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Decoherence and measurement-induced correlations

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Decoherence arises from the interaction of a quantum system with the environment or, more precisely, from the measurement in which the environment "measures" the quantum system and establishes correlations with it. In this work, we first quantify measurement-induced correlations from both classical and quantum perspectives. Then we quantify decoherence via measurement-induced classical correlations. By virtue of an intrinsic and powerful link between entanglement (as quantified by entanglement of formation) and classical correlations (as quantified by the difference between the total correlations and quantum discord) in pure tripartite systems, we present general analytical formulas for measurement-induced correlations and decoherence measures for qubit systems and, furthermore, reveal a conservation relation for the information-disturbance tradeoff.

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I. INTRODUCTION

When a quantum system is coupled with the environment, correlations are established between them, and decoherence is essentially a manifestation of such correlations [1–3]. This can be completely cast into the von Neumann scheme of quantum measurement in which correlations are established between a system and a measurement apparatus if we interpret the environment as the measurement apparatus [4,5]. The fundamental question arises of how to characterize and quantify such measurement-induced correlations. This primary issue is of both theoretical significance and practical interest since measurement plays a pivotal role in quantum theory. The present work is devoted to an investigation of this problem and its implications for decoherence. Several basic figures of merit characterizing measurement-induced correlations are proposed and their fundamental significance and properties are illustrated.

For this purpose, recall that a general quantum measurement $\mathcal{M} = \{M_j\}$ on a state ρ induces, in the selective case, an ensemble $\{p_j,\rho_j\}$ consisting of the outcome probabilities $p_j := \mathrm{tr} M_j \rho M_j^\dagger$ and the corresponding post-measurement states $\rho_j := \frac{1}{p_j} M_j \rho M_j^\dagger$. In the nonselective case, the measurement results are ignored and the overall post-measurement state

$$\mathcal{M}(\rho) := \sum_j p_j \rho_j = \sum_j M_j \rho M_j^\dagger \tag{1}$$

is the average of the ensemble $\{p_j,\rho_j\}$. The measurement $\mathcal{M}$ defined by Eq. (1) is also usually called an operation or a channel, and M_j are the so called Kraus operators, which satisfy the normalization condition $\sum_j M_j^\dagger M_j = \mathbf{1}$. The family $\{M_j^\dagger M_j\}$ constitutes a positive operator-valued measure [6,7]. For a given measurement, there are two canonical ways to embed it into bipartite states: The first is via the Jamiolkoski-Choi isomorphism between an operation and a bipartite state with the help of an ancilla which is a "ghost" copy of the initial system (channel-state duality) [8], the second is to couple the system with an apparatus (or environment) by virtue of a unitary operator. We will combine these two ways in order to reveal and evaluate measurement-induced correlations.

Our approach is as follows: Consider a quantum system with a Hilbert space H^b. Any measurement $\mathcal{M}$ as described by Eq. (1) on this system has the following unitary-reduction representation [6,7]:

$$\mathcal{M}(\rho) = \mathrm{tr}_c U(\rho \otimes |c\rangle\langle c|)U^\dagger.$$

Here tr_c denotes the partial trace over H^c, U is a unitary operator on $H^b \otimes H^c$ with H^c interpreted as an apparatus space, and $|c\rangle$ is an initial state of the apparatus. In the conventional system-environment approach to decoherence, H^c is usually interpreted as an environment, which, of course, can also be regarded as a measurement apparatus. We call the pair $(U,|c\rangle)$ a unitary realization of the measurement $\mathcal{M}$. In general, a measurement has infinitely many different unitary realizations, which are connected by local unitary operators on the apparatus space [7].

Now the correlations established between the system and the measurement apparatus are fully embodied in the unitary operator U, and it is intuitive and natural to quantify the correlations between the system and the apparatus induced by the measurement in terms of correlations in the final system-apparatus state $U(\rho \otimes |c\rangle\langle c|)U^\dagger$. We will exploit this idea in order to capture measurement-induced correlations. It turns out that such measurement-induced correlations are intimately related to entanglement in the ancilla-system state $\mathcal{I}^a \otimes \mathcal{M}(|\Psi^{ab}\rangle\langle\Psi^{ab}|)$. Here, $|\Psi^{ab}\rangle$ is a purification of the initial system state ρ, and $\mathcal{I}^a$ is the identity operation on the ancilla space $H^a = H^b$ used for purifying ρ. Before embarking on this task, we recall the notions of classical and quantum correlations [9–12] in Sec. II and employ these notions to define measurement-induced correlations in Sec. III. Several examples are worked out explicitly. As applications, we quantify decoherence and information-disturbance tradeoff in Sec. IV and Sec. V respectively. Finally, Sec. VI is devoted to discussion, and the appendix is devoted to the proof of Theorem 1 and detailed calculations for examples 1, 2, and 3.

II. CLASSICAL VERSUS QUANTUM CORRELATIONS

Given a bipartite state σ^{bc} shared by two parties b and c with marginal states $\sigma^b := \mathrm{tr}_c \sigma^{bc}$ and $\sigma^c := \mathrm{tr}_b \sigma^{bc}$, respectively,

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its amount of total correlations is usually quantified by the quantum mutual information [13–16]:

$$I(\sigma^{bc}) := S(\sigma^b) + S(\sigma^c) - S(\sigma^{bc}).$$

Here, $S(\sigma^b) := -\mathrm{tr}\sigma^b \log_2 \sigma^b$ is the von Neumann entropy.

