Bound-state properties of four-body muonic quasiatoms

Alexei M. Frolov[*] and David M. Wardlaw[†]

Department of Chemistry, University of Western Ontario, London, Ontario, Canada N6H 5B7

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Total energies and various bound-state properties are determined for the ground states in all six four-body muonic $a^+b^+\mu^-e^-$ quasiatoms. These quasiatoms contain two nuclei of the hydrogen isotopes p^+,d^+,t^+, one negatively charged muon μ^-, and one electron e^-. In general, each of the four-body muonic $a^+b^+\mu^-e^-$ quasiatoms, where $(a,b) = (p,d,t)$, can be considered as the regular one-electron (hydrogen) atom with the complex nucleus $a^+b^+\mu^-$ that has a finite number of bound states. Furthermore, all properties of such quasinuclei $a^+b^+\mu^-$ are determined from highly accurate computations performed for the three-body muonic ions $a^+b^+\mu^-$ with the use of pure Coulomb interaction potentials between particles. It is shown that the bound-state spectra of such quasiatoms are similar to the spectrum of the regular hydrogen atom, but there are a few important differences. Such differences can be used in future experiments to improve the overall accuracy of current evaluations of various properties of hydrogenlike systems, including the lowest-order relativistic and quantum electrodynamics corrections to the total energies.

I. INTRODUCTION

Recently, by performing variational calculations of the bound states in three-body muonic molecular ions [1], we have rediscovered a class of four-body systems that are of interest in some applications. The existence of these neutral four-body atomic systems follows from the analysis of actual experimental conditions that can be found in liquid hydrogen or deuterium. Indeed, in actual experiments with muons it is hard to imagine a positively charged muonic molecular ion, e.g., the $(pd\mu)^+$ ion, which moves as a separate and stable system in liquid hydrogen or deuterium with density $\rho \approx 0.213$ g cm^{-3}. It is clear that such a system will take an additional electron e^- from surrounding atoms or molecules and form the neutral muonic quasiatom $pd\mu e$. Later such a four-body quasiatom $pd\mu e$ will react with a protium or deuterium molecule and form a stable six-body quasimolecule $(pd\mu)pe_2$ or $(pd\mu)de_2$. All transformations of muonic four-body quasiatoms and six-body quasimolecules occur on a time scale of $\tau \approx 1 \times 10^{-10}$ s, which is significantly shorter than the muon lifetime $\tau_\mu \approx 2 \times 10^{-6}$ s and/or the reaction time for the nuclear (p,d) reaction. This means that the muonic quasiatoms $pd\mu e$, $pd\mu e$, $dt\mu e$, etc., and muonic quasimolecules $(pd\mu)pe_2$ and $(pd\mu)de_2$ can be observed in actual experiments, and it is therefore interesting to investigate the properties of such systems. In this study, we determine the bound-state properties of the four-body $ab\mu e$ quasiatoms, and we discuss some interesting experiments for these systems.

For conciseness of presentation, we confine our attention to the $pd\mu e$ quasiatom; extension to the other five hydrogen isotopes is straightforward. In atomic units $\hbar = 1, m_e = 1, e = 1$, the Hamiltonian of the four-body $pd\mu e$ quasiatom is

$$H = -\frac{1}{2m_p}\Delta_1 - \frac{1}{2m_d}\Delta_2 - \frac{1}{2m_\mu}\Delta_3 - \frac{1}{2}\Delta_4$$
$$+ \frac{1}{r_{12}} - \frac{1}{r_{13}} - \frac{1}{r_{14}} - \frac{1}{r_{23}} - \frac{1}{r_{24}} + \frac{1}{r_{34}}, \quad (1)$$

