^1\frac{f(v)dv}{1-v^2}\left[4\pi\delta(\mathbf{r})-\frac{4m_e^2}{(1-v^2)r}e^{-\frac{2m_er}{\sqrt{1-v^2}}}\right].\tag{31}$$

The corresponding $(2P\text{-}2S)$ shift is the following:

$$\Delta E^B_{\rm two-loopVP}(2P\text{-}2S)=-0.0002\ \text{meV}.\tag{32}$$

Other two-loop contributions to the Breit potential are omitted because they give energy corrections with an accuracy thatlies outside the calculation in this work.

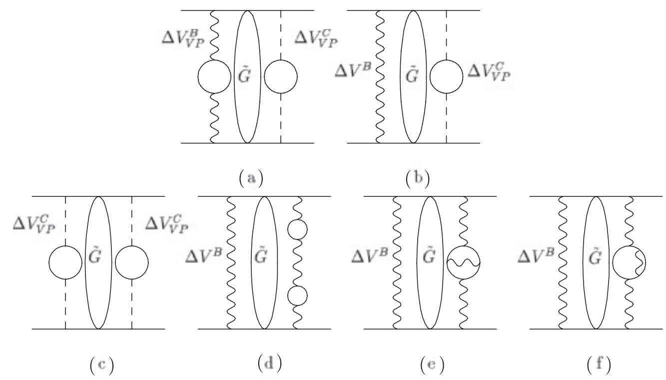


FIG. 4. Effects of one-loop and two-loop vacuum polarization in second-order perturbation theory (SOPT). The dashed line shows the Coulomb photon. $\tilde G$ is the reduced Coulomb Green's function (34). The potentials ΔV^B, $\Delta V^C_{\rm VP}$, and $\Delta V^B_{\rm VP}$ are determined, respectively, by Eqs. (1), (3), and (22).

In second-order perturbation theory (SOPT) we have a number of electron vacuum polarization contributions in orders $\alpha^2(Z\alpha)^2$ and $\alpha(Z\alpha)^4$, shown in Figs. 4(b) and 4(c):

$$\Delta E^{\rm VP}_{\rm SOPT}=\langle\psi|\Delta V^C_{\rm VP}\tilde G\Delta V^C_{\rm VP}|\psi\rangle+2\langle\psi|\Delta V^B\tilde G\Delta V^C_{\rm VP}|\psi\rangle.\tag{33}$$

The abbreviation SOPT is used in Tables I and II for the contributions obtained in second-order PT. The second-order perturbation theory corrections in the energy spectrum of the hydrogen-like system are determined by use of the reduced Coulomb Green's function $\tilde G$ (RCGF). It has a partial-wave expansion [45] as follows:

$$\tilde G_n(\mathbf{r},\mathbf{r}')=\sum_{l,m}\tilde g_{nl}(r,r')Y_{lm}(\mathbf{n})Y^*_{lm}(\mathbf{n}').\tag{34}$$

The radial function $\tilde g_{nl}(r,r')$ was presented in Ref. [45] in a form of the Sturm expansion in the Laguerre polynomials. For a calculation of the Lamb shift $(2P\text{-}2S)$ in muonic deuterium it is convenient to use the compact representation for the RCGF of $2S$ and $2P$ states, which was obtained in Refs. [4,46]:

$$\tilde G(2S)=-\frac{Z\alpha\mu^2}{4x_1x_2}e^{-\frac{x_1+x_2}{2}}\frac{1}{4\pi}g_{2S}(x_1,x_2),\tag{35}$$

$$\begin{aligned}g_{2S}(x_1,x_2)=&\,8x_<-4x_<^2+8x_>+12x_<x_>-26x_<^2x_>\\&+2x_<^3x_>-4x_>^2-26x_<x_>^2+23x_<^2x_>^2\\&-x_<^3x_>^2+2x_<x_>^3-x_<^2x_>^3+4e^x(1-x_<)\\&\times(x_>-2)x_>+4(x_<-2)x_<(x_>-2)x_>\\&\times[-2C+Ei(x_<)-\ln(x_<)-\ln(x_>)],\end{aligned}\tag{36}$$

$$\tilde G(2P)=-\frac{Z\alpha\mu^2}{36x_1^2x_2^2}e^{-\frac{x_1+x_2}{2}}\frac{3}{4\pi}\frac{(\mathbf{x}_1\mathbf{x}_2)}{x_1x_2}g_{2P}(x_1,x_2),\tag{37}$$

$$\begin{aligned}g_{2P}(x_1,x_2)=&\,24x_<^3+36x_<^3x_>+36x_<^3x_>^2+24x_>^3\\&+36x_<x_>^3+36x_<^2x_>^3+49x_<^3x_>^3-3x_<^4x_>^3\\&-12e^{x_<}(2+x_<+x_<^2)x_>^3-3x_<^3x_>^4+12x_<^3x_>^3\\&\times[-2C+Ei(x_<)-\ln(x_<)-\ln(x_>)],\end{aligned}\tag{38}$$

TABLE II. Fine structure of the $2P$ state in the muonic deuterium atom.

Contribution to fine splitting $\Delta E^{\rm fs}$	Numerical value (meV)	Equation, reference
Contribution of order $(Z\alpha)^4\,\frac{\mu^3(Z\alpha)^4}{32m_1^2}\left(1+\frac{2m_1}{m_2}\right)$	8.83848	(82), [2,6]
Muon AMM contribution $\frac{\mu^3(Z\alpha)^4}{16m_1^2}a_\mu\left(1+\frac{m_1}{m_2}\right)$	0.01957	(82), [2,6]
Contribution of order $(Z\alpha)^6$	0.00031	(82), [2,6]
Contribution of order $(Z\alpha)^6 m_1/m_2$	−0.00001	(82), [2,6]
Contribution of order $\alpha(Z\alpha)^4$ in first-order PT $\langle\Delta V^{\rm fs}_{\rm VP}\rangle$	0.00346	(84)
Contribution of order $\alpha(Z\alpha)^4$ in second-order PT $\langle\Delta V^C_{\rm VP}\tilde G\Delta V^{\rm fs}\rangle$	0.00229	(85)
Contribution of order $\alpha(Z\alpha)^6\,\frac{\alpha(Z\alpha)^6\mu^3}{32\pi m_1^2}\left[\ln\frac{\mu(Z\alpha)^2}{m_1}+\frac{1}{5}\right]$	−0.00001	(82), [6]
VP Contribution from 1γ interaction of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^{\rm fs}_{\rm VP-VP}\rangle$	0.000003	(87)
VP Contribution from 1γ interaction of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^{\rm fs}_{\rm two-loop,VP}\rangle$	0.00002	(89)
VP Contribution in second-order PT of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^C_{\rm VP}\tilde G\Delta V^{\rm fs}_{\rm VP}\rangle$	0.000002	Fig. 4(a), $\Delta V^B\to\Delta V^{\rm fs}$
VP Contribution in second-order PT of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^C_{\rm VP-VP}\tilde G\Delta V^{\rm fs}\rangle$	−0.000001	Fig. 4(d), $\Delta V^B\to\Delta V^{\rm fs}$
VP Contribution in second-order PT of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^C_{\rm 2-loop,VP}\tilde G\Delta V^{\rm fs}\rangle$	0.000026	(90), Figs. 4(e) and 4(f), $\Delta V^B\to\Delta V^{\rm fs}$
Nuclear-structure correction in 1γ interaction	−0.00028	(91), [22]
Summary contribution	8.86386	

