# Weak versus approximate values in quantum state determination

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We generalize the concept of the weak value of a quantum observable to cover arbitrary real positive operator measures. We show that the definition is operationally meaningful in the sense that it can be understood within the quantum theory of sequential measurements. In particular, we show that the weak value can be obtained from a single measurement scheme. We then present a detailed analysis of the recent experiment [J. S. Lundeen *et al.*, Nature (London) **474**, 188 (2011)] concerning the reconstruction of the state of a photon using weak measurements. We compare their method with the reconstruction method through informationally complete phase space measurements. In particular, we show that unlike with phase space measurements, the reconstruction of a completely unknown state is not always possible using the method of weak measurements.

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## I. INTRODUCTION

The weak value of a quantum observable, as defined by Aharonov *et al.* [1], has always remained a somewhat controversial concept despite the vast amount of attention it has received over the years (see, e.g., Ref. [2] and references therein). The purpose of this paper is to clarify this topic and our goal is twofold. On one hand, we wish to show that the weak values can be exhaustively explained within the quantum theory of sequential measurements. On the other hand, in view of the recent paper of Lundeen *et al.* [3], we wish to compare the role of weak and approximate measurements in quantum state determination. An additional motivation for the paper comes from the desire to give a mathematically rigorous treatment to this subject.

To this end we briefly recall the quantum theory of measurement (Sec. II). We generalize the standard measurement model of von Neumann [4] (for a survey, see Ref. [5]), so that it can be applied to arbitrary real observables (as positive operator measures) (Sec. III). We then generalize the concept of the weak value of an observable to cover arbitrary real observables and compare it with its approximative values obtained from the relevant standard measurement (Sec. IV). In this way our definition generalizes, not only the original idea [1], but also the more recent works where positive operator measures with discrete or even finite outcome spaces have been considered [6,7]. As the next step, we use the theory of sequential measurements to construct measurement schemes that give the real and imaginary parts of the weak value as limits of sequences of conditional averages (Sec. V). The result for this general framework is thus in agreement with the previous works [8–11]. We then take a step further and show that it is possible to obtain the whole weak value from a single measurement scheme by using a phase space observable as the probe observable. Finally, we analyze the method of Lundeen *et al.* for obtaining the pointwise values of the wavefunction from weak measurements (Sec. VI). We discuss some of its shortcomings and present as an alternative approach the known method of approximate sequential measurements.

Throughout the paper we denote by $\mathcal{H}$ the Hilbert space associated with a physical system, and by $\mathcal{L}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ the sets of bounded and trace class operators acting on $\mathcal{H}$. The concepts of states, observables, and the statistical duality they define form the rudimentary frame of the description of the system: a state $\rho$ is (represented as) a positive ($\rho \geqslant 0$) trace one ($\mathrm{tr}[\rho] = 1$) operator, an observable is (represented as) a normalized positive operator measure (POM) $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ (defined on a $\sigma$-algebra $\mathcal{A}$ of subsets of a set $\Omega$ of the values of the observable), the probability measure $\mathcal{A} \ni X \mapsto \mathsf{p}^{\mathsf{E}}_{\rho}(X) = \mathrm{tr}[\rho\mathsf{E}(X)]$ giving the measurement outcome statistics for the observable $\mathsf{E}$ in the state $\rho$. Usually the value space $(\Omega, \mathcal{A})$ of an observable is just the real Borel space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, which we use subsequently. For a pure state ($\rho = \rho^2$) we use also a unit vector (vector state) representation $\varphi \in \mathcal{H}$, $\rho\varphi = \varphi$.

## II. QUANTUM MEASUREMENTS

Quantum theory of measurement operates on a hierarchy of three levels of description reflecting the options of restricting one's attention to the outcome probabilities at the level of the measured system, or taking into account the system's conditional state changes, or adopting the most comprehensive level of modeling the interaction and information transfer between the system and the probe. We shall briefly describe these three levels since all of them play a role in our analysis; for a more comprehensive survey, see, e.g., Ref. [12].