[*]afrolov@uwo.ca
[†]dwardlaw@uwo.ca

where the notation 1 designates the protium nucleus (p or ^{1}H), the notation 2 (or $+$) means the deuterium (d) nucleus, while 3 and 4 stand for the negatively charged muon μ^- and electron e^-, respectively. This system of notation will be used everywhere below in this study. Note that the three heavy particles p^+,d^+, and μ^- in this system have very large masses in comparison with the electron mass m_e (in atomic units $m_e = 1$) and opposite electric charges. Therefore, we can expect that inside of the $pd\mu e$ quasiatom, these three heavy particles (p^+,d^+, and μ^-) will form a separate three-body cluster $pd\mu$ that is spatially compact and has a positive electric charge $+1$. The fourth particle (electron e^-) moves at very large distances from this central, heavy cluster. If a_0 is the Bohr radius, then the radius of the electron orbit in the $pd\mu e$ quasiatom is $R_e \approx a_0$ while the spatial radius of the central heavy cluster is $R_\mu \approx (\frac{m_e}{m_\mu})a_0 \approx (\frac{1}{206.768262})a_0 \ll a_0$. Analogous relations between radii of the electron and muonic orbits can be found in other four-body quasiatoms $ab\mu e$. In fact, all bound-state properties in this family of quasiatoms $ab\mu e$ can be separated into two different groups: electronic properties and muonic properties. Sometimes it is convenient to split the muonic properties into two additional subgroups: muonic and nuclear properties. The reason for such a separation is obvious, since each of the nuclear masses M_p, M_d, M_t is substantially larger than the muon mass $m_\mu = 206.768262m_e$.

II. TOTAL ENERGIES

Based on the cluster structure of the four-body quasiatom $pd\mu e$, we can predict that this quasiatom is a bound four-body system that is very similar to the regular hydrogen atom. Moreover, its binding energy must be very close to the total energy of the one-electron hydrogen atom. For instance, the binding energy of the ground $1S(L = 0)$ state in the $pd\mu e$ atom must be close to -0.5 a.u., while the analogous energy of the $2S(L = 0)$ state (electron state) in the $pd\mu e$ quasiatom must be close to -0.125 a.u., etc. In general, the total energy of the bound state with the principal quantum number n in the

$pd\mu e$ quasiatom is approximately equal to

$$E \approx E(pd\mu) + E(\text{H}) \approx -106.012\,527\,069\,515\,844$$
$$- \frac{1}{2n^2} \text{ a.u.,} \qquad (2)$$

where the value $-106.012\,527\,069\,515\,844$ a.u. is the best-to-date energy of the ground state in the $pd\mu$ ion [1]. Thus, the ground state ($n = 1$) in this four-body quasiatom $pd\mu e$ has the total energy $E \approx -106.512\,527\,069\,515\,844(2)$ a.u. [1]. Analogous relations can be found for other $ab\mu e$ quasiatoms. The formula, Eq. (2), is valid in those cases in which we can neglect all electron-muon, electron-deuteron and electron-proton correlations. In reality, such correlations contribute to the total energy, and the value of E given by Eq. (2) must be corrected.

It is clear that the total energy E depends on particle masses. In this study, we shall assume that $m_p = 1836.152\,701 m_e$, $m_d = 3670.483\,014 m_e$, $m_t = 5496.921\,58 m_e$, and $m_\mu = 206.768\,262 m_e$. These masses are often used in accurate computations of the muonic molecular ions. With these masses, the total energy of the four-body quasiatom $pd\mu e$ in its ground state is $E \approx -106.512\,439\,401$ a.u., i.e., it is slightly above the value $-106.512\,527\,069\,52$ a.u. mentioned above. It is clear that the nonrelativistic energy E from Eq. (2) is only an approximation to the exact total energy of the $pd\mu e$ quasiatom, since this value of E does not include all contributions from the lowest-order relativistic and quantum electrodynamics (QED) corrections. A large number of other corrections, e.g., the finite-size corrections, corrections on nuclear interactions, etc., to the total energy have been ignored as well.