Following Henderson and Vedral [9], the amount of classical correlations in σ^{bc} is well quantified by

$$C(\sigma^{bc}) := \max_{\Pi} \left[S(\sigma^b) - \sum_j q_j S(\sigma_j^b) \right],$$

where the max is over all measurements $\Pi = \{\Pi_j\}$ on system c and $q_j := \mathrm{tr}(\mathbf{1} \otimes \Pi_j)\sigma^{bc}(\mathbf{1} \otimes \Pi_j^\dagger)$,

$$\sigma_j^b := \mathrm{tr}_c(\mathbf{1} \otimes \Pi_j)\sigma^{bc}(\mathbf{1} \otimes \Pi_j^\dagger)/q_j.$$

Since $I(\sigma^{bc})$ quantifies the total correlations, the amount of quantum correlations can be defined as

$$Q(\sigma^{bc}) := I(\sigma^{bc}) - C(\sigma^{bc}).$$

In particular, if the measurements are restricted to the von Neumann measurements (orthogonal, one-dimensional projections) in the above definition, then one gets the quantum discord introduced by Ollivier and Zurek [10], which has operational interpretations and interesting applications in quantum information theory [17–22]. Except for some particular states such as the Bell-diagonal states [12], it is usually difficult to evaluate the classical correlations and the quantum discord, even for two-qubit states [23,24]. Some other measures of correlations are also introduced and studied, such as the information deficit [25], the measurement-induced disturbance [11], the geometric discord [26,27], the measurement-induced nonlocality [28], etc.

Another important and by far the best-studied measure of a particular kind of quantum correlations (entanglement) is the entanglement of formation [29]:

$$E(\sigma^{bc}) := \min \sum_k r_k E\big(|\Psi_k^{bc}\rangle\langle\Psi_k^{bc}|\big).$$

Here, the min is over all pure state decompositions $\sigma^{bc} = \sum_k r_k |\Psi_k^{bc}\rangle\langle\Psi_k^{bc}|$, and $E(|\Psi_k^{bc}\rangle\langle\Psi_k^{bc}|) = S(\mathrm{tr}_c|\Psi_k^{bc}\rangle\langle\Psi_k^{bc}|)$ is the entanglement entropy of the pure state $|\Psi_k^{bc}\rangle$. In particular, for any two-qubit state σ^{bc}, its entanglement of formation can be explicitly evaluated as [30]

$$E(\sigma^{bc}) = H\left(\frac{1 - \sqrt{1 - \lambda^2}}{2}\right). \tag{2}$$

Here, $H(x) := -x\log_2 x - (1-x)\log_2(1-x)$ is the Shannon entropy function, $\lambda := \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$ is the concurrence of σ^{bc}, λ_j are the eigenvalues of $\sigma^{bc}\tilde{\sigma}^{bc}$ in decreasing order, $\tilde{\sigma}^{bc} := (\sigma_y \otimes \sigma_y)\bar{\sigma}^{bc}(\sigma_y \otimes \sigma_y)$ is the time-reversed version of σ^{bc}, while $\bar{\sigma}^{bc}$ is the complex conjugate of σ^{bc} (in matrix form), and σ_y is the Pauli spin matrix. With the above preparation, we are now ready to quantify measurement-induced correlations.

III. MEASUREMENT-INDUCED CORRELATIONS

By virtue of the classical correlations and quantum correlations of the final system-apparatus state

$$\sigma^{bc} := U(\rho \otimes |c\rangle\langle c|)U^\dagger,$$

arising from the unitary realization $(U,|c\rangle)$ of the measurement $\mathcal{M}$ on the system b, we define measurement-induced total, classical, and quantum correlations as

$$I_\rho(\mathcal{M}) := I(\sigma^{bc}), \quad C_\rho(\mathcal{M}) := C(\sigma^{bc}),$$
$$Q_\rho(\mathcal{M}) := Q(\sigma^{bc}),$$

respectively. When $\rho = \mathbf{1}/\dim H^b$ is the maximally mixed state, we denote the corresponding measurement-induced correlations as $I(\mathcal{M})$, $C(\mathcal{M})$, and $Q(\mathcal{M})$, respectively.

Theorem 1.The measurement-induced correlations $I_\rho(\mathcal{M})$, $C_\rho(\mathcal{M})$, and $Q_\rho(\mathcal{M})$ are all independent of the unitary realizations $(U,|c\rangle)$ of $\mathcal{M}$, and thus are intrinsic properties of ρ and $\mathcal{M}$.

This follows from the local unitary freedom of unitary realizations of a measurement which is implied by Theorem 8.2 in Ref. [7], page 372. For the detailed proof, see the appendix.

In general, if the number of the Kraus operators of the measurement $\mathcal{M}$ is large, it will be impossible to evaluate the classical correlations and quantum correlations between the system and the apparatus. Fortunately, in this circumstance, by exploiting the elegant Koashi-Winter relation [31,32]

$$S(\sigma^b) = C(\sigma^{bc}) + E(\sigma^{ab}) \tag{3}$$

between the classical correlations in the bc system and the entanglement of formation in the ab system for any *pure* tripartite state σ^{abc}, we can derive the following alternative expressions for the measurement-induced correlations, which link the measurement-induced correlations to the entanglement between the system and the ancilla.