The crudest statistical level is the one described in the Introduction. On the next level, one also describes the conditional state changes of the system due to a measurement, conditioned with respect to the pointer values. These changes are most conveniently captured in the concept of an *instrument* [13,14], that is, an operation valued measure $\mathcal{I} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$. This means that each $X \in \mathcal{B}(\mathbb{R})$ determines a positive contractive linear map $\mathcal{I}(X) : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ and $\mathcal{I}(X)(\rho)$ is the unnormalized conditional output state. Each instrument determines uniquely the associated observable via the formula $\mathrm{tr}[\rho\mathsf{E}(X)] = \mathrm{tr}[\mathcal{I}(X)(\rho)]$ or equivalently via the dual instrument as $\mathsf{E}(X) = \mathcal{I}(X)^*(I)$.

The most detailed descriptions of measurements are given by the *measurement schemes* where the coupling of the

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physical system and the probe (aka measuring apparatus) is considered. More precisely, a measurement scheme is a 4-tuple $\mathcal{M} = \langle \mathcal{K}, \sigma, \Phi, \mathsf{Z} \rangle$, where $\mathcal{K}$ is the Hilbert space of the probe, $\sigma$ its initial state, $\Phi : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{T}(\mathcal{H} \otimes \mathcal{K})$ a state transformation (i.e., a trace-preserving operation) modeling the interaction between the system and the probe, and $\mathsf{Z} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{K})$ the pointer observable (POM). Each measurement scheme determines a unique instrument via

$$\mathcal{I}(X)(\rho) = \mathrm{tr}_{\mathcal{K}}[\Phi(\rho \otimes \sigma) I \otimes \mathsf{Z}(X)], \tag{1}$$

where $\mathrm{tr}_{\mathcal{K}}[\cdot]$ denotes partial trace over $\mathcal{K}$. As a consequence, a measurement scheme also determines uniquely the measured observable $\mathsf{E}$, and one has

$$\mathrm{tr}[\rho \mathsf{E}(X)] = \mathrm{tr}[\mathcal{I}(X)(\rho)] = \mathrm{tr}[\Phi(\rho \otimes \sigma) I \otimes \mathsf{Z}(X)]. \tag{2}$$

In most cases, $\Phi$ can be given by a unitary operator $U$ on $\mathcal{H} \otimes \mathcal{K}$ so that $\Phi(\rho \otimes \sigma) = U(\rho \otimes \sigma)U^*$. It is often convenient to allow the pointer observable $\mathsf{Z}$ to have different values than the measured observable $\mathsf{E}$ has; in such a case one needs to introduce a (measurable) pointer function $f : \mathbb{R} \to \mathbb{R}$ and adjust Eqs. (1) and (2) accordingly, that is, replacing the set $X$ with $f^{-1}(X)$ for the pointer observable $\mathsf{Z}$. It is a fundamental result of the quantum measurement theory that each observable $\mathsf{E}$ admits a measurement scheme in the sense of Eq. (2) [15].

We wish to emphasize that the correspondences between measurement schemes and instruments, and instruments and observables, are many-to-one, reflecting the obvious fact that a given observable may be measured in various ways, as well as that a given instrument may be realized by various measurement schemes.

Now suppose that one wants to measure a pair of observables, $\mathsf{E}_1$ and $\mathsf{E}_2$, by performing their measurements sequentially, for instance, first $\mathsf{E}_1$ and then $\mathsf{E}_2$. On the level of instruments this leads to the sequentially composed instrument $\mathcal{I}_{12}$ defined by $\mathcal{I}_{12}(X \times Y) = \mathcal{I}_2(Y) \circ \mathcal{I}_1(X)$ [13,16]. On the statistical level such a sequential measurement defines the (sequential joint) observable $\mathsf{M}$ given by $\mathsf{M}(X \times Y) = \mathcal{I}_1(X)^*[\mathcal{I}_2(Y)^*(I)]$. In particular, the Cartesian margins of $\mathsf{M}$ are