The total energy E given by Eq. (2) corresponds to the case in which the central quasinucleus (or heavy three-body cluster) $pd\mu$ is in its ground $S(L = 0)$ state. However, this three-body quasinucleus $pd\mu$ can also be detected in its bound $P(L = 1)$ state with the total energy $-101.453\,777\,548\,914$ a.u. [1]. In this case, the total energy of the $pd\mu e$ quasiatom with the central nucleus in its bound $P(L = 1)$ state is $\approx -101.953\,777\,548\,914$ a.u., i.e., it is substantially different from the value $E \approx -106.512\,439\,401$ a.u. mentioned above for the ground state, and the main difference arises from the energy of the central quasinucleus $pd\mu$.

As is well known, the six muonic molecular ions $pp\mu, pd\mu, pt\mu, dd\mu, dt\mu$, and $tt\mu$ have 22 bound states (see, e.g., [1] and references therein). These are the $S(L = 0)$, $P(L = 1)$, $D(L = 2)$, and $F(L = 3)$ states, where the notation L designates the total angular momentum of this three-body ion $ab\mu$, where $(a,b) = (p,d,t)$. There are nine (bound) $S(L = 0)$ states, nine $P(L = 1)$ states, three $D(L = 2)$ states, and one $F(L = 3)$ state. The four-body muonic quasiatoms, which contain two nuclei of hydrogen isotopes, can be considered as a family of similar one-electron (or hydrogenlike) atoms. One of the six muonic molecular ions $pp\mu, pd\mu, pt\mu, dd\mu, dt\mu$, and $tt\mu$ plays the role of the nucleus in each of these atoms. In general, such a nucleus can be either in the ground $S(L = 0)$ state or in one of its "rotationally" and/or "vibrationally" excited states. Here, to designate the excited states in the $ab\mu$ ion, we use the (L, ν) notation, where L denotes the rotational state while ν stands for the vibrational state.

All bound-states properties, including the lowest-order relativistic and QED corrections, determined for such hydrogenlike quasiatoms depend substantially on the hydrogen-isotope composition and excitation of the central three-body "nucleus" $ab\mu$. The spectrum of the excited states is very well known for each of the six muonic molecular ions $ab\mu$. Therefore, the $ab\mu e$ quasiatoms with different hydrogen isotopes $a = (p,d,t)$ and $b = (p,d,t)$ can be considered as model hydrogenlike atoms. The "nuclear" spectra in such atoms are known to very high accuracy. This allows one to consider possible interactions between "atomic" and "nuclear" bound states in the $ab\mu e$ quasiatoms. Note again that all "nuclear" properties of the $ab\mu$ quasinucleus can be obtained from highly accurate Coulomb calculations performed for three-body systems.

In general, the electron motion in the $ab\mu e$ quasiatom is well separated from the motion of the three heavy particles $a^+ b^+ \mu^-$. For instance, the expectation values of the kinetic energies of the electron and muon in the $ab\mu e$ quasiatom differ by a factor of 50–100. However, as is well known (see, e.g., [2,3] and references therein), the "vibrationally" excited $P(L = 1)$ states [or (1,1) states] in the $dd\mu$ and $dt\mu$ three-body ions are very weakly bound. The binding energies of the three-body ions $dd\mu$ and $dt\mu$ in their excited (1,1) states are $\approx -1.974\,988\,1$ and $\approx -0.660\,338\,7$ eV, respectively [1]. These "nuclear" binding energies are comparable with the corresponding atomic energies. In such cases, one can certainly observe a strong interference between the "nuclear" and "atomic" bound states. In reality, for bound states with very weakly bound nuclei (or quasinuclei), we cannot discuss their nuclear and electron spectra separately. The analysis of atomic systems with weakly bound nuclei is very complex, but its results are of great interest in a number of applications.