Theorem 2. Let $|\Psi^{ab}\rangle$ be a purification of ρ in the space $H^a \otimes H^b$ with $H^a = H^b$ being an ancilla, and $\sigma^{ab} := \mathcal{I}^a \otimes \mathcal{M}(|\Psi^{ab}\rangle\langle\Psi^{ab}|)$ with $\mathcal{I}^a$ being the identity measurement on H^a, then

$$I_\rho(\mathcal{M}) = S(\mathcal{M}(\rho)) + S(\sigma^{ab}) - S(\rho), \tag{4}$$

$$C_\rho(\mathcal{M}) = S(\mathcal{M}(\rho)) - E(\sigma^{ab}), \tag{5}$$

$$Q_\rho(\mathcal{M}) = S(\sigma^{ab}) - S(\rho) + E(\sigma^{ab}). \tag{6}$$

In particular, if $\mathcal{M}$ is a measurement on a qubit system, then all the above quantities can be evaluated explicitly in view of Eq. (2).

For the proof, consider the overall initial ancilla-system-apparatus tripartite state $|\Psi^{abc}\rangle := |\Psi^{ab}\rangle \otimes |c\rangle$ and the final state $|\Phi^{abc}\rangle := (\mathcal{I}^a \otimes U)(|\Psi^{ab}\rangle \otimes |c\rangle)$ where $(U,|c\rangle)$ is a unitary realization of $\mathcal{M}$. This constitutes in some sense a tripartite purification of both the initial system state ρ and the measurement $\mathcal{M}$; see Fig. 1. Noting that $\sigma^{abc} := |\Phi^{abc}\rangle\langle\Phi^{abc}|$ is a *pure* tripartite state, $\sigma^b = \mathcal{M}(\rho)$, $S(\sigma^c) = S(\sigma^{ab})$, and $S(\sigma^{bc}) = S(\sigma^a) = S(\rho^a) = S(\rho^b) = S(\rho)$, we obtain Eq. (4) from the definition $I_\rho(\mathcal{M}) = I(\sigma^{bc})$. Moreover, Eq. (5) follows from Eq. (3), and Eq. (6) follows from a combination of Eqs. (4) and (5).

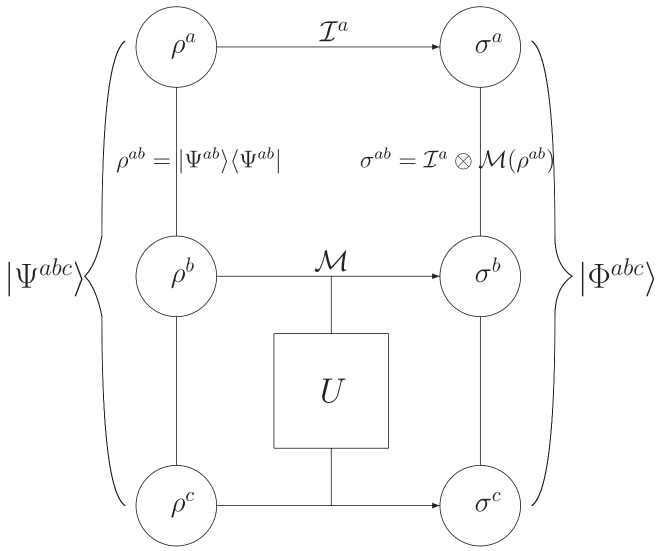

FIG. 1. Combination of purification of the system state ρ and unitary representation of the measurement $\mathcal{M}$ yields a tripartite joint purification of both ρ and $\mathcal{M}$. The overall initial state is $\rho^{abc} := |\Psi^{abc}\rangle\langle\Psi^{abc}|$, and the overall final state is $\sigma^{abc} := |\Phi^{abc}\rangle\langle\Phi^{abc}|$ with $|\Phi^{abc}\rangle := \mathcal{I}^a \otimes U(|\Psi^{abc}\rangle)$. Here $\mathcal{I}^a$ is the identity operation on H^a, and U is a unitary operator on $H^b \otimes H^c$. Furthermore, noting that $\rho = \rho^b, \rho^c = |c\rangle\langle c|$ and $\sigma^b = \mathcal{M}(\rho)$.

In order to gain an intuitive understanding of measurement-induced correlations, let us illustrate them by several fundamental and important examples.

Example 0. Consider a system state with nondegenerate spectral decomposition $\rho = \sum_k \lambda_k |k\rangle\langle k|$ on H^b. For the von Neumann measurement $\mathcal{M} = \{|k\rangle\langle k|\}$ along the eigenbase of ρ, we have a unitary realization $(U,|c\rangle)$ with $U|k\rangle \otimes |c\rangle = |k\rangle \otimes |k\rangle$ and $|c\rangle$ any pure state of the measurement apparatus space $H^c = H^b$. Consequently,

$$\sigma^{bc} = U(\rho \otimes |c\rangle\langle c|)U^\dagger = \sum_k \lambda_k |k\rangle\langle k| \otimes |k\rangle\langle k|,$$

and we readily (or directly from Theorem 2) get $I_\rho(\mathcal{M}) = C_\rho(\mathcal{M}) = S(\rho)$, $Q_\rho(\mathcal{M}) = 0$, which means that here the measurement-induced correlations are purely classical.