$$\mathsf{M}_1(X) = \mathsf{M}(X \times \mathbb{R}) = \mathsf{E}_1(X),$$
$$\mathsf{M}_2(Y) = \mathsf{M}(\mathbb{R} \times Y) = \mathcal{I}_1(\mathbb{R})^*(\mathsf{E}_2(Y)),$$

which shows the characteristic quantum feature that the first measurement typically disturbs the subsequent one: the first margin is the observable measured first, the second margin is a smeared (or disturbed) version of the observable measured second, smearing depending on the first measurement. It is to be emphasized that the structure of a sequential joint observable $\mathsf{M}$ does not depend on any details of the second measurement.

## III. GENERALIZED STANDARD MODEL

Let $\mathsf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be an observable to be measured. By a special version [15] of the Naimark dilation theorem there exists a Hilbert space $\mathcal{H}_0$, a unit vector $\psi \in \mathcal{H}_0$, and a spectral measure $\mathsf{P}^A : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H}_0)$, with the corresponding selfadjoint (first moment) operator $A = \int x \, d\mathsf{P}^A(x)$, such that

$$\mathsf{E}(X) = V_\psi^* \, \mathsf{P}^A(X) \, V_\psi,$$

where $V_\psi : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_0$ is the embedding $V_\psi(\varphi) = \varphi \otimes \psi$, $\varphi \in \mathcal{H}$. In particular, for any state $\rho$ we have $\mathrm{tr}[\rho \mathsf{E}(X)] = \mathrm{tr}[\rho \otimes |\psi\rangle\langle\psi| \mathsf{P}^A(X)]$.

Let $\mathcal{K} = L^2(\mathbb{R})$ be the Hilbert space of the probe. For each $\lambda > 0$ define the state transformation $\Phi^\lambda : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{T}(\mathcal{H} \otimes \mathcal{K})$ via

$$\Phi^\lambda(\rho \otimes \sigma) = \mathrm{tr}_{\mathcal{H}_0}[e^{-i\lambda A \otimes P} \rho \otimes |\psi\rangle\langle\psi| \otimes \sigma \, e^{i\lambda A \otimes P}],$$

where $P$ is the momentum operator in $\mathcal{K}$. Since $P$ generates spatial translations, it is natural to choose as the pointer observable the position of the probe, represented by the spectral measure $\mathsf{P}^Q : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{K})$ of the position operator $Q$. Due to the coupling constant $\lambda$ it is now convenient to choose a pointer function $f^\lambda(x) = \lambda^{-1}x$. The 5-tuple $\mathcal{M}^\lambda = \langle \mathcal{K}, \sigma, \Phi^\lambda, \mathsf{P}^Q, f^\lambda \rangle$ constitutes a measurement scheme with the intention to measure the system observable $\mathsf{E}$. We call this a *generalized standard model for* $\mathsf{E}$.

The instrument as well as the observable actually measured are now easily computed from Eqs. (1) and (2) with the adoption of the pointer function $f^\lambda$. Since we consider their $\lambda$-dependence we explicitly parametrize the instrument as well as the observable by that only. For notational simplicity, we assume that the initial probe state $\sigma$ is a pure state given by a unit vector $\phi$. One obtains the associated instrument and its dual

$$\mathcal{I}^\lambda(X)(\rho) = \int_X \mathrm{tr}_{\mathcal{H}_0}[K_x V_\psi \rho V_\psi^* K_x^*] \, dx, \quad \rho \in \mathcal{T}(\mathcal{H}),$$
$$\mathcal{I}^\lambda(X)^*(B) = \int_X V_\psi^* K_x^* (B \otimes I) K_x V_\psi \, dx, \quad B \in \mathcal{L}(\mathcal{H}),$$