III. VARIATIONAL WAVE FUNCTIONS

To determine the total energies and bound-state properties, one needs to construct approximate wave functions for the four-body quasiatoms. In this study, these wave functions are approximated by the variational expansion written in the basis of the six-dimensional (or four-body) gaussoids of relative (or interparticle) scalar coordinates r_{ij}. This variational expansion was originally proposed 30 years ago in [4] for accurate variational calculations of few-nucleon nuclei and Λ nuclei. For the bound $S(L = 0)$ states, the variational ansatz of fully correlated six-dimensional (or four-body) gaussoids is written in the form [4]

$$\Psi_{L=0} = (1 + \delta_{ab}\epsilon_{ab}\mathcal{P}_{ab}) \sum_{k=1}^{N} C_k \exp\big(-\alpha_{12}^{(k)} r_{12}^2 - \alpha_{13}^{(k)} r_{13}^2$$
$$- \alpha_{23}^{(k)} r_{23}^2 - \alpha_{14}^{(k)} r_{14}^2 - \alpha_{24}^{(k)} r_{24}^2 - \alpha_{34}^{(k)} r_{34}^2\big), \qquad (3)$$

where C_k are the linear coefficients (or linear variational parameters), while $\alpha_{ij}^{(k)}$ are the optimized nonlinear parameters. The notation $\epsilon_{ab}\mathcal{P}_{ab}$ means the appropriate symmetrizer (or antisymmetrizer), i.e., a projection operator that produces the wave function with the correct permutation symmetry in those cases in which $a = b$. This case is designated in Eq. (3) with the use of the δ function. The operator $\mathcal{P}_{ab}$ is the pair-permutation

operator for all coordinates, i.e., for the spatial, spin, isospin, etc., coordinates.

By using some effective strategies for optimization of the nonlinear parameters $\alpha_{ij}^{(k)}$ in Eq. (3), one can obtain very accurate variational wave functions with a relatively small number of terms $N \approx 400$–600 in Eq. (3). The generalization of the variational expansion, Eq. (3), to the bound states $L \geqslant 1$ is straightforward. It is clear that for the quasiatoms $ab\mu e$ with the "rotationally" excited "nuclei" $ab\mu$, the minimal number of terms in Eq. (3) must be larger to achieve better overall accuracy. However, in this study we shall not discuss numerical computations of bound states with $L \geqslant 1$, and we restrict ourselves to the analysis of the bound $S(L = 0)$ states.

IV. BINDING ENERGIES AND OTHER BOUND-STATE PROPERTIES

As we have shown above, the total energy of the $pd\mu e$ atom is approximately equal to the sum of the total energies of the hydrogen atom and the $pd\mu$ ion, i.e., $E(pd\mu e) \approx E(pd\mu) + E(\text{H}; n\ell)$, where n and ℓ are the principal and angular quantum numbers of the hydrogen atom. The corresponding binding energy is the difference between this value and the total energy of the three-body $pd\mu$ ion. This means that the binding energy of the ground state of the four-body atom $pd\mu e$ is approximately equal to the total energy of the ground state of the hydrogen atom, which equals -0.5 a.u. In other words, the total energy of the ground state in the four-body $pd\mu e$ system is approximately equal to the sum of the total energies of the three-body muonic molecular ion $pd\mu$ and the ground-state energy of the hydrogen atom with the infinitely heavy nucleus, i.e., -0.5 a.u. The use of the finite masses for different hydrogen isotopes p,d,t and muon μ slightly decreases the absolute value of the total energy of the $pd\mu e$ system. The corresponding binding energies of all six $ab\mu e$ quasiatoms in their ground states ε_H can be found in Table I. These values have been determined with the use of our data from Table I [1] and the following formula:

$$\varepsilon_H = E(ab\mu e) - E(ab\mu). \qquad (4)$$

As follows from Table I, these (electron) binding energies are very close to the expected value -0.5 a.u. A very accurate evaluation of ε_H is given by the formula

$$\varepsilon_H \approx \varepsilon_A = -\frac{0.5}{1 + \frac{1}{m_a + m_b + m_\mu}}. \qquad (5)$$

The values of ε_H and ε_A can also be found in Table I. They correspond to the total energies obtained with the use of $N = 600$ basis wave functions in Eq. (3).