Example 1. For the measurement (phase-damping channel) $\mathcal{M}_{\text{phase}}(\rho) := M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger$ with

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\gamma} \end{pmatrix}, \quad 0 \leqslant \gamma \leqslant 1, \tag{7}$$

it can be evaluated that (see the appendix)

$$I(\mathcal{M}_{\text{phase}}) = H\left(\frac{1-\sqrt{1-\gamma}}{2}\right),$$

$$C(\mathcal{M}_{\text{phase}}) = 1 - H\left(\frac{1-\sqrt{\gamma}}{2}\right),$$

$$Q(\mathcal{M}_{\text{phase}}) = H\left(\frac{1-\sqrt{1-\gamma}}{2}\right) + H\left(\frac{1-\sqrt{\gamma}}{2}\right) - 1.$$

Example 2. For the measurement (amplitude-damping channel) $\mathcal{M}_{\text{amplitude}}(\rho) = M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger$ with

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \quad 0 \leqslant \gamma \leqslant 1, \tag{8}$$

it can be evaluated that (see the appendix)

$$I(\mathcal{M}_{\text{amplitude}}) = H\left(\frac{1-\gamma}{2}\right) + H\left(\frac{\gamma}{2}\right) - 1,$$

$$C(\mathcal{M}_{\text{amplitude}}) = H\left(\frac{1-\gamma}{2}\right) - H\left(\frac{1-\sqrt{\gamma}}{2}\right),$$

$$Q(\mathcal{M}_{\text{amplitude}}) = H\left(\frac{\gamma}{2}\right) + H\left(\frac{1-\sqrt{\gamma}}{2}\right) - 1.$$

The explicit expressions for measurement-induced correlations in examples 1 and 2 can be readily derived from Theorem 2 with the help of Eq. (2), as well as from the original definitions. It is interesting to compare the measurement-induced correlations for the phase and the amplitude-damping channels, as illustrated in Fig 2. Since the phase-damping channel is actually the complete decoherent channel, with γ characterizing the intensity of decoherence, it is just natural that $C(\mathcal{M}_{\text{phase}})$ is increasing with respect to γ. On the other hand, it is interesting to observe that, for the amplitude-damping channel, when $\gamma = 1$, it transforms any state into a pure state, and there are no measurement-induced correlations at all. In particular, there is no decoherence in this instance. Thus the measurement-induced correlations reveal some deeper insight concerning the difference between the phase-damping and amplitude-damping channels.

Example 3. Consider the measurement (Pauli channel) $\mathcal{M}_{\text{Pauli}}(\rho) = \sum_{j=0}^3 M_j \rho M_j^\dagger = p_0 \rho + \sum_{j=1}^3 p_j \sigma_j \rho \sigma_j$, with $M_0 = \sqrt{p_0}\mathbf{1}$, $M_j = \sqrt{p_j}\sigma_j$ for $j = 1,2,3$ and (p_0, p_1, p_2, p_3) being a probability distribution, $\{\sigma_j\}$ are the Pauli spin matrices. In this case, the system-apparatus space is 2×4 dimensional and it is difficult to calculate the measurement-induced correlations directly from the final system-apparatus state σ^{bc}. However, in view of Theorem 2, we may calculate these quantities indirectly by using the explicit formula for the entanglement of formation in σ^{ab}, which is a 2×2 dimensional state. By this method, we get (see the appendix)

$$I(\mathcal{M}_{\text{Pauli}}) = -\sum_j p_j \log_2 p_j,$$

$$C(\mathcal{M}_{\text{Pauli}}) = 1 - H\left(\frac{1-\sqrt{1-\lambda^2}}{2}\right),$$

$$Q(\mathcal{M}_{\text{Pauli}}) = -\sum_j p_j \log_2 p_j + H\left(\frac{1-\sqrt{1-\lambda^2}}{2}\right) - 1.$$

IV. QUANTIFYING DECOHERENCE VIA MEASUREMENT-INDUCED CORRELATIONS

The decoherent effect of a measurement (channel) is directly related to the measurement-induced classical correlations because only the classical correlations can be "read out" from the apparatus (information gain). Thus we define

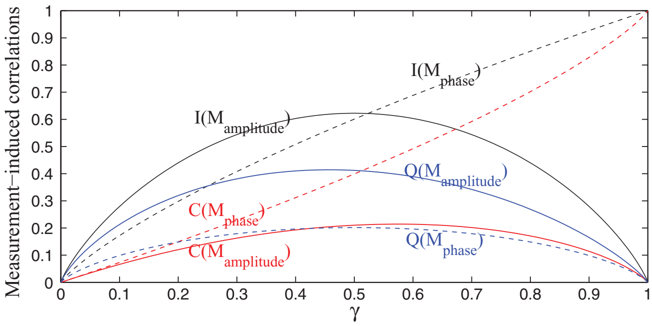

FIG. 2. (Color online) Solid lines depict the measurement-induced total (black), classical (red), and quantum (blue) correlations for the amplitude-damping channel. The dashed lines depict those for the phase-damping channel. We see that the amount of measurement-induced classical correlations $C(\mathcal{M}_{\rm phase})$ for the phase-damping channel is larger than $C(\mathcal{M}_{\rm amplitude})$, which is the amount of measurement-induced classical correlations for the amplitude-damping channel.

the decoherence caused by a measurement $\mathcal{M}$ on a state ρ as $\mathrm{De}(\mathcal{M},\rho) := C_\rho(\mathcal{M})$. This decoherence measure captures nicely our intuition about decoherence exemplified by the decay of off-diagonal elements of density matrices, and goes much beyond. In general, this quantity synthesizes two effects related to coherence and decoherence whenever mixed states are involved: The first is the "local" decoherent effect of $\mathcal{M}$ on ρ itself arising from the transformation $\rho \to \mathcal{M}(\rho)$, and the second is the "global" decoherent effect of $\mathcal{M}$ on the purification $|\Psi^{ab}\rangle$ of ρ arising from the transformation $|\Psi^{ab}\rangle \to \mathcal{I}^a \otimes \mathcal{M}(|\Psi^{ab}\rangle\langle\Psi^{ab}|)$ (e.g., destruction of the entanglement in $|\Psi^{ab}\rangle$). If ρ is a pure state, then $\mathrm{De}(\mathcal{M},\rho) = S(\mathcal{M}(\rho))$ is the increase of entropy [since $S(\rho) = 0$]. On the other hand, consider Example 0; the "local" decoherent effect of $\mathcal{M}$ on ρ should be regarded as zero since the measurement does not change the state. However, if we regarded ρ as the marginal state of a pure bipartite state $|\Psi^{ab}\rangle$, then the entanglement loss between the system and the ancilla (used for purifying ρ) caused by the measurement is exactly the entropy of ρ, which should be regarded as some kind of "global" decoherent effect. Thus the overall decoherence is $S(\rho)$, which is in accordance with the calculation result of Example 0.