where $x_< = \min(x_1,x_2)$, $x_> = \max(x_1,x_2)$, and $C = 0.57721566\ldots$ is the Euler constant. As a result, the two-loop vacuum polarization contribution to the first term of Eq. (33) can be presented originally in the integral form [Fig. 4(c)]. The subsequent numerical integration gives the following results:

$$\Delta E^{\rm VP,VP}_{\rm SOPT}(2S) = -\frac{\mu\alpha^2(Z\alpha)^2}{72\pi^2}\int_1^\infty \rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\int_0^\infty\left(1-\frac{x}{2}\right)e^{-x\left(1+\frac{2m_e\xi}{W}\right)}dx\int_0^\infty\left(1-\frac{x'}{2}\right)e^{-x'\left(1+\frac{2m_e\eta}{W}\right)}dx'g_{2S}(x,x')$$
$$= -0.1750 \text{ meV}, \tag{39}$$

$$\Delta E^{\rm VP,VP}_{\rm SOPT}(2P) = -\frac{\mu\alpha^2(Z\alpha)^2}{7776\pi^2}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\int_0^\infty e^{-x\left(1+\frac{2m_e\xi}{W}\right)}dx\int_0^\infty e^{-x'\left(1+\frac{2m_e\eta}{W}\right)}dx'g_{2P}(x,x')$$
$$= -0.0030 \text{ meV}, \tag{40}$$

where the superscript (VP,VP) designates the second-order PT contribution when each of the perturbation potentials contains VP correction. The results, Eqs. (39) and (40), agree with the calculation in Ref. [31]. Changing the one-electron VP potential by the muon VP potential, we find that corresponding correction to the Lamb shift is very small:

$$\Delta E^{\rm VP,MVP}_{\rm SOPT}(2P\text{-}2S) = 0.0001 \text{ meV}. \tag{41}$$

The second term in Eq. (33) has a similar structure [see Fig. 4(b)]. A transformation of different matrix elements in it is carried out with the use of algebraic relations of the following form:

$$\langle\psi|\frac{\mathbf{p}^4}{(2\mu)^2}\sum_m{}'\frac{|\psi_m\rangle\langle\psi_m|}{E_2-E_m}\Delta V^C_{\rm VP}|\psi\rangle = \langle\psi|\left(E_2+\frac{Z\alpha}{r}\right)\left(\hat H_0+\frac{Z\alpha}{r}\right)\sum_m{}'\frac{|\psi_m\rangle\langle\psi_m|}{E_2-E_m}\Delta V^C_{\rm VP}|\psi\rangle$$
$$= \langle\psi|\left(E_2+\frac{Z\alpha}{r}\right)^2\tilde G\Delta V^C_{\rm VP}|\psi\rangle - \langle\psi|\frac{Z\alpha}{r}\Delta V^C_{\rm VP}|\psi\rangle + \langle\psi|\frac{Z\alpha}{r}|\psi\rangle\langle\psi|\Delta V^C_{\rm VP}|\psi\rangle. \tag{42}$$

Omitting further details of the calculation of numerous matrix elements in Eq. (42), we present here the summary numerical contribution from the second term in Eq. (33) to the shift $(2P\text{-}2S)$:

$$\Delta E^{B,\rm VP}_{\rm SOPT}(2P\text{-}2S) = 0.0530 \text{ meV}. \tag{43}$$

Other contributions of second-order PT [see Figs. 4(d)–4(f)] have the general structure similar to Eqs. (39) and (40). They appear after the replacements $\Delta V^C_{\rm VP}\to\Delta V^B$ and $\Delta V^C_{\rm VP}\to\Delta V^C_{\rm VP,VP}$ in the basic amplitude shown in Fig. 4(c). The estimate of this contribution of order $\alpha^2(Z\alpha)^4$ to the shift $(2P\text{-}2S)$ can be derived if we take into account in the Breit potential the leading-order term in the ratio m_1/m_2. Its numerical value is

$$\Delta E^{\rm VP,VP;\Delta V^B}_{\rm SOPT}(2P\text{-}2S) = 0.0004 \text{ meV}. \tag{44}$$

The two-loop vacuum polarization contribution is determined also by the amplitude in Fig. 4(a). To obtain its numerical value in the energy spectrum we have to use Eqs. (3) and (22). In the leading order in the ratio m_1/m_2 we take again the potential (22), which leads to very small correction of order $\alpha^2(Z\alpha)^4$ as follows:

$$\Delta E_{\rm SOPT}^{{\rm VP},\Delta V_{\rm VP}^B}(2P\text{-}2S) = -0.00001\ \text{meV}. \quad (45)$$

Three-loop vacuum polarization contributions to the energy spectrum in second-order perturbation theory are presented in Fig. 5. Respective potentials required for their calculation are obtained earlier in Eqs. (3), (8), and (12). Considering an accuracy of the calculation we can restrict our analysis by a shift of the $2S$ level, which can be written in the following form:

$$\Delta E_{\rm SOPT}^{\rm VP-VP,VP}(2S) = -\frac{\mu\alpha^3(Z\alpha)^2}{108\pi^3}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\int_1^\infty \rho(\zeta)d\zeta\int_0^\infty dx\left(1-\frac{x}{2}\right)$$
$$\times\int_0^\infty dx'\left(1-\frac{x'}{2}\right)e^{-x'(1+\frac{2m_e\zeta}{W})}\frac{1}{\xi^2-\eta^2}\left[\xi^2 e^{-x(1+\frac{2m_e\xi}{W})}-\eta^2 e^{-x(1+\frac{2m_e\eta}{W})}\right]g_{2S}(x,x') = -0.0007\ \text{meV}, \quad (46)$$

$$\Delta E_{\rm SOPT}^{\rm 2-loopVP,VP}(2S) = -\frac{\mu\alpha^3(Z\alpha)^2}{18\pi^3}\int_0^1 \frac{f(v)dv}{1-v^2}\int_1^\infty \rho(\xi)d\xi$$
$$\times\int_0^\infty dx\left(1-\frac{x}{2}\right)e^{-x(1+\frac{2m_e}{\sqrt{1-v^2}W})}\int_0^\infty dx'\left(1-\frac{x'}{2}\right)e^{-x'(1+\frac{2m_e\xi}{W})}g_{2S}(x,x') = -0.0018\ \text{meV}. \quad (47)$$

In third-order perturbation theory (TOPT), the three-loop VP contribution to the Lamb shift consists of two terms. One part of it is shown in Fig. 2(c). This contribution can be calculated by means of Eqs. (3) and (35)–(38) [29,31]. We carry out the coordinate integration analytically and the integration over three spectral parameters numerically. The result,

$$\Delta E_{\rm TOPT}^{\rm VP,VP,VP}(2P\text{-}2S) = 0.0001\ \text{meV}, \quad (48)$$

is in agreement with Refs. [29,31].