where

$$K_x = \sqrt{\lambda}\phi(-\lambda(A - x)) = \int \sqrt{\lambda}\phi[-\lambda(y - x)] \, d\mathsf{P}^A(y),$$

for all $x \in \mathbb{R}$. Note that the form of the dual instrument may be considered as an analog of the Stinespring-Kraus representation of a completely positive map [17]. The observable actually measured by this scheme is a smeared version $\mu^\lambda * \mathsf{E}$ of the desired one, $\mathsf{E}$, where the convolution is defined as

$$\mu^\lambda * \mathsf{E}(X) = \int \mu^\lambda(X - x) \, d\mathsf{E}(x), \quad X \in \mathcal{B}(\mathbb{R}),$$

and the convolving probability measure is defined through $\mu^\lambda(X) = \langle \phi | \mathsf{P}^Q(\lambda X) \phi \rangle$. In particular, the convolution may be expressed more explicitly as

$$\mu^\lambda * \mathsf{E}(X) = \int_{\mathbb{R}} \int_X |\phi[\lambda(y - x)]|^2 \, dy \, d\mathsf{E}(x).$$

Note that for a state $\rho$ the corresponding probability measure is the usual convolution $\mu^\lambda * \mathsf{p}_\rho^{\mathsf{E}}$ of probability measures. In particular, if the associated measures have densities $g$ and $h$, respectively, the density of the convolution is $g * h$.

We wish to emphasize that although the generalized standard measurement model as well as the associated instrument depend on the used Naimark dilation of $\mathsf{E}$ (aka ancilla) the actually measured observable $\mu^\lambda * \mathsf{E}$ is independent of the used dilation. In particular, this result remains valid if $\mathsf{E}$ is a spectral measure (in which case no ancilla is needed).

The (generalized) standard model for an observable $\mathsf{E}$ constitutes its *approximate measurement* in the sense that the

actually measured observable $\mu^\lambda * \mathsf{E}$ is a convolution of $\mathsf{E}$ with the probability measure $\mu^\lambda$. The approximation depends on $\mu^\lambda$ and the degree of approximation can be quantified in many different ways, for instance, by the standard deviation of $\mu^\lambda$.

The set of possible values (i.e., measurement outcomes) of the observable $\mu^\lambda * \mathsf{E}$, namely the support of the POM, $\mathrm{supp}(\mu^\lambda * \mathsf{E})$, is typically much bigger than the set of possible values of $\mathsf{E}$, $\mathrm{supp}(\mathsf{E})$. For instance, if $\mathsf{E}$ is a two-valued spectral measure, $\mathrm{supp}(\mu^\lambda * \mathsf{E})$ can be anything from this two-point set till the whole of $\mathbb{R}$, depending on the support of $\mu^\lambda$. We call the numbers in $\mathrm{supp}(\mu^\lambda * \mathsf{E})$ the *approximate values* of $\mathsf{E}$ obtained from the (generalized) standard model. We also note that if one chooses the "smearing" measure $\mu^\lambda$ such that its expectation (or average) value is zero then the actually measured observable $\mu^\lambda * \mathsf{E}$ and the observable $\mathsf{E}$ are statistically indistinguishable at the level of first moments (expectation values). However, the two observables (POMs) coincide if and only if $\mu^\lambda$ is a point measure concentrated at the origin, a condition which is clearly impossible for our specific measurement scheme $\mathcal{M}^\lambda$.

## IV. GENERALIZED WEAK VALUES

The sequential measurement scheme can be used as a motivation for defining what Aharonov *et al*. called the weak value of an observable [1]. The intuitive idea behind the weak value is that by letting the strength of the interaction between the object and probe in the first measurement become sufficiently weak, the disturbance caused by the first measurement on the subsequent one becomes negligible. The price to be paid is that the first measurement becomes very poor. In other words, the observable becomes more and more smeared. However, by a clever choice of the probe state it is possible to control the measurement so that the average value of the first measurement remains the same.