From Examples 1 and 2 we also see that the decoherent effect of the phase-damping channel is indeed stronger than that of the amplitude-damping channel. It should be emphasized that here it is the amount of measurement-induced *classical* correlations, rather than that of the measurement-induced *quantum* correlations, that captures the decoherence. It would be interesting to further investigate the role of measurement-induced *quantum* correlations in decoherence and measurement.

V. INFORMATION-DISTURBANCE TRADEOFF

Theorem 2 has interesting implications for information-disturbance tradeoff in quantum measurements [33–40]. The measurement-induced classical correlations $C_\rho(\mathcal{M})$ can be interpreted as the information gained by the measurement $\mathcal{M}$ [41]. How to quantify the disturbance on ρ caused by $\mathcal{M}$? First, there is the entropy change $S(\mathcal{M}(\rho)) - S(\rho)$. Second, there is the ancilla-system entanglement loss $E(\rho^{ab}) - E(\sigma^{ab}) = S(\rho) - E(\sigma^{ab})$. If we combine the two effects and define the disturbance

$$D_\rho(\mathcal{M}) := S(\mathcal{M}(\rho)) - S(\rho) + E(\rho^{ab}) - E(\sigma^{ab}),$$

then Eq. (5) may be interpreted as an information-disturbance tradeoff identity; that is,

$$C_\rho(\mathcal{M}) = D_\rho(\mathcal{M}),$$

which of course is a particular manifestation of information conservation. Another intrinsic interpretation of Eq. (5) is as follows: Since $S(\mathcal{M}(\rho)) = E(\sigma^{ac:b})$ is precisely the entanglement of the pure state σ^{abc} with the partition $ac : b$, the quantity

$$S(\mathcal{M}(\rho)) - E(\sigma^{ab}) = E(\sigma^{ac:b}) - E(\sigma^{ab}),$$

which represents the difference between the entanglement in the partition $ac : b$ and $a : b$, respectively, may be roughly regarded as some kind of "virtual entanglement" between system b and measurement apparatus c caused by the measurement. Equation (5) states that this "virtual entanglement" coincides with the measurement-induced classical correlations!

VI. DISCUSSION

By analyzing and separating the correlations in the system-apparatus state after a measurement, we have proposed three notions to characterize and quantify measurement-induced correlations. These quantities reveal more precisely the nature and characteristics of information transfer from a system to a measurement apparatus. This framework also encapsulates naturally the decoherence scheme when we interpret the environment as an apparatus measuring the quantum system. The notion of measurement-induced classical correlations can be used for quantifying decoherence and has fundamental implications for information-disturbance tradeoff.

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APPENDIX

Here we present the proof of Theorem 1, as well as the detailed calculations for examples 1, 2 and 3.

Proof of Theorem 1. Assume that there exist two different unitary realizations $(U_1,|c_1\rangle)$ and $(U_2,|c_2\rangle)$ for a given measurement $\mathcal{M}$, with corresponding apparatus spaces H^{c_1} and H^{c_1}, respectively, then for any state ρ on H^b,

$$\mathcal{M}(\rho) = \mathrm{tr}_{c_1} U_1(\rho \otimes |c_1\rangle\langle c_1|)U_1^\dagger = \mathrm{tr}_{c_2} U_2(\rho \otimes |c_2\rangle\langle c_2|)U_2^\dagger.$$

Without loss of generality, we assume $\dim H^{c_1} \geqslant \dim H^{c_2}$ and embed H^{c_2} into H^{c_1}. Let $\{|e_j\rangle\}$ be an orthonormal basis of H^{c_1} which is formed by an extension of an orthonormal basis $\{|e_k\rangle\}$ of H^{c_2}. Let $M_j := \langle e_j|U_1|c_1\rangle$, $N_k := \langle e_k|U_2|c_2\rangle$, then $\{M_j\}$ and $\{N_k\}$ are two different Kraus operator representations for the same measurement $\mathcal{M}$; namely,

$$\mathcal{M}(\rho) = \sum_j M_j \rho M_j^\dagger = \sum_k N_k \rho N_k^\dagger.$$

Moreover, for any state ρ on H^b,

$$U_1(\rho \otimes |c_1\rangle\langle c_1|)U_1^\dagger = \sum_{jj'} M_j \rho M_{j'}^\dagger \otimes |e_j\rangle\langle e_{j'}|,$$

$$U_2(\rho \otimes |c_2\rangle\langle c_2|)U_2^\dagger = \sum_{kk'} N_k \rho N_{k'}^\dagger \otimes |e_k\rangle\langle e_{k'}|.$$