IV. NUCLEAR-STRUCTURE AND VACUUM POLARIZATION EFFECTS

An influence of nuclear structure on the muon motion in muonic deuterium is determined in the leading order by the root-mean-square (rms) radius of the deuteron (charge radius). We present all charge radius corrections at two values of r_d: $r_d = 2.1424(21)$ fm (CODATA 2010) and $r_d = 2.130(9)$ fm [34] [Fig. 6(a)]:

$$\Delta E_{\rm str}(2P\text{-}2S) = -\frac{\mu^3(Z\alpha)^4}{12}r_d^2 = -6.07313\times r_d^2$$
$$= -27.8749(-27.5532)\ \text{meV}, \quad (49)$$

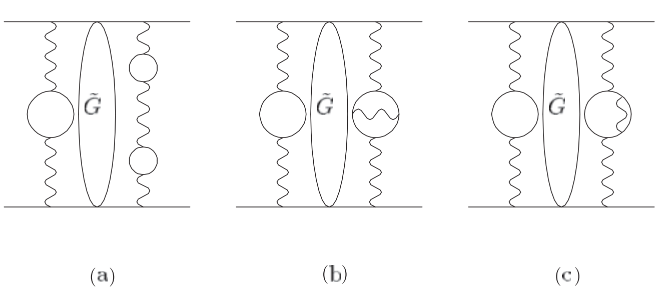


FIG. 5. The three-loop vacuum polarization corrections in second-order perturbation theory. $\tilde{G}$ is the reduced Coulomb Green's function.

where the subscript "str" designates the structure correction. The precise value of the deuteron charge radius is needed for the interpretation of new data on transitions in the muonic deuterium atom.

There are vacuum polarization corrections connected with the deuteron structure in first- and second-order perturbation theory [see diagrams in Figs. 6(b) and 6(c)]. The potential corresponding to the amplitude in Fig. 6(b) can be written as follows:

$$\Delta V_{\rm str}^{\rm VP}(r) = \frac{2Z\alpha^2}{9}r_d^2\int_1^\infty \rho(\xi)d\xi\left[\delta(\mathbf{r})-\frac{m_e^2\xi^2}{\pi r}e^{-2m_e\xi r}\right]. \quad (50)$$

Its contribution to the $2P$-$2S$ Lamb shift is determined by the following formula:

$$\Delta E_{\rm str}^{\rm VP}(2P\text{-}2S)$$
$$= -\frac{\mu^3\alpha(Z\alpha)^4}{36\pi}r_d^2\int_1^\infty \rho(\xi)d\xi\left[1-\frac{16m_e^4\xi^4}{(2m_e\xi+W)^2}\right]$$
$$= -0.01350r_d^2 = -0.0620(-0.0612)\ \text{meV}. \quad (51)$$

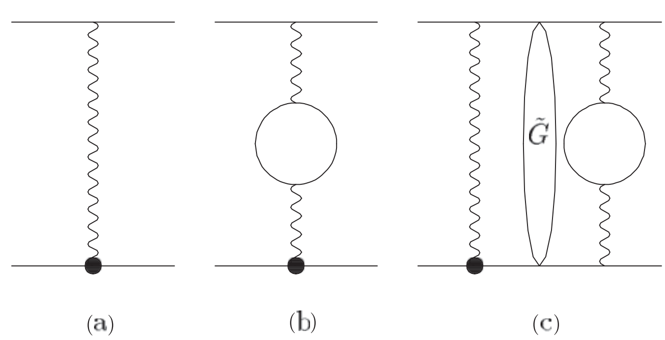


FIG. 6. The leading-order nuclear-structure and vacuum polarization corrections. The thick point represents the nuclear vertex operator.

The contribution of the same order $\alpha(Z\alpha)^4$ is specified by the amplitude in the second-order perturbation theory in Fig. 6(c):

$$\Delta E^{\rm VP}_{\rm str,SOPT}(2P\text{-}2S) = -\frac{\mu^3\alpha(Z\alpha)^4}{36\pi}r_d^2\int_1^\infty \rho(\xi)d\xi\,\frac{-12+23b_1-8b_1^2-4b_1^3+4b_1^4+4b_1\left(3-4b_1+2b_1^2\right)\ln b_1}{b_1^5}$$
$$= -0.020487 r_d^2 \text{ meV} = -0.0940(-0.0929) \text{ meV}, \quad b_1 = 1+\frac{2m_e}{W}\xi. \tag{52}$$

Factorizing r_d^2 in Eqs. (49), (51), and (52), we obtain the finite-size correction in the following form:

$$\Delta E_{\rm str}(2P\text{-}2S) + \Delta E^{\rm VP}_{\rm str}(2P\text{-}2S) + \Delta E^{\rm VP}_{\rm str,SOPT}(2P\text{-}2S)$$
$$= -6.10712 r_d^2 = -28.0309(-27.7074) \text{ meV}. \tag{53}$$

The next important correction of order $(Z\alpha)^5$ is described by one-loop exchange diagrams (Fig. 7). An investigation of the elastic contribution to the Lamb shift and the deuteron polarizability contribution was performed in Refs. [22,47–50]. A recent detailed calculation of the nuclear-structure and polarizability corrections which improves the previous theoretical results is presented in Ref. [51]. In Table I we have included the value of the $(2P\text{-}2S)$ shift 1.680(16) meV from Ref. [51].

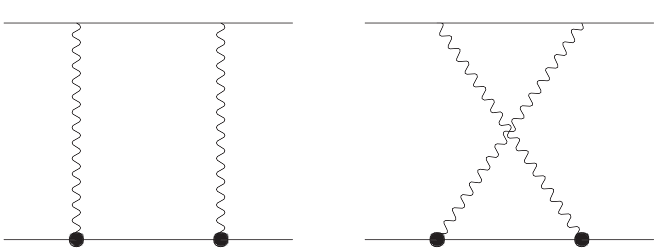
Two-loop vacuum polarization corrections with an account of the nuclear structure are presented in Figs. 8(a)–8(c). The interaction operators constructed by means of Eq. (7) are determined by the following integral formulas:

$$\Delta V^{\rm VP-VP}_{\rm str}(r)$$
$$= \frac{2Z\alpha^3}{27\pi^2}r_d^2\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta$$
$$\times\left[\pi\delta(\mathbf{r}) - \frac{m_e^2}{r(\xi^2-\eta^2)}\left(\xi^4 e^{-2m_e\xi r} - \eta^4 e^{-2m_e\eta r}\right)\right], \tag{54}$$

$$\Delta V^{\rm two-loopVP}_{\rm str}(r)$$
$$= \frac{4Z\alpha^3}{9\pi^2}r_d^2\int_0^1 \frac{f(v)dv}{1-v^2}\left[\pi\delta(\mathbf{r}) - \frac{m_e^2}{r(1-v^2)}e^{-\frac{2m_e r}{\sqrt{1-v^2}}}\right]. \tag{55}$$

The sum of corrections from Eqs. (54) and (55) to the Lamb shift $(2P\text{-}2S)$ is equal to

$$\Delta E^{\rm VP,VP}_{\rm str}(2P\text{-}2S) = (-10.5\times 10^{-5})r_d^2$$
$$= -0.0005(0.0005) \text{ meV}. \tag{56}$$



(a) (b)

FIG. 7. Nuclear-structure corrections of order $(Z\alpha)^5$. The thick point is the deuteron vertex operator.