By using our total energies computed with the use of $N = 400$ and 600 basis functions, we can extrapolate our results to the infinite number of basis functions. The corresponding energy $E(N = \infty)$ must be closer to the actual total energy than each of the $E(N = 400)$ and $E(N = 600)$ energies. In general, the following extrapolation formula is often used for this purpose:

$$E(N_i) = E(N = \infty) + \frac{A}{N_i^\gamma}. \qquad (6)$$

From a series of calculations performed for four-body muonic quasiatoms, we have found that for such systems, the parameter γ in Eq. (6) varies between 3.5 and 4. Below, we shall assume that $\gamma = 3.5$ in Eq. (6). Now the formula, Eq. (6), can be applied to all four-body muonic quasiatoms mentioned in Table I. It should be mentioned, however, that this extrapolation formula can be applied only in those cases in which the internal (or nonlinear) parameters of this method are not changed (or not varied). In variational expansion, Eq. (3), the nonlinear parameters are always varied to produce the results of good numerical accuracy. Therefore, the parameters $E(N = \infty)$ and A become N-dependent. This means that by using different values of $E(N = 400)$ and/or $E(N = 600)$, one finds a number of different $E(N = \infty)$ values. Formally, we have a distribution of $E(N = \infty)$ values that can be written in the form $E(N = \infty) = \tilde{E}(N = \infty) \pm \Delta$, where Δ is the corresponding uncertainty. The values of $\tilde{E}(N = \infty)$ and Δ can be found in Table I for each muonic quasiatom.

Other bound-state properties of the $ab\mu e$ quasiatoms computed with our approximate wave functions (see Table II), e.g., the $\langle r_{ij} \rangle, \langle r_{ij}^2 \rangle, \langle \delta(\mathbf{r}_{ij}) \rangle$ expectation values, coincide well either with the known properties of the hydrogen atom (in those cases in which one of the indexes i or j designates the electron), or with the bound-state properties known for the three-body muonic molecular ion $ab\mu$ (see, e.g., [1] and references therein). This uniformly indicates that each of the $ab\mu e$ quasiatoms has the two-shell cluster structure. The electron moves at a substantial (atomic) distance from the compact central cluster $ab\mu$. This central cluster $ab\mu$ has a structure that is similar to a molecular (or two-center) structure. It is different from the expected "pure nuclear" (or one-center) structure. The effective spatial radius of the central cluster is $m_\mu \approx 206.768$ times smaller than the radius of the electron orbit. All these conclusions directly follow from Table II.

Note that the set of operators included in Table II is not exhaustive and was selected for illustrative purposes. Most of the bound-state properties from Table II have been determined to relatively high accuracy, which can be be estimated by comparing the corresponding expectation values computed with the use of 400 and 600 basis functions. The overall accuracy of the $\langle r_{ij}^{-1} \rangle$ expectation values (seven to eight stable decimal digits) can be estimated by using the virial theorem. The expectation values $\langle r_{ij}^n \rangle$, where $n = -2,1,2$, have close overall accuracy. Analogously, the number of correct decimal digits in the expectation values of the partial kinetic energies (or single-particle kinetic energies) $\langle -\frac{1}{2}\nabla_i^2 \rangle$ can also be evaluated with the use of the virial theorem. The expectation values of the electron-nuclear δ functions contain approximately six stable decimal digits, while the muon-nuclear and electron-muon δ functions are slightly less accurate (approximately four to five stable decimal digits). The expectation values of the δ functions between two nuclei of hydrogen isotopes include only one to two accurate decimal digits. These expectation values must be improved in future calculations. The computed expectation values can be compared directly with the expectation values obtained earlier for the three-particle $pd\mu, pt\mu$, and $dt\mu$ ions and for the one-electron hydrogen atom. In general, such a comparison is of great interest, since it allows us to compare directly the overall qualities of the different variational wave

TABLE I. The total nonrelativistic energies E of the ground states of the six four-body muonic quasiatoms $ab\mu e$ (in atomic units). N designates the total number of basis functions used in Eq. (3).