The original line of reasoning can be generalized to cover arbitrary pairs of observables. In order to make it rigorous, we need some technical details concerning the operator integral [18]. Let $\mathsf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be an observable. For all $\psi,\varphi \in \mathcal{H}$ denote by $\mathsf{E}_{\psi,\varphi}$ the complex measure $X \mapsto \langle\psi|\mathsf{E}(X)\varphi\rangle$. Let $\mathcal{D}(x,\mathsf{E}) \subset \mathcal{H}$ denote the subspace of those $\varphi$ for which the identity map $x \mapsto x$ is $\mathsf{E}_{\psi,\varphi}$-integrable for all $\psi \in \mathcal{H}$. The first moment operator of $\mathsf{E}$ is then the linear operator $\mathsf{E}[1] : \mathcal{D}(x,\mathsf{E}) \to \mathcal{H}$ defined as

$$\langle\psi|\mathsf{E}[1]\varphi\rangle = \int x\, d\mathsf{E}_{\psi,\varphi}(x), \quad \varphi \in \mathcal{D}(x,\mathsf{E}), \psi \in \mathcal{H}.$$

With these considerations, we can now define the weak value of an observable.

*Definition 1*. *Let* $\mathsf{E},\mathsf{F} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ *be observables,* $\varphi \in \mathcal{D}(x,\mathsf{E})$, $\|\varphi\| = 1$ *and let* $Y \in \mathcal{B}(\mathbb{R})$ *be such that* $\langle\varphi|\mathsf{F}(Y)\varphi\rangle \neq 0$. *The weak value of* $\mathsf{E}$ *in a vector state* $\varphi$ *conditioned by* $\mathsf{F}(Y)$ *is*

$$\mathsf{E}_w[\varphi,\mathsf{F}(Y)] = \frac{\langle\varphi|\mathsf{F}(Y)\mathsf{E}[1]\varphi\rangle}{\langle\varphi|\mathsf{F}(Y)\varphi\rangle}. \tag{3}$$

Note that the domain $\mathcal{D}(x,\mathsf{E})$ of the first moment operator $\mathsf{E}[1]$ contains as a subspace the square-integrability domain $\widetilde{\mathcal{D}}(x,\mathsf{E})$ consisting of those $\varphi \in \mathcal{H}$ for which the function $x \mapsto x^2$ is integrable with respect to the positive measure $\mathsf{E}_{\varphi,\varphi}$. In general, these two domains may be different, that is, the inclusion $\widetilde{\mathcal{D}}(x,\mathsf{E}) \subset \mathcal{D}(x,\mathsf{E})$ may be proper. In a typical case one has either $\widetilde{\mathcal{D}}(x,\mathsf{E}) = \mathcal{D}(x,\mathsf{E})$ or $\widetilde{\mathcal{D}}(x,\mathsf{E}) = \{0\}$. For instance, if $\mathsf{E} = \mu * \mathsf{P}^A$ for some spectral measure $\mathsf{P}^A$ and probability measure $\mu$ such that the first moment $\mu[1] = \int x\, d\mu(x)$ exists, then $\mathcal{D}(x,\mathsf{E}) = \mathcal{D}(A)$ and $\mathsf{E}[1] = A + \mu[1]I$. Furthermore, $\widetilde{\mathcal{D}}(x,\mathsf{E})$ equals either $\mathcal{D}(A) = \mathcal{D}(x,\mathsf{E})$ or $\{0\}$ depending on whether or not the second moment $\mu[2] = \int x^2\, d\mu(x)$ is finite [19]. As for the weak value, if $\mu[1] = 0$ and $\mathsf{F}(Y)$ is a one-dimensional projection, that is, $\mathsf{F}(Y) = |\eta\rangle\langle\eta|$ for some unit vector $\eta \in \mathcal{H}$, we obtain the familiar expression

$$\mathsf{E}_w[\varphi,\mathsf{F}(Y)] = \frac{\langle\varphi|\mathsf{F}(Y)\mathsf{E}[1]\varphi\rangle}{\langle\varphi|\mathsf{F}(Y)\varphi\rangle} = \frac{\langle\eta|A\varphi\rangle}{\langle\eta|\varphi\rangle}.$$