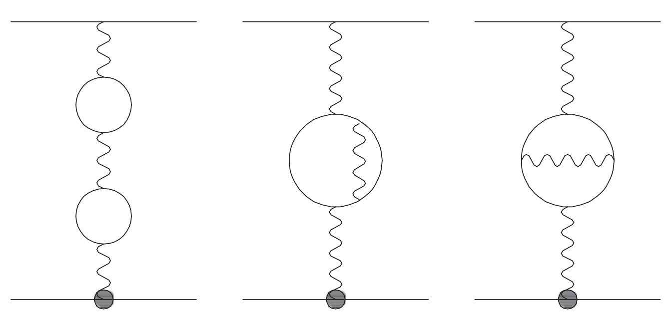
The two-loop vacuum polarization and nuclear-structure corrections of order $\alpha^2(Z\alpha)^4$ in second-order PT shown in Figs. 9 (a)–9(d) also can be calculated by means of relations discussed in Sec. III. The summary shift is equal to

$$\Delta E^{\rm VP,VP}_{\rm str,SOPT}(2P\text{-}2S) = (-9.5\times 10^{-5})r_d^2$$
$$= -0.0004(0.0004) \text{ meV}. \tag{57}$$

In addition, there is the nuclear-structure correction of order $\alpha(Z\alpha)^5$ coming from the two-photon exchange diagrams with the electron vacuum polarization insertion (see Fig. 10). It can be calculated as the elastic contribution of order $(Z\alpha)^5$ [50]. However, there is no need to calculate it because in this case we have the same cancellation between the elastic two-photon correction and a part of the deuteron excited states correction as for the contribution of order $(Z\alpha)^5$ [51]. Indeed, using the notations of Ref. [51], we can present the muon matrix element $P_{\rm VP}$ for nonrelativistic two-photon exchange with an account of the vacuum polarization in the following form:

$$P_{\rm VP} = \frac{2\alpha^3}{3\pi}\phi^2(0)\int_1^\infty \rho(\xi)d\xi$$
$$\times\int\frac{d\mathbf{q}}{(2\pi)^3}\frac{(4\pi)^2}{q^2\left(q^2+4m_e^2\xi^2\right)}\frac{1}{E+\frac{q^2}{2m_1}}$$
$$\times\left[e^{i\mathbf{q}(\mathbf{R}-\mathbf{R}')} - 1 + \frac{q^2}{6}(\mathbf{R}-\mathbf{R}')^2\right], \tag{58}$$

where $\mathbf{R}$ is the position of the proton with respect to the nuclear mass center. Integrating (58) over q and expanding the resulting expression over small parameter $\sqrt{2m_1E}|\mathbf{R}-\mathbf{R}'|$



(a) (b) (c)

FIG. 8. Nuclear-structure and two-loop vacuum polarization effects in one-photon interaction. The thick point is the nuclear vertex operator.

we obtain

$$P_{\rm VP} = \frac{32\alpha^3}{3} m_1 \phi^2(0) |\mathbf{R} - \mathbf{R}'|^3 \int_1^\infty \rho(\xi) d\xi \left[\frac{a_\xi^3 - 3a_\xi^2 + 6a_\xi + 6e^{-a_\xi} - 6}{12 a_\xi^4} - 2m_1 E |\mathbf{R} - \mathbf{R}'|^2 \frac{a_\xi^4 - 4a_\xi^3 + 12a_\xi^2 - 24a_\xi - 24e^{-a_\xi} + 24}{48 a_\xi^6} \right], \quad a_\xi = 2m_e \xi |\mathbf{R} - \mathbf{R}'|. \tag{59}$$

It follows from Eq. (59) that in the leading order in $\sqrt{2m_1 E}|\mathbf{R} - \mathbf{R}'|$ the elastic correction to atomic energy is canceled by the deuteron excited states correction (see more detailed discussion in Ref. [51]). An estimation of the second term contribution in the square brackets of Eq. (59) to the energy spectrum can be derived if we take into account that the integral over ξ is determined by the region near $\xi \approx 1$. Expanding the second term in Eq. (59) at small a_ξ we obtain $(-\pi/240 a_\xi)$. Performing an analytical integration over ξ and summing over excited deuteron states, we then obtain the contribution to the Lamb shift as follows:

$$\delta E^{\rm VP}_{\rm pol}(2P\text{-}2S) = -\frac{m_1^2 \alpha^3 \phi^2(0)}{1024 m_e} \left[\frac{1}{3} \langle \phi_D | R^2 H_D R^2 | \phi_D \rangle - \frac{4}{5} \langle \phi_D | R_i H_D R^2 R_i | \phi_D \rangle + \frac{2}{5} \langle \phi_D | \left(R_i R_j - \frac{1}{3} \delta_{ij} R^2 \right) H_D (R_i R_j - \frac{1}{3} \delta_{ij} R^2) | \phi_D \rangle \right] = -0.0001 \text{ meV}, \tag{60}$$

where ϕ_D is the deuteron wave function. We make all integrations in Eq. (60) analytically using the deuteron wave function in the zero-range approximation [52]

$$\phi_D(r) = \sqrt{\frac{\kappa}{2\pi}} \frac{1}{r} e^{-\kappa r}, \tag{61}$$

where $\kappa = 0.0457$ GeV is the inverse deuteron size.

Another term in the Lamb shift of order $\alpha(Z\alpha)^5$ is determined by a muon-line radiative correction to the nuclear size effect. It was obtained in Ref. [53] in a suitable form for a subsequent numerical estimate as follows:

$$\Delta E^{\alpha(Z\alpha)^5}_{\rm str}(2P\text{-}2S) = 1.985 \frac{\alpha (Z\alpha)^5 \mu^3}{8} r_d^2 = (9.62 \times 10^{-4}) r_d^2 = 0.0044(0.0044) \text{ meV}. \tag{62}$$

In addition, there is the correction of order $\alpha(Z\alpha)^5$ with a muon vacuum polarization [see Fig. 10]. Accounting for the result of its calculation from Ref. [6], the total coefficient in Eq. (62) should be changed as follows: $1.985 \to 1.485$. However, we can consider the muon VP and nuclear-structure amplitudes in Fig. 10 together with the contribution of the deuteron excited states. Calculating this summary contribution by means of equations similar to Eqs. (58)–(60) (see also Ref. [51]), we observe the cancellation of the elastic correction and excited states correction.