$E(N)$	$pd\mu e$	$pt\mu e$	$dt\mu e$
$E(N = 400)$	-106.51224604	-107.99443095	-111.86418078
$E(N = 600)$	-106.51229235	-107.99446242	-111.86418855
$\tilde{E}(N = \infty)(\Delta)$	$-106.5125(2)$	$-107.9947(2)$	$-111.8645(2)$
ε_H	-0.49976528	-0.49975981	-0.49984161
ε_A	-0.49991250	-0.49993369	-0.49994667
$E(N)$	$pp\mu e$	$dd\mu e$	$tt\mu e$
$E(N = 400)$	-102.72328481	-110.31675855	-113.47258632
$E(N = 600)$	-102.72329280	-110.31677881	-113.47259711
$\tilde{E}(N = \infty)(\Delta)$	$-102.72335(3)$	$-110.31685(3)$	$-113.47265(3)$
ε_H	-0.49978922	-0.49985243	-0.49974808
ε_A	-0.49987114	-0.49993376	-0.49995536

functions. On the other hand, it is very interesting to find electronic properties of the four-body $ab\mu e$ quasiatoms that are substantially different from the known properties of the hydrogen atom(s).

The expectation values of different bound-state properties determined to high numerical accuracy allow one to estimate a large number of fundamental atomic properties that can be directly measured in actual experiments. Here, by "fundamental property" we mean some combination of atomic expectation values that determines the results of direct experimental observations. In other words, any fundamental property leads to some experimental effects. Below, we discuss only the following two effects: (a) the evaluation of the field component of the total isotope shift for the $ab\mu e$ quasiatoms, and (b) the hyperfine structure splittings in these quasiatoms. As is well known (see, e.g., [5,6] and references therein), the field shift in atomic systems is related to the extended nuclear charge distribution, which produces the non-Coulomb field

at distances close to the nucleus, i.e., at $r \approx R_N \approx 10^{-13}$ cm ($=1$ fm), where R_N is the radius of the nucleus. It is clear that the largest deviations between the Coulomb and actual potentials can be found close to the atomic nucleus, i.e., for distances $r \approx R_N \approx r_e \ll \Lambda \ll a_0$, where $r_e = \alpha^2 a_0$ is the classical electron radius and $\Lambda = \alpha a_0$ is the Compton wavelength. Here and below, $\alpha = 7.297\,352\,568 \times 10^{-3}$ is the fine-structure constant and $a_0 \approx 5.291\,772\,49 \times 10^{-11}$ m is the Bohr radius. An approximate formula for the field shift E_M^{fs} in light atoms takes the form [7]

$$E_M^{\text{fs}} = \frac{8\pi}{5} Q\alpha^4 \xi \left(\frac{R_N}{r_e}\right)^2, \quad (7)$$

where Q is the nuclear charge and ξ is an additional factor that is uniformly related to the charge distribution in the nucleus [7]. An essentially equivalent formula was given long ago by Cooper and Henley [8]. Now note that all "nuclei" in $ab\mu e$

TABLE II. The expectation values $\langle X_{ij} \rangle$ in atomic units ($m_e = 1, \hbar = 1, e = 1$) of some properties for the ground states of the $pd\mu e, pd\mu e$, and $dt\mu e$ ions. Below, the notations 1 and 2 designate the two heavy hydrogen nuclei, while 3 stands for the negatively charged muon and 4 denotes the electron.