By Theorem 8.2 in Ref. [7] (page 372), we know that there exists a unitary matrix $W = \{w_{jj'}\}$ on H^{c_1} such that $M_j = \sum_k w_{jk} N_k$. Since $\{|f_k\rangle := \sum_j w_{jk}|e_j\rangle\}$ constitutes an orthonormal base, there exists a unitary operator X such that $|f_k\rangle = X|e_k\rangle$, and we have

$$\begin{aligned} &U_1(\rho \otimes |c_1\rangle\langle c_1|)U_1^\dagger \\ &= \sum_{jj'} M_j \rho M_{j'}^\dagger \otimes |e_j\rangle\langle e_{j'}| \\ &= \sum_{jj'} \left(\sum_k w_{jk} N_k\right) \rho \left(\sum_{k'} \bar{w}_{j'k'} N_{k'}^\dagger\right) \otimes |e_j\rangle\langle e_{j'}| \\ &= \sum_{kk'} N_k \rho N_{k'}^\dagger \otimes \sum_j w_{jk}|e_j\rangle \sum_{j'} \bar{w}_{j'k'}\langle e_{j'}| \\ &= \sum_{kk'} N_k \rho N_{k'}^\dagger \otimes |f_k\rangle\langle f_{k'}| \\ &= \sum_{kk'} N_k \rho N_{k'}^\dagger \otimes X|e_k\rangle\langle e_{k'}|X^\dagger \\ &= (\mathbf{1} \otimes X)\, U_2(\rho \otimes |c_2\rangle\langle c_2|)U_2^\dagger(\mathbf{1} \otimes X^\dagger). \end{aligned}$$

Since quantum mutual information is invariant under local unitary operations, we conclude that $I_\rho(\mathcal{M})$ is independent of the unitary realizations of $\mathcal{M}$. Similarly, from the definition of classical correlations, we see that $C_\rho(\mathcal{M})$ is also invariant under local unitary operations. This implies that $C_\rho(\mathcal{M})$ and $Q_\rho(\mathcal{M})$ are independent of the unitary realizations of $\mathcal{M}$. ■

For examples 1 and 2, we will present two methods to evaluate the measurement-induced correlations: The first is by direct calculations; the second is by use of Theorem 2, which is much easier.

Example 1. For the measurement (phase-damping channel)

$$\mathcal{M}_{\text{phase}}(\rho) := M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger,$$

with

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\gamma} \end{pmatrix}, \quad 0 \leqslant \gamma \leqslant 1, \tag{A1}$$

one of its unitary realizations $(U,|c\rangle)$ can be represented as $|c\rangle := |0\rangle$ and

$$U := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & -i\sqrt{\gamma} \\ 0 & 0 & \sqrt{\gamma} & i\sqrt{1-\gamma} \end{pmatrix},$$

in the standard base $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$. Clearly, U satisfies

$$M_j = \langle j|U|c\rangle, \quad j = 0,1.$$

After performing the measurement $\mathcal{M}_{\text{phase}}$ on the system state $\rho = \rho^b = \mathbf{1}/2$, the final system-apparatus state

$$\sigma^{bc} := U(\rho \otimes |0\rangle\langle 0|)U^\dagger$$

can be expressed as

$$\sigma^{bc} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\gamma & \sqrt{\gamma(1-\gamma)} \\ 0 & 0 & \sqrt{\gamma(1-\gamma)} & \gamma \end{pmatrix}.$$

Therefore,

$$\sigma^b = \frac{\mathbf{1}}{2}, \quad \sigma^c = \frac{1}{2}\begin{pmatrix} 2-\gamma & \sqrt{\gamma(1-\gamma)} \\ \sqrt{\gamma(1-\gamma)} & \gamma \end{pmatrix},$$

and

$$S(\sigma^b) = 1, \quad S(\sigma^c) = H\left(\frac{1-\sqrt{1-\gamma}}{2}\right), \quad S(\sigma^{bc}) = 1,$$

from which we obtain

$$I(\mathcal{M}_{\text{phase}}) = H\left(\frac{1-\sqrt{1-\gamma}}{2}\right).$$

To evaluate $C(\mathcal{M}_{\text{phase}})$, we consider another unitary realization $((\mathbf{1} \otimes Z)U,|0\rangle)$ of $\mathcal{M}$ with

$$Z = \begin{pmatrix} \sqrt{\frac{1+\sqrt{1-\gamma}}{2}} & \sqrt{\frac{1-\sqrt{1-\gamma}}{2}} \\ -\sqrt{\frac{1-\sqrt{1-\gamma}}{2}} & \sqrt{\frac{1+\sqrt{1-\gamma}}{2}} \end{pmatrix}.$$

Note that Z diagonalizes σ^c; namely,

$$Z\sigma^c Z^\dagger = \begin{pmatrix} \frac{1+\sqrt{1-\gamma}}{2} & 0 \\ 0 & \frac{1-\sqrt{1-\gamma}}{2} \end{pmatrix}.$$

Let

$$\tau^{bc} := (\mathbf{1} \otimes Z)\sigma^{bc}(\mathbf{1} \otimes Z^\dagger) = \frac{1}{4}\begin{pmatrix} 1+\sqrt{1-\gamma} & -\sqrt{\gamma} & 0 & 0 \\ -\sqrt{\gamma} & 1-\sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & 1+\sqrt{1-\gamma} & \sqrt{\gamma} \\ 0 & 0 & \sqrt{\gamma} & 1-\sqrt{1-\gamma} \end{pmatrix},$$

then $C(\mathcal{M}_{\text{phase}}) = C(\tau^{bc})$.