Nuclear-structure corrections of order $(Z\alpha)^6$ can be derived with the use of relativistic corrections to nonrelativistic wave functions in matrix element (49) [6,22,54]. We present here the total contribution to the Lamb shift $(2P\text{-}2S)$, including an additional state-independent correction obtained in Refs. [22,54] as follows:

$$\begin{aligned} \Delta E^{(Z\alpha)^6}_{\rm str}&(2P\text{-}2S) \\ &= \frac{(Z\alpha)^6}{12} \mu^3 \left\{ r_d^2 \left[\langle \ln \mu Z\alpha r \rangle + C - \frac{3}{2} \right] - \frac{1}{2} r_d^2 + \frac{1}{3} \langle r^3 \rangle \left\langle \frac{1}{r} \right\rangle - I_2^{\rm rel} - I_3^{\rm rel} - \mu^2 F_{\rm NR} + \frac{1}{40} \mu^2 \langle r^4 \rangle \right\} \\ &= (-21.28 \times 10^{-4}) r_d^2 + 0.0029 \\ &= -0.0069(-0.0068) \text{ meV}, \end{aligned} \tag{63}$$

where the quantities $I^{\rm rel}_{2,3}$ and $F_{\rm NR}$ are written explicitly in Refs. [22,54]. In the square brackets we have extracted the frequently used quantity (main term) for an estimation of the contribution to the $(2P\text{-}2S)$ Lamb shift in the hydrogen atom because other corrections are very small (near 1%) and could be safely omitted. In the case of muonic deuterium they give the contribution near 25% of the main term and should be taken into account. Another separation of the terms corresponding to the contributions of the $2S$ and $2P$ states in the curly brackets is the following: $\{\frac{3}{16} r_d^2 + \frac{1}{40} \mu^2 \langle r^4 \rangle + r_d^2 [\langle \ln \mu Z\alpha r \rangle + C - \frac{35}{16}] + \frac{1}{3} \langle r^3 \rangle \langle \frac{1}{r} \rangle - I_2^{\rm rel} - I_3^{\rm rel} - \mu^2 F_{\rm NR}\}$. A numerical estimate is given on the basis of an exponential parametrization for the charge distribution from Ref. [22].

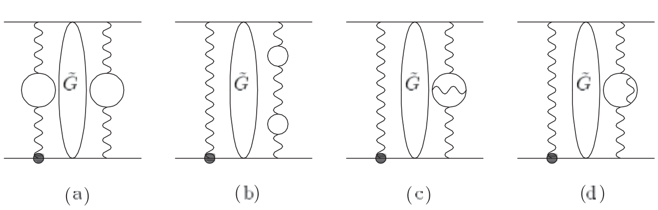


FIG. 9. Nuclear-structure and two-loop vacuum polarization effects in second-order perturbation theory. The thick point is the nuclear vertex operator. $\tilde{G}$ is the reduced Coulomb Green's function.

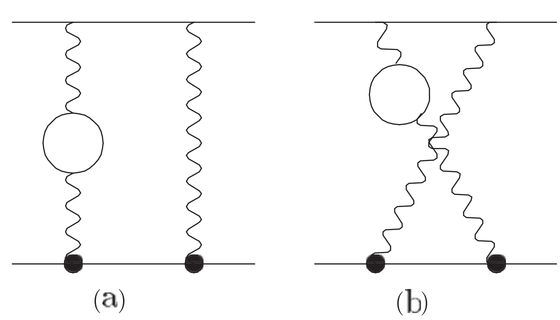


FIG. 10. The nuclear-structure and electron vacuum polarization effects in the two-photon exchange diagrams. The thick point is the nuclear vertex operator.

V. RECOIL CORRECTIONS, MUON SELF-ENERGY, AND VACUUM POLARIZATION EFFECTS

Research of different order corrections to the Lamb shift ($2P$-$2S$) of electronic hydrogen has been performed for many years. Modern analysis of the advances in the solution of this problem is presented in several review articles [6,43,55,56]. Most of the results were obtained in analytical form, so they can be used directly in the muonic deuterium atom. In this section we analyze different contributions to the energy spectrum of μD up to sixth order in α and derive their numerical estimations in the Lamb shift ($2P$-$2S$).

There are several recoil corrections of different orders in α which give important contributions in order to attain the necessary accuracy of the calculation. The recoil correction of order $(Z\alpha)^4\mu^3/m_2^2$ to the Lamb shift appears in the matrix element of the Breit potential with functions (2). It is calculated for muonic deuterium in Refs. [5,37] as follows:

$$\Delta E_{\rm rec}(2P\text{-}2S)=\frac{\mu^3(Z\alpha)^4}{12m_2^2}=0.0672\ \text{meV}. \tag{64}$$

The recoil correction of fifth order in $(Z\alpha)$ is determined by use of the following expression [6,55]:

$$\Delta E_{\rm rec}^{(Z\alpha)^5}=\frac{\mu^3(Z\alpha)^5}{m_1m_2\pi n^3}\left[\frac{2}{3}\delta_{l0}\ln\frac{1}{Z\alpha}-\frac{8}{3}\ln k_0(n,l)-\frac{1}{9}\delta_{l0}-\frac{7}{3}a_n-\frac{2}{m_2^2-m_1^2}\delta_{l0}\left(m_2^2\ln\frac{m_1}{\mu}-m_1^2\ln\frac{m_2}{\mu}\right)\right], \tag{65}$$

where $\ln k_0(n,l)$ is the Bethe logarithm, which is as follows:

$$\ln k_0(2S)=2.811769893120563, \tag{66}$$

$$\ln k_0(2P)=-0.030016708630213, \tag{67}$$

$$a_n=-2\left[\ln\frac{2}{n}+\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)+1-\frac{1}{2n}\right]\delta_{l0}+\frac{(1-\delta_{l0})}{l(l+1)(2l+1)}. \tag{68}$$

Equation (65) gives the following numerical correction to the Lamb shift:

$$\Delta E_{\rm rec}^{(Z\alpha)^5}(2P\text{-}2S)=-0.0266\ \text{meV}. \tag{69}$$

The recoil correction of the sixth order in $(Z\alpha)$ was calculated analytically in Refs. [32,57–60] as follows:

$$\Delta E_{\rm rec}^{(Z\alpha)^6}(2P\text{-}2S)=\frac{(Z\alpha)^6m_1^2}{8m_2}\left(\frac{23}{6}-4\ln 2\right)=0.0001\ \text{meV}. \tag{70}$$

Omitting the explicit form of the radiative-recoil corrections of orders $\alpha(Z\alpha)^5$ and $(Z^2\alpha)(Z\alpha)^4$ from Table 9 in Ref. [6], we present their numerical value to the Lamb shift ($2P$-$2S$) of the muonic deuterium atom as follows:

$$\Delta E_{\rm rad-rec}(2P\text{-}2S)=-0.0026\ \text{meV}. \tag{71}$$

The energy contributions obtained in Refs. [6,61,62] from radiative corrections to the lepton line, the Dirac and Pauli form factors and muon vacuum polarization are given by

$$\Delta E_{\rm MVP,MSE}(2S)=\frac{\alpha(Z\alpha)^4}{8\pi}\frac{\mu^3}{m_1^2}\left\{\frac{4}{3}\ln\frac{m_1}{\mu(Z\alpha)^2}-\frac{4}{3}\ln k_0(2S)+\frac{38}{45}+\frac{\alpha}{\pi}\left[-\frac{9}{4}\zeta(3)+\frac{3}{2}\pi^2\ln 2-\frac{10}{27}\pi^2-\frac{2179}{648}\right]+4\pi Z\alpha\left(\frac{427}{384}-\frac{\ln 2}{2}\right)\right\}=0.7647\ \text{meV}, \tag{72}$$

$$\Delta E_{\rm MVP,MSE}(2P)=\frac{\alpha(Z\alpha)^4}{8\pi}\frac{\mu^3}{m_1^2}\left\{-\frac{4}{3}\ln k_0(2P)-\frac{m_1}{6\mu}-\frac{\alpha}{3\pi}\frac{m_1}{\mu}\left[\frac{3}{4}\zeta(3)-\frac{\pi^2}{2}\ln 2+\frac{\pi^2}{12}+\frac{197}{144}\right]\right\}=-0.0100\ \text{meV}. \tag{73}$$