Formally, the generalization of Definition 1 to mixed states is straightforward. Indeed, for a state $\rho$ such that $\mathrm{tr}[\mathsf{F}(Y)\rho] \neq 0$ we can define the weak value of $\mathsf{E}$ conditioned by $\mathsf{F}(Y)$ as

$$\mathsf{E}_w[\rho,\mathsf{F}(Y)] = \frac{\mathrm{tr}[\mathsf{F}(Y)\mathsf{E}[1]\rho]}{\mathrm{tr}[\mathsf{F}(Y)\rho]}, \tag{4}$$

provided that $\mathsf{F}(Y)\mathsf{E}[1]\rho$ is a trace class operator. However, since typically $\mathsf{E}[1]$ is an unbounded operator, this requirement is in general highly nontrivial. To avoid these technical difficulties, we restrict our attention to pure states keeping in mind that on a formal level the calculations can be carried out using, e.g., the spectral decomposition $\rho = \sum_{j=0}^{\infty} w_j|\varphi_j\rangle\langle\varphi_j|$ for the state.

## V. WEAK VALUES FROM SEQUENTIAL MEASUREMENTS

We will now proceed by showing that the general definition of a weak value is operationally meaningful. As the first step, we will show that the real and imaginary parts can be obtained as conditional averages in two different sequential measurement schemes in the limit of zero interaction strength. This generalizes the known results into this more general situation. In the second part, we will show how a single scheme using a phase space observable for the pointer can be used to obtain the whole weak value. We prove our results in the case that the initial state $\varphi$ belongs to the square-integrability domain $\widetilde{\mathcal{D}}(x,\mathsf{E})$. Whether or not similar results hold for an arbitrary $\varphi \in \mathcal{D}(x,\mathsf{E})$ remains an open question.

### A. Real and imaginary parts from separate measurements

Consider the generalized standard measurement scheme $\mathcal{M}^\lambda$ and suppose that after this measurement, realizing the observable $\mu^\lambda * \mathsf{E}$, we perform (an exact) measurement of $\mathsf{F}$ thus obtaining the (sequential joint) observable $\mathsf{M}^\lambda : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$. If we then postselect only the values $(x,y)$ with $y \in Y$ [for a fixed $Y$ for which $\langle\varphi|\mathsf{M}^\lambda_2(Y)\varphi\rangle \neq 0$] and normalize the probabilities, we end up with a conditional probability measure

$$X \mapsto \frac{\langle\varphi|\mathcal{I}^\lambda(X)^*[\mathsf{F}(Y)]\varphi\rangle}{\langle\varphi|\mathcal{I}^\lambda(\mathbb{R})^*[\mathsf{F}(Y)]\varphi\rangle} = \frac{\langle\varphi|\mathsf{M}^\lambda(X \times Y)\varphi\rangle}{\langle\varphi|\mathsf{M}^\lambda_2(Y)\varphi\rangle}.$$

We now claim that the real part of the weak value is obtained by taking the average of the above measure and then taking the limit of zero interaction strength, that is,

$$\mathrm{Re}\,\{\mathsf{E}_w[\varphi,\mathsf{F}(Y)]\} = \lim_{\lambda\to 0}\int x\,\frac{\langle\varphi|\mathcal{I}^\lambda(dx)^*[\mathsf{F}(Y)]\varphi\rangle}{\langle\varphi|\mathcal{I}^\lambda(\mathbb{R})^*[\mathsf{F}(Y)]\varphi\rangle}. \quad (5)$$