By the results in Refs. [32,42] for Bell-diagonal states or two-qubit states of rank 2, it can be shown that the amount of classical correlations can be evaluated by restricting the measurements on the apparatus space H^c to the von Neumann measurements $\Pi := \{\Pi_j\}$. Noting that τ^{bc} is of rank 2, thus, in this case,

$$C(\tau^{bc}) := \max_{\Pi}\left[S(\tau^b) - \sum_j q_j S(\tau_j^b)\right] = S(\tau^b) - \min_{\Pi}\sum_j q_j S(\tau_j^b),$$

with

$$q_j := \mathrm{tr}(\mathbf{1}\otimes\Pi_j)\tau^{bc}, \quad \tau_j^b := \frac{1}{q_j}\mathrm{tr}_c(\mathbf{1}\otimes\Pi_j)\tau^{bc}(\mathbf{1}\otimes\Pi_j).$$

A generic von Neumann measurement on H^c in this context can be parametrized by a unitary operator

$$V = \begin{pmatrix} u & -v \\ \bar{v} & \bar{u} \end{pmatrix}, \quad u,v \in C, \quad |u|^2 + |v|^2 = 1,$$

as

$$\Pi = \{\Pi_0 := V|0\rangle\langle 0|V^\dagger,\ \Pi_1 := V|1\rangle\langle 1|V^\dagger\}.$$

Now straightforward calculations lead to

$$q_0 = \mathrm{tr}(\mathbf{1}\otimes\Pi_0)\tau^{bc} = \frac{1+\sqrt{1-\gamma}(|u|^2-|v|^2)}{2},$$
$$q_1 = \mathrm{tr}(\mathbf{1}\otimes\Pi_1)\tau^{bc} = \frac{1-\sqrt{1-\gamma}(|u|^2-|v|^2)}{2},$$

and

$$\tau_0^b = \frac{1}{q_0}\mathrm{tr}_c(\mathbf{1}\otimes\Pi_0)\tau^{bc}(\mathbf{1}\otimes\Pi_0) = \frac{1}{4q_0}\begin{pmatrix} 1+\sqrt{1-\gamma}(|u|^2-|v|^2)-\sqrt{\gamma}(uv+\bar{u}\bar{v}) & 0 \\ 0 & 1+\sqrt{1-\gamma}(|u|^2-|v|^2)+\sqrt{\gamma}(uv+\bar{u}\bar{v}) \end{pmatrix},$$
$$\tau_1^b = \frac{1}{q_1}\mathrm{tr}_c(\mathbf{1}\otimes\Pi_1)\tau^{bc}(\mathbf{1}\otimes\Pi_1) = \frac{1}{4q_1}\begin{pmatrix} 1-\sqrt{1-\gamma}(|u|^2-|v|^2)+\sqrt{\gamma}(uv+\bar{u}\bar{v}) & 0 \\ 0 & 1-\sqrt{1-\gamma}(|u|^2-|v|^2)-\sqrt{\gamma}(uv+\bar{u}\bar{v}) \end{pmatrix}.$$

Consequently, $q_0S(\tau_0^b)+q_1S(\tau_1^b)$ is a symmetric function of u and v, and it can be shown by elementary but tedious analysis that its minimum value is $H(\frac{1-\sqrt{\gamma}}{2})$, which can be achieved when $u = v = 1/\sqrt{2}$. From this we conclude that

$$C(\mathcal{M}_{\text{phase}}) = 1 - H\left(\frac{1-\sqrt{\gamma}}{2}\right),$$

and

$$Q(\mathcal{M}_{\text{phase}}) = H\left(\frac{1-\sqrt{1-\gamma}}{2}\right) + H\left(\frac{1-\sqrt{\gamma}}{2}\right) - 1.$$

An alternative, indirect, and yet easier way to get the measurement-induced correlations $C(\mathcal{M}_{\text{phase}})$ and $Q(\mathcal{M}_{\text{phase}})$ is to invoke Theorem 2. Let $|\Psi^{ab}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ be a purification of the system state $\rho = \rho^b = \mathbf{1}/2$, then the final ancilla-system state is

$$\sigma^{ab} := \mathcal{I}^a \otimes \mathcal{M}_{\text{phase}}(|\Psi^{ab}\rangle\langle\Psi^{ab}|) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-\gamma} & 0 & 0 & 1 \end{pmatrix}.$$

The entanglement of formation of this state is

$$E(\sigma^{ab}) = H\left(\frac{1-\sqrt{\gamma}}{2}\right).$$

Now by Theorem 2, we readily get

$$C(\mathcal{M}_{\text{phase}}) = 1 - H\left(\frac{1-\sqrt{\gamma}}{2}\right),$$
$$Q(\mathcal{M}_{\text{phase}}) = H\left(\frac{1-\sqrt{1-\gamma}}{2}\right) + H\left(\frac{1-\sqrt{\gamma}}{2}\right) - 1.$$

We see that the two methods lead to the same results.

Example 2. For the measurement (amplitude-damping channel)

$$\mathcal{M}_{\text{amplitude}}(\rho) = M_0\rho M_0^\dagger + M_1\rho M_1^\dagger,$$
with
$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \quad 0 \leqslant \gamma \leqslant 1, \tag{A2}$$

one of its unitary realizations $(U,|c\rangle)$ can be represented as $|c\rangle := |0\rangle$, and

$$U := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & \sqrt{\gamma} & 0 \\ 0 & -\sqrt{\gamma} & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

in the standard basis. After performing the measurement $\mathcal{M}_{\text{amplitude}}$ on the system state $\rho = \mathbf{1}/2$, the final system-apparatus state

$$\sigma^{bc} := U(\rho\otimes|0\rangle\langle 0|)U^\dagger$$

can be expressed as

$$\sigma^{bc} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & \sqrt{\gamma(1-\gamma)} & 0 \\ 0 & \sqrt{\gamma(1-\gamma)} & 1-\gamma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

from which we obtain

$$\sigma^b = \frac{1}{2}\begin{pmatrix} 1+\gamma & 0 \\ 0 & 1-\gamma \end{pmatrix}, \quad \sigma^c = \frac{1}{2}\begin{pmatrix} 2-\gamma & 0 \\ 0 & \gamma \end{pmatrix},$$

and

$$S(\sigma^b) = H\left(\frac{1-\gamma}{2}\right), \quad S(\sigma^c) = H\left(\frac{\gamma}{2}\right), \quad S(\sigma^{bc}) = 1.$$