The diagram in Fig. 11(b) with the electron loop polarization insertion in the radiative photon gives the contribution to the energy spectrum, which can be expressed in terms of the slope of the Dirac form factor F_1' and the Pauli form factor F_2 [6] as follows:

$$\Delta E_{\rm rad+VP}(nS)=\frac{\mu^3}{m_1^2}\frac{(Z\alpha)^4}{n^3}\left[4m_1^2F_1'(0)\delta_{l0}+F_2(0)\frac{C_{jl}}{2l+1}\right], \tag{74}$$

$$C_{jl}=\delta_{l0}+(1-\delta_{l0})\frac{\left[j(j+1)-l(l+1)-\frac{3}{4}\right]}{l(l+1)}\frac{m_1}{\mu}. \tag{75}$$

The two-loop contribution to the form factors $F_1'(0)$ and $F_2(0)$ was calculated in Ref. [63] (see also Refs. [1,6]) as follows:

$$m_1^2F_1'(0)=\left(\frac{\alpha}{\pi}\right)^2\left[\frac{1}{9}\ln^2\frac{m_1}{m_e}-\frac{29}{108}\ln\frac{m_1}{m_e}+\frac{1}{9}\zeta(2)+\frac{395}{1296}\right], \tag{76}$$

$$F_2(0)=\left(\frac{\alpha}{\pi}\right)^2\left[\frac{1}{3}\ln\frac{m_1}{m_e}-\frac{25}{36}+\frac{\pi^2}{4}\frac{m_e}{m_1}-4\frac{m_e^2}{m_1^2}\ln\frac{m_1}{m_e}+3\frac{m_e^2}{m_1^2}\right]. \tag{77}$$

The correction to the Lamb shift then is equal to

$$\Delta E_{\rm rad+VP}(2P\text{-}2S)=-0.0018\ \text{meV}. \tag{78}$$

To estimate the muon self-energy and electron vacuum polarization contribution in Fig. 11(a), we use the relation obtained in Ref. [4] as follows:

$$\Delta E^{\rm VP}_{\rm MSE}=\frac{\alpha}{3\pi m_1^2}\ln\frac{m_1}{\mu(Z\alpha)^2}\biggl[\langle\psi_n|\Delta\Delta V^C_{\rm VP}|\psi_n\rangle +2\langle\psi_n|\Delta V^C_{\rm VP}\tilde{G}\Delta\left(-\frac{Z\alpha}{r}\right)|\psi_n\rangle\biggr]. \quad (79)$$

The sum of all matrix elements which appear in Eq. (79) leads to the following shift ($2P$-$2S$):

$$\Delta E^{\rm VP}_{\rm MSE}(2P\text{-}2S)=-0.0047\ \text{meV}. \quad (80)$$

The hadron vacuum polarization (HVP) contribution which can be taken into account on the basis of the numerical result obtained for muonic hydrogen in Refs. [64,65] is included in Table I. The error of the measurement of the cross section $\sigma(e^+e^-\to\pi^+\pi^-)$ was decreased to a few percentage points. Therefore, we estimate a corresponding theoretical error for the HVP correction of 5% (±0.0006 meV).

VI. FINE STRUCTURE OF THE 2P STATE

The leading-order $(Z\alpha)^4$ contribution to the fine structure is determined by the operator $\Delta V^{\rm fs}$ as follows:

$$\Delta V^{\rm fs}(r)=\frac{Z\alpha}{4m_1^2r^3}\left[1+\frac{2m_1}{m_2}+2a_\mu\left(1+\frac{m_1}{m_2}\right)\right](\mathbf{L}\boldsymbol{\sigma}_1), \quad (81)$$

where $\Delta V^{\rm fs}$ includes the recoil correction and muon anomalous magnetic moment a_μ correction. The fine-structure interval ($2P_{3/2}$-$2P_{1/2}$) for muonic deuterium can be written in the following form [66–68]:

$$\begin{aligned}\Delta E^{\rm fs}&=E(2P_{3/2})-E(2P_{1/2})\\&=\frac{\mu^3(Z\alpha)^4}{32m_1^2}\left[1+\frac{2m_1}{m_2}+2a_\mu\left(1+\frac{m_1}{m_2}\right)\right]+\frac{5m_1(Z\alpha)^6}{256}-\frac{m_1^2(Z\alpha)^6}{64m_2}+\frac{\alpha(Z\alpha)^6\mu^3}{32\pi m_1^2}\left[\ln\frac{\mu(Z\alpha)^2}{m_1}+\frac{1}{5}\right]\\&\quad+\alpha(Z\alpha)^4A_{\rm VP}+\alpha^2(Z\alpha)^4B_{\rm VP}+A_{\rm str}(Z\alpha)^6\mu^2r_d^2.\end{aligned} \quad (82)$$

This expression includes the relativistic correction of order $(Z\alpha)^6$, which can be calculated on the basis of the Dirac equation, relativistic recoil effects of order $m_1(Z\alpha)^6/m_2$, a correction of order $\alpha(Z\alpha)^6$ enhanced by the factor $\ln(Z\alpha)$ [6], and a number of terms of fifth and sixth order in α which are determined by effects of the vacuum polarization and nuclear structure. The recoil correction $(-m_1^3(Z\alpha)^4/32m_2^2)$ (the Barker-Glover correction [69]) is also taken into account in Eq. (82). This is evident from the expansion of first term in Eq. (82) over the mass ratio m_1/m_2 up to second-order terms: $m_1(Z\alpha)^4(1-m_1/m_2)/32$. The contributions to the coefficients $A_{\rm VP}$ and $B_{\rm VP}$ arise in the first and second orders of perturbation theory. Numerical values of terms in Eq. (82), which are presented in analytical form, are quoted in Table II for an accuracy of 0.00001 meV. The fine structure interval (82) in the energy spectrum of electronic hydrogen has been considered for a long time a basic test of quantum electrodynamics.