First notice that $\lim_{\lambda\to 0}\langle\varphi|\mathcal{I}^\lambda(\mathbb{R})^*[\mathsf{F}(Y)]\varphi\rangle = \langle\varphi|\mathsf{F}(Y)\varphi\rangle$, so it is sufficient to consider only the numerator for which we define $\Lambda^\lambda(X) = \langle\varphi|\mathcal{I}^\lambda(X)^*[\mathsf{F}(Y)]\varphi\rangle$. Now by assuming that $\phi\in\mathcal{D}(Q)$ and $\varphi\in\widetilde{\mathcal{D}}(x,\mathsf{E})$ [the latter being equivalent to $\varphi\otimes\psi\in\mathcal{D}(A)=\mathcal{D}(x,\mathsf{P}^A)$], and by using the translation covariance of position we can show that $e^{-i\lambda A\otimes P}\varphi\otimes\psi\otimes\phi\in\mathcal{D}(x,I\otimes I\otimes\mathsf{P}^Q)$ for all $\lambda\geqslant 0$. It follows that the required average value is given by

$$\Lambda^\lambda[1] = \frac{1}{\lambda}\big\langle e^{-i\lambda A\otimes P}\varphi\otimes\psi\otimes\phi\big|\mathsf{F}(Y)\otimes I\otimes Q e^{-i\lambda A\otimes P} \times\varphi\otimes\psi\otimes\phi\big\rangle.$$

If we then make the assumptions that $\langle\phi|Q\phi\rangle = 0$ and $\phi\in\mathcal{D}(QP)\cap\mathcal{D}(PQ)$, we can calculate the limit $\lambda\to 0$. Using again the translation covariance of position, we can actually calculate the limit in a rigorous manner and obtain

$$\lim_{\lambda\to 0}\Lambda^\lambda[1] = i\,\langle\mathsf{E}[1]\varphi|\mathsf{F}(Y)\varphi\rangle\langle\phi|PQ\phi\rangle - i\,\langle\mathsf{F}(Y)\varphi|\mathsf{E}[1]\varphi\rangle\langle\phi|QP\phi\rangle. \quad (6)$$

Here we have used the fact that the operator identity $V_\psi^* A V_\psi = \mathsf{E}[1]$ holds on the square-integrability domain $\widetilde{\mathcal{D}}(x,\mathsf{E})$ [20].

We immediately see that a sufficient condition for the above limit to be proportional to the real part of the weak value is that $\langle\phi|\{Q,P\}\phi\rangle = 0$, where $\{\cdot,\cdot\}$ denotes the anticommutator. A similar condition was also found in Ref. [21], where it was expressed as vanishing current density. However, we wish to obtain the strict equality and therefore we pose the more restrictive condition $\langle\phi|QP\phi\rangle = \frac{i}{2}$. Such a condition is satisfied for instance by a Gaussian $\phi(x) = \frac{1}{\sqrt{\Delta\sqrt{2\pi}}}e^{-\frac{x^2}{4\Delta^2}}$. Under these assumptions Eq. (5) is clearly valid, which shows that at least the real part of the (generalized) weak value is accessible via measurements. We summarize these considerations in the form of a proposition.

*Proposition 1*. Let $\mathcal{I}^\lambda$ be the instrument defined by the measurement scheme $\mathcal{M}^\lambda = \langle\mathcal{K},\phi,\Phi^\lambda,\mathsf{P}^Q,f^\lambda\rangle$ where $\phi\in\mathcal{D}(QP)\cap\mathcal{D}(PQ)$ is such that $\langle\phi|Q\phi\rangle = 0$ and $\langle\phi|QP\phi\rangle = \frac{i}{2}$. Then

$$\lim_{\lambda\to 0}\int x\,\frac{\langle\varphi|\mathcal{I}^\lambda(dx)^*[\mathsf{F}(Y)]\varphi\rangle}{\langle\varphi|\mathcal{I}^\lambda(\mathbb{R})^*[\mathsf{F}(Y)]\varphi\rangle} = \mathrm{Re}\,\frac{\langle\varphi|\mathsf{F}(Y)\mathsf{E}[1]\varphi\rangle}{\langle\varphi|\mathsf{F}(Y)\varphi\rangle} \quad (7)$$

for all $\varphi\in\widetilde{\mathcal{D}}(x,\mathsf{E})$.