$\langle X_{ij} \rangle$	$pd\mu e$	$pt\mu e$	$dt\mu e$	$\langle X_{ij} \rangle$	$pd\mu e$	$pt\mu e$	$dt\mu e$
$\langle r_{12}^{-2} \rangle$	6844.24	7006.69	7860.90	$\langle r_{12}^{-1} \rangle$	76.31734	77.53007	83.49840
$\langle r_{13}^{-2} \rangle$	38979.3	37946.9	48009.2	$\langle r_{13}^{-1} \rangle$	132.5684	130.9652	149.4313
$\langle r_{23}^{-2} \rangle$	52479.2	56036.3	52600.6	$\langle r_{23}^{-1} \rangle$	155.7732	161.5537	156.7953
$\langle r_{14}^{-2} \rangle$	1.99803	1.99926	1.99821	$\langle r_{14}^{-1} \rangle$	0.999936	1.000105	0.999971
$\langle r_{24}^{-2} \rangle$	1.99773	1.99894	1.99805	$\langle r_{24}^{-1} \rangle$	0.999946	1.000120	0.999977
$\langle r_{34}^{-2} \rangle$	1.99637	1.99740	1.99714	$\langle r_{34}^{-1} \rangle$	0.999893	1.000065	0.999932
$\langle r_{12} \rangle$	0.0149961	0.0146856	0.0132898	$\langle r_{12}^2 \rangle$	0.253289×10^{-3}	0.242031×10^{-3}	0.193839×10^{-3}
$\langle r_{13} \rangle$	0.0118563	0.0119035	0.0102429	$\langle r_{13}^2 \rangle$	0.187901×10^{-3}	0.187857×10^{-3}	0.137575×10^{-3}
$\langle r_{23} \rangle$	0.0100972	0.0096824	0.0097874	$\langle r_{23}^2 \rangle$	0.137921×10^{-3}	0.126410×10^{-3}	0.126237×10^{-3}
$\langle r_{14} \rangle$	1.498820	1.498714	1.499141	$\langle r_{14}^2 \rangle$	2.990077	2.989908	2.992869
$\langle r_{24} \rangle$	1.498797	1.498686	1.499141	$\langle r_{24}^2 \rangle$	2.989995	2.989814	2.992831
$\langle r_{34} \rangle$	1.498817	1.498708	1.499158	$\langle r_{34}^2 \rangle$	2.990058	2.989881	2.992882
$\langle -\frac{1}{2}\nabla_1^2 \rangle$	11997.364	12048.584	16719.815	$\langle \delta_{12} \rangle$	168.0	87.2	9.71
$\langle -\frac{1}{2}\nabla_2^2 \rangle$	15710.188	17057.831	18036.979	$\langle \delta_{13} \rangle$	1.0352×10^6	9.9989×10^5	1.3589×10^6
$\langle -\frac{1}{2}\nabla_3^2 \rangle$	19683.906	20227.987	21406.215	$\langle \delta_{23} \rangle$	1.5244×10^6	1.6635×10^6	1.5331×10^6
$\langle -\frac{1}{2}\nabla_4^2 \rangle$	0.5000089	0.50017715	0.50002151	$\langle \delta_{14} \rangle$	0.30981	0.31007	0.31145
$\langle \delta_{24} \rangle$	0.30838	0.30849	0.31041	$\langle \delta_{34} \rangle$	0.30650	0.30729	0.30895

quasiatoms are three-body muonic molecular ions $ab\mu$ with the spatial radius $a_\mu \approx a_0/m_\mu \gg r_e = \alpha^2 a_0$. Therefore, we can predict that the field component of the total isotope shift in the $ab\mu e$ quasiatoms must be substantially larger (10 000 times larger) than for the usual light atoms. Furthermore, this component of the total isotope shift will be a- and b-dependent, since the spatial radius of the $ab\mu e$ quasiatom depends on the two nuclei of the hydrogen isotopes a and b.