The fine-structure potential with the leading-order vacuum polarization and its contribution to the coefficient $A_{\rm VP}$ are given by [4]:

$$\Delta V^{\rm fs}_{\rm VP}(r)=\frac{\alpha(Z\alpha)}{12\pi m_1^2r^3}\int_1^\infty\rho(s)ds\left[1+\frac{2m_1}{m_2}+2a_\mu\left(1+\frac{m_1}{m_2}\right)\right]e^{-2m_esr}(1+2m_esr)(\mathbf{L}\boldsymbol{\sigma}_1), \quad (83)$$

$$\Delta E^{\rm fs}_1=\frac{\mu^3\alpha(Z\alpha)^4}{96\pi m_1^2}\left[1+\frac{2m_1}{m_2}+2a_\mu\left(1+\frac{m_1}{m_2}\right)\right]\int_1^\infty\rho(\xi)d\xi\frac{1+6\frac{m_e}{W}\xi}{\left(1+2\frac{m_e}{W}\xi\right)^3}=0.00346\ \text{meV}. \quad (84)$$

Higher-order corrections $\alpha^2(Z\alpha)^4$ contributing to a_μ, as well as recoil effects, are taken into account in this expression. The same order $O(\alpha(Z\alpha)^4)$ contribution can be obtained in second-order perturbation theory in the following form:

$$\begin{aligned}\Delta E^{\rm fs}_{\rm VP,SOPT}&=\frac{\alpha(Z\alpha)^4\mu^3}{1728\pi m_1^2}\left[1+2a_\mu+(1+a_\mu)\frac{2m_1}{m_2}\right]\int_1^\infty\frac{\rho(\xi)d\xi}{\left(1+2\frac{m_e}{W}\xi\right)^5}\\&\quad\times\left[18\frac{2m_e\xi}{W}\left(\frac{8m_e\xi}{W}+11\right)+4\left(1+\frac{2m_e\xi}{W}\right)\ln\left(1+\frac{2m_e\xi}{W}\right)+3\right]=0.00229\ \text{meV}.\end{aligned} \quad (85)$$

Let us consider two-loop vacuum polarization contributions in the one-photon interaction shown in Fig. 1. They give corrections to fine-structure splitting of the P-wave levels of order $\alpha^2(Z\alpha)^4$. In the coordinate representation, the interaction operator has the form [28,35]:

$$\begin{aligned}\Delta V^{\rm fs}_{\rm VP-VP}(r)&=\frac{Z\alpha}{r^3}\left[\frac{1+2a_\mu}{4m_1^2}+\frac{1+a_\mu}{2m_1m_2}\right](\mathbf{L}\boldsymbol{\sigma}_1)\left(\frac{\alpha}{3\pi}\right)^2\int_1^\infty\rho(\xi)d\xi\int_1^\infty\frac{\rho(\eta)d\eta}{(\xi^2-\eta^2)}\\&\quad\times[\xi^2(1+2m_e\xi r)e^{-2m_e\xi r}-\eta^2(1+2m_e\eta r)e^{-2m_e\eta r}].\end{aligned} \quad (86)$$

Averaging Eq. (86) over the wave functions in Eq. (2), we obtain the following correction to the interval (82):

$$\Delta E^{\rm fs}_{\rm VP-VP}=\frac{\mu^3\alpha^2(Z\alpha)^4}{288\pi^2m_1^2}\left[1+2a_\mu+\frac{2m_1}{m_2}(1+a_\mu)\right]\int_1^\infty\rho(\xi)d\xi \times\int_1^\infty\rho(\eta)d\eta\frac{1}{(\xi^2-\eta^2)}\left[\xi^2\frac{6\frac{m_e\xi}{W}+1}{\left(\frac{2m_e\xi}{W}+1\right)^3}-\eta^2\frac{6\frac{m_e\eta}{W}+1}{\left(\frac{2m_e\eta}{W}+1\right)^3}\right]=0.000003\ \text{meV}. \quad (87)$$

The two-loop vacuum polarization potential and the correction to the fine structure $(2P_{3/2}-2P_{1/2})$ are given by

$$\Delta V^{\rm fs}_{\rm two-loopVP}(r)=\frac{2Z\alpha^3}{3\pi^2r^3}\left[\frac{1+2a_\mu}{4m_1^2}+\frac{1+a_\mu}{2m_1m_2}\right]\int_0^1\frac{f(v)dv}{1-v^2}e^{-\frac{2m_er}{\sqrt{1-v^2}}}\left(1+\frac{2m_er}{\sqrt{1-v^2}}\right)(\mathbf{L}\boldsymbol{\sigma}_1), \quad (88)$$

$$\Delta E^{\rm fs}_{\rm two-loopVP}=\frac{\mu^3\alpha^2(Z\alpha)^4}{48\pi^2m_1^2}\left[1+2a_\mu+\frac{2m_1}{m_2}(1+a_\mu)\right]\int_0^1\frac{f(v)dv}{1-v^2}\frac{\left(6\frac{m_e}{W\sqrt{1-v^2}}+1\right)}{\left(1+\frac{2m_e}{W\sqrt{1-v^2}}\right)^3}=0.00002\ \text{meV}. \quad (89)$$

The two-loop vacuum polarization contributions in second-order perturbation theory, shown in Figs. 4(a) and 4(d)–4(f) $(\Delta V^B\to\Delta V^{\rm fs})$, have the same order $\alpha^2(Z\alpha)^4$. For their calculation it is necessary to employ the Coulomb potential modified by two-loop vacuum polarization effects [27,28]. The amplitude in Figs. 4(e) and 4(f) gives the following correction of order $\alpha^2(Z\alpha)^4$ to fine-structure splitting:

$$\Delta E^{\rm fs}_{\rm two-loopVP,SOPT}=\frac{\mu^3\alpha^2(Z\alpha)^4}{3\pi^2m_1m_2}\left[1+a_\mu+\frac{m_2}{2m_1}(1+2a_\mu)\right]\int_0^1\frac{f(v)dv}{1-v^2}\frac{1}{\left(1+\frac{2m_e}{W\sqrt{1-v^2}}\right)^6} \times\left[5\frac{2m_e}{W\sqrt{1-v^2}}+4\left(1+\frac{2m_e}{W\sqrt{1-v^2}}\right)\ln\left(1+\frac{2m_e}{W\sqrt{1-v^2}}\right)\right]=0.000026\ \text{meV}. \quad (90)$$

Two other contributions from amplitudes in Figs. 4(a) and 4(d) have the similar integral structure. Their numerical values are included in Table II.

There is also the correction to fine-structure splitting due to nuclear structure. In the 1γ interaction it is related to the charge form factor of the deuteron. The fine-structure potential (81) is obtained for the point deuteron. In the case of a deuteron of a finite size we can express the contribution of the nuclear-structure to fine-structure splitting in terms of the charge radius [68] as follows:

$$\Delta E^{\rm fs}_{\rm str}=-\frac{\mu^5(Z\alpha)^6}{64m_1^2}r_d^2\left[1+\frac{2m_1}{m_2}+2a_\mu\left(1+\frac{m_1}{m_2}\right)\right] =-0.00028\ \text{meV}. \quad (91)$$

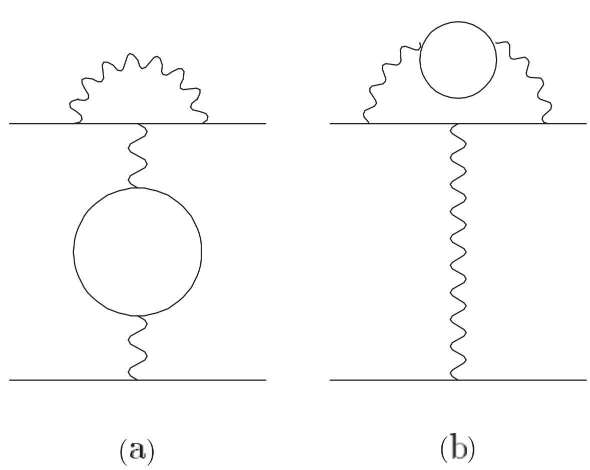


FIG. 11. Radiative corrections with the vacuum polarization effects.