Consequently,

$$I(\mathcal{M}_{\text{amplitude}}) = H\left(\frac{1-\gamma}{2}\right) + H\left(\frac{\gamma}{2}\right) - 1.$$

To evaluate the classical correlations $C(\mathcal{M}_{\text{amplitude}})$, we can restrict the measurements to von Neumann measurements by the same reason as in Example 1 since σ^{bc} is of rank 2. Now by direct use of Theorem 1 in Ref. [43], we readily get

$$C(\mathcal{M}_{\text{amplitude}}) = H\left(\frac{1-\gamma}{2}\right) - H\left(\frac{1-\sqrt{\gamma}}{2}\right),$$

$$Q(\mathcal{M}_{\text{amplitude}}) = H\left(\frac{\gamma}{2}\right) + H\left(\frac{1-\sqrt{\gamma}}{2}\right) - 1.$$

We can also derive the above results more easily by use of Theorem 2. Let $|\Psi^{ab}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ be a purification of the system state $\rho = \rho^b = \mathbf{1}/2$, then the final ancilla-system state is

$$\sigma^{ab} := \mathcal{I}^a \otimes \mathcal{M}_{\text{amplitude}}(|\Psi^{ab}\rangle\langle\Psi^{ab}|) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \sqrt{1-\gamma} & 0 & 0 & 1-\gamma \end{pmatrix},$$

with entanglement of formation

$$E(\sigma^{ab}) = H\left(\frac{1-\sqrt{\gamma}}{2}\right).$$

Consequently,

$$C(\mathcal{M}_{\text{amplitude}}) = H\left(\frac{1-\gamma}{2}\right) - H\left(\frac{1-\sqrt{\gamma}}{2}\right),$$

$$Q(\mathcal{M}_{\text{amplitude}}) = H\left(\frac{\gamma}{2}\right) + H\left(\frac{1-\sqrt{\gamma}}{2}\right) - 1,$$

which are the same as those obtained by the direct method.

Example 3. For the measurement (Pauli channel)

$$\mathcal{M}_{\text{Pauli}}(\rho) = \sum_{j=0}^{3} M_j \rho M_j^\dagger = p_0\rho + \sum_{j=1}^{3} p_j \sigma_j \rho \sigma_j,$$

with $M_0 = \sqrt{p_0}\mathbf{1}$, $M_j = \sqrt{p_j}\sigma_j$ for $j = 1,2,3$ and (p_0,p_1,p_2,p_3) is a probability distribution, $\{\sigma_j\}$ are the Pauli spin matrices, the system-apparatus space is 2×4 dimensional, and it is difficult to calculate the measurement-induced correlations directly from the final system-apparatus state σ^{bc}. However, in view of Theorem 2, we may calculate these quantities indirectly by using the explicit formula for the entanglement of formation in σ^{ab}, which is 2×2 dimensional. To do this, let $|\Psi^{ab}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ be a purification of the system state $\rho = \rho^b = \mathbf{1}/2$, then the final ancilla-system state is

$$\sigma^{ab} := \mathcal{I}^a \otimes \mathcal{M}_{\text{Pauli}}(|\Psi^{ab}\rangle\langle\Psi^{ab}|) = \frac{1}{2}\begin{pmatrix} p_0+p_3 & 0 & 0 & p_0-p_3 \\ 0 & p_1+p_2 & p_1-p_2 & 0 \\ 0 & p_1-p_2 & p_1+p_2 & 0 \\ p_0-p_3 & 0 & 0 & p_0+p_3 \end{pmatrix}.$$

The entanglement of formation of this state is

$$E(\sigma^{ab}) = H\left(\frac{1-\sqrt{1-\lambda^2}}{2}\right),$$

with $\lambda := \max\{0, 2p_{\max} - 1\}$ being the concurrence, and $p_{\max} := \max\{p_0,p_1,p_2,p_3\}$.

The final states of the ancilla, the system and the apparatus after the measurement $\mathcal{M}_{\text{pauli}}$ are $\sigma^a = \mathbf{1}/2$, $\sigma^b = \mathbf{1}/2$, and $\sigma^c = \text{diag}(p_0,p_1,p_2,p_3)$, respectively. The marginal entropies are given by $S(\sigma^a) = 1$, $S(\sigma^b) = 1$, and $S(\sigma^c) = -\sum_{j=0}^{3} p_j \log_2 p_j$. By Theorem 2, we have

$$C(\sigma^{bc}) = S(\rho^b) - E(\sigma^{ab}) = 1 - H\left(\frac{1-\sqrt{1-\lambda^2}}{2}\right).$$

Consequently, the measurement-induced correlations are

$$I(\mathcal{M}_{\text{Pauli}}) = -\sum_{j=0}^{3} p_j \log_2 p_j,$$

$$C(\mathcal{M}_{\text{Pauli}}) = 1 - H\left(\frac{1-\sqrt{1-\lambda^2}}{2}\right),$$

$$Q(\mathcal{M}_{\text{Pauli}}) = -\sum_{j=0}^{3} p_j \log_2 p_j + H\left(\frac{1-\sqrt{1-\lambda^2}}{2}\right) - 1.$$

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