Another interesting problem involves predicting the hyperfine structure splitting of the four-body $pd\mu e$ and other similar quasiatoms. Later the computed hyperfine structure splitting must be compared with the corresponding experimental results. For instance, consider the hyperfine structure splitting of the four-body $pd\mu e$ quasiatom. The central quasiparticle, i.e., the $pd\mu$ ion, has two bound states with different angular momenta. These states are also bound in the analogous $pt\mu$ and $pp\mu$ ions. In each of the $dd\mu$ and $dt\mu$ ions, one finds five bound states: two $S(L = 0)$ states, two $P(L = 1)$ states, and one $D(L = 2)$ state. The heaviest $tt\mu$ ion has six bound states: two $S(L = 0)$ states, two $P(L = 1)$ states, one $D(L = 2)$ state, and one $F(L = 3)$ state. We can define the angular moment L of the central "nucleus" $ab\mu$ and the magnetic moment $\mathbf{S}$ (or "spin" for short) associated with L. The "nuclear" spin $\mathbf{S}$ of the $ab\mu$ quasinucleus is combined with the electron spin $\mathbf{s}_e$, and this produces the hyperfine structure splitting of the four-body $ab\mu e$ quasiatom. By determining the expectation values of the corresponding δ functions and a few other properties, one can evaluate the hyperfine structure splitting in all $S(L = 0)$ states of the six four-body $ab\mu e$ systems, where $(a,b) = (p,d,t)$. Accurate numerical evaluation of the hyperfine splitting in the $ab\mu e$ systems will be our goal in an upcoming study. The computed values of the hyperfine structure splittings must be compared with the actual experimental values. Analysis of the hyperfine structure splitting for the rotationally excited states is more complicated.

In some sense, the four-body muonic quasiatoms are the two-shell atomic systems that are similar to the helium-muonic atoms discussed in [9–11] (see also references therein). In particular, these four-body muonic quasiatoms, e.g., $pd\mu e$, have a very compact central cluster $(pd\mu)$ and one electron moving in the electric field of this cluster. The fundamental difference between the $pd\mu e$ quasiatom and the $^4\mathrm{He}\mu e$ atom follows from the different nature of their central clusters. A very heavy nucleus $^4\mathrm{He}$ (or $^3\mathrm{He}$) is essentially the center of the $^4\mathrm{He}\mu e$ atom, while the two positively charged particles p and d in the $pd\mu$ quasinucleus are not bound to each other without the muon. Briefly, we can say that such a central cluster $pd\mu$ has a "molecular" structure.

V. CONCLUSIONS

We have discussed the bound-state spectra and properties of the muonic four-body quasiatoms $ab\mu e$, where $(a,b) = (p,d,t)$. The energy spectra of such four-body quasiatoms are of interest in some applications. Briefly, an arbitrary bound state in the four-body muonic quasiatom $ab\mu e$ is represented as motion of the negatively charged electron e^- in the field of a compact "central cluster" $ab\mu$, which has positive electric charge $+1$. The spatial radius of the central cluster in the $ab\mu e$ quasiatom is $\approx$206.77 times smaller than the actual radius of the electron orbit $r_e \approx a_0$, where a_0 is the Bohr (atomic) radius. In general, the study of bound-state spectra in four-body muonic quasiatoms $ab\mu e$ may lead to some important results and conclusions that can also be useful in applications to other atomic and quasiatomic systems. In reality, for each four-body muonic quasiatom $ab\mu e$, we know the bound-state spectra of its central quasinucleus $ab\mu$. Moreover, the probabilities of all possible electromagnetic transitions in this quasinucleus $(ab\mu)$ can be determined to very good accuracy. This simplifies the analysis of many related phenomena, including internal radiative transitions, in the four-body quasiatom $ab\mu e$.

There are a number of related problems that are of special interest in various applications, such as an accurate computation of some bound-state properties for the four-body muonic quasiatoms $ab\mu e$, including the lowest-order relativistic, and QED corrections to the total and binding energies. There is great interest in determining such corrections from direct calculations and comparing them with the results of approximate evaluations based on the one-electron model of the $ab\mu e$ quasiatom. In general, the theoretical and experimental analysis of four-body quasiatoms $ab\mu e$ is a broad area of study. Analysis and solutions of this problem will certainly lead to a substantial improvement of our current knowledge about the bound-state spectra in four-body systems with arbitrary particle masses.

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