The calculation of the nuclear-structure corrections to the energies of the P levels of order $(Z\alpha)^6$ was performed in Ref. [22]. Our numerical result (91) for the fine-structure splitting agrees with the calculation in Ref. [22].

VII. SUMMARY AND CONCLUSION

In this work, various corrections of orders α^3, α^4, α^5, and α^6 are calculated for the Lamb shift $(2P_{1/2}$-$2S_{1/2})$ and fine-structure splitting $(2P_{3/2}$-$2P_{1/2})$ in the muonic deuterium atom. Contrary to earlier investigations of the energy spectra of light muonic atoms in Refs. [1,2,18], we have used the three-dimensional quasipotential approach for the description of the two-particle bound state. Our analysis of the different contributions to the Lamb shift accounts for the terms of two groups. The first group contains the specific corrections for muonic deuterium, connected with the electron vacuum polarization effects, nuclear-structure and recoil effects in first- and second-order perturbation theory. As a rule, the contributions of this group are obtained in integral form over auxiliary parameters and calculated numerically. The necessary order corrections of the second group include analytical results known from the corresponding calculation in the electronic hydrogen Lamb shift. Recent advances in the physics of the energy spectra of simple atoms are presented in Refs. [6,55,56] which we have used in this study. Numerical values of all corrections are written in Tables I and II, which also contain basic references on the earlier performed investigations (other references can be found in Refs. [1,2,6]). We compare our intermediate results for the different corrections with calculations given in Ref. [2]. Most of the results, including the Uehling, Källen-Sabry, and Wichmann-Kroll corrections, the muon Lamb shift contribution, the nuclear size and VP corrections, and the recoil terms, agree well. Our results for the relativistic contributions to the vacuum polarization are in agreement with those obtained in Ref. [5]. The second-order VP correction [Eqs. (39) and (40)] agrees with the result of Ref. [31] just as the three-loop VP contribution which is determined in Table I by the three lines corresponding to the one-photon interaction (0.0060 meV), the second-order PT (0.0025 meV), and the third-order PT (0.0001 meV) agree. The total numerical value, 202.4139 meV, of the Lamb shift ($2P$-$2S$) in the muonic deuterium atom from Table I is in good agreement with the theoretical result, 202.263 meV, reported in Ref. [2]. Our result differs from that of Ref. [2] with regard to the calculation of new contributions of higher order in α and m_1/m_2, the proton structure and polarizability correction [51], and the slightly different numerical value of the charge radius of the deuteron r_d used in this work. The two-loop vacuum polarization contribution, 0.1720 meV, of order $\alpha^2(Z\alpha)^2$ in the second-order PT is absent in Ref. [2]. In Ref. [2], the value of the charge radius used is $r_d = 2.139(3)$ fm. The fine-structure splitting ($2P_{3/2}$-$2P_{1/2}$) in Table II (8.86386 meV) agrees also with the result (8.864 meV) from Ref. [2]. An improved, recent analysis of the different corrections to the Lamb shift in μD is performed in Ref. [3]. The total value of the Lamb shift ($2P_{1/2}$-$2S_{1/2}$) for $r_d = 2.130$ fm, according to Table 4 from Ref. [3], is 202.9440 meV. This value exceeds our result (202.7375 meV) by 0.2065 meV. In our opinion, the only two essential differences between our Table I and Ref. [3] are related by the Zemach correction (0.4329 meV) and the polarizability correction (1.5 meV) [3]. It was shown in Ref. [51] that the Zemach correction is canceled by the deuteron excited states contribution. As a result, the nuclear-structure and polarizability contribution is equal to 1.680 meV [51], which we use in our work.

As mentioned above, the numerical values of the corrections are obtained with an accuracy of 0.0001 meV because certain contributions to the Lamb shift ($2P$-$2S$) of order α^6 attain the value of 0.1 μeV. The theoretical error caused by uncertainties in the fundamental parameters (e.g., the fine-structure constant and particle masses) entering the leading-order contributions is around 10^{-5} meV. The other part of the theoretical error is related to the QED corrections of higher order. This part can be estimated from the leading contribution of higher order in α: $m_1\alpha(Z\alpha)^6 \ln(Z\alpha)/\pi n^3 \approx 0.0001$ meV. The theoretical uncertainty connected with the nuclear-structure and polarizability contributions is equal to 0.0160 meV [51]. We have also a small theoretical uncertainty determined from the HVP contribution which we estimate to be 5% (± 0.0006 meV). This estimation is based on the experimental uncertainty in the cross section of e^+e^- annihilation into hadrons. The rounding errors can amount to $0.0001 \div 0.0002$ meV. Finally, the biggest theoretical error, ± 0.0550 meV [for $r_d = 0.1424(21)$ fm], is related to the uncertainty of the deuteron charge radius. Thereby, the total theoretical error of the calculation is equal to ± 0.0573 meV. To obtain this estimate, we add the above-mentioned uncertainties in quadrature.

Let us summarize the basic particularities of the Lamb shift calculation performed above.

(1) The numerical value of the specific parameter $m_e/\mu Z\alpha = 0.7$ in the muonic deuterium atom is sufficiently large, so the electron vacuum polarization effects play an essential role in the interaction of the bound particles. We have considered the one-loop, two-loop, and three-loop VP contributions to the Lamb shift ($2P_{1/2}$-$2S_{1/2}$). A number of important vacuum polarization contributions from the 1γ interaction agree with the results obtained in Refs. [2,29–31].

(2) Nuclear-structure effects are expressed in the Lamb shift of the muonic deuterium atom in terms of the deuteron charge radius r_d. We analyze complex effects due to nuclear structure and vacuum polarization in the first and second orders of perturbation theory. The elastic nuclear structure contribution from two-photon exchange amplitudes is canceled by part of the deuteron polarizability correction [51].

(3) Nuclear-structure and polarizability effects give the largest theoretical uncertainty in the total value of the Lamb shift ($2P$-$2S$). It is useful to express the final theoretical value of the ($2P$-$2S$) Lamb shift in the form $\Delta E^{\mathrm{Ls}}(2P\text{-}2S) = (230.4511 - 6.108485 r_d^2)$ meV with the value of the deuteron charge radius defined in fm. Then, comparing this expression with the experimental value of the Lamb shift measured to 0.01 meV (50 ppm), we can obtain a more accurate value of r_d with an accuracy of 0.0005 fm.

ACKNOWLEDGMENTS

The authors are grateful to A. Antognini, R. N. Faustov, F. Kottmann, T. Nebel, and R. Pohl for information regarding experimental results with muonic hydrogen, the new muonic helium project of the CREMA collaboration, and useful discussions. We thank U. D. Jentschura for reading our manuscript, valuable remarks about the recoil correction (64), and comments on our calculations. We express our gratitude to the referee of Physical Review A for constructive criticism and numerous useful remarks which helped us to improve the presentation and discussion of the obtained results. This work is supported by the Russian Foundation for Basic Research (Grant No. 11-02-00019) and the Federal Program "Scientific and pedagogical personnel of innovative Russia" (Grant No. NK-20P/1).

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