In order to obtain the imaginary part, consider the measurement scheme $\mathcal{N}^\lambda = \langle\mathcal{K},\phi,\Phi^\lambda,\mathsf{P}^P,f^\lambda\rangle$, which differs from $\mathcal{M}^\lambda$ by the pointer observable: instead of monitoring shifts in the probe's position we now observe the boosts the probe obtains. Note that at this point the initial probe state is also arbitrary. This change has a significant effect on the scheme. In particular, the measured observable becomes a trivial one, $X\mapsto\langle\phi|\mathsf{P}^P(\lambda X)\phi\rangle I$, meaning that no information is gained about the object system. However, this does not mean that the measurement does not affect the state of the system. Indeed, the instrument $\mathcal{J}^\lambda$ associated with this scheme is nontrivial, as can be seen from the dual form

$$\mathcal{J}^\lambda(X)^*(B) = \int_X|\sqrt{\lambda}\hat{\phi}(\lambda x)|^2\,V_\psi^* L_x^*(B\otimes I)L_x V_\psi\,dx,$$

where $L_x = e^{-i\lambda^2 xA}$ and $B\in\mathcal{L}(\mathcal{H})$.

Now suppose that we again perform a sequential measurement where the second observable is $\mathsf{F}$. Then the calculations for the conditional average value and the limit of zero interaction are performed as before. In particular, by denoting $\Gamma^\lambda(X) = \langle\varphi|\mathcal{J}^\lambda(X)^*[\mathsf{F}(Y)]\varphi\rangle$ and by assuming that $\phi\in\mathcal{D}(P)$ with $\langle\phi|P\phi\rangle = 0$ we find that

$$\lim_{\lambda\to 0}\Gamma^\lambda[1] = 2\,\langle\phi|P^2\phi\rangle\mathrm{Im}\,\langle\mathsf{F}(Y)\varphi|\mathsf{E}[1]\varphi\rangle,$$

so that by assuming that $\langle\phi|P^2\phi\rangle = \frac{1}{2}$ we get the desired result. For the Gaussians we have $\langle\phi|P^2\phi\rangle = \frac{1}{4\Delta^2}$, so that the above condition is satisfied with the choice $\Delta^2 = \frac{1}{2}$ for the variance. Once again we summarize this as a proposition.

*Proposition 2*. Let $\mathcal{J}^\lambda$ be the instrument defined by the measurement scheme $\mathcal{N}^\lambda = \langle\mathcal{K},\phi,\Phi^\lambda,\mathsf{P}^P,f^\lambda\rangle$ where $\phi\in\mathcal{D}(P^2)$ is such that $\langle\phi|P\phi\rangle = 0$ and $\langle\phi|P^2\phi\rangle = \frac{1}{2}$. Then

$$\lim_{\lambda\to 0}\int x\,\frac{\langle\varphi|\mathcal{J}^\lambda(dx)^*[\mathsf{F}(Y)]\varphi\rangle}{\langle\varphi|\mathcal{J}^\lambda(\mathbb{R})^*[\mathsf{F}(Y)]\varphi\rangle} = \mathrm{Im}\,\frac{\langle\varphi|\mathsf{F}(Y)\mathsf{E}[1]\varphi\rangle}{\langle\varphi|\mathsf{F}(Y)\varphi\rangle} \quad (8)$$

for all $\varphi\in\widetilde{\mathcal{D}}(x,\mathsf{E})$.

### B. Weak value from a single measurement

At the heart of quantum mechanics lies the fact that not all pairs of observables admit joint measurements. In particular, since position and momentum do not have a joint observable, the above schemes cannot be simply combined by choosing a pointer observable $\mathsf{M}:\mathcal{B}(\mathbb{R}^2)\to\mathcal{L}(\mathcal{H})$, which would give $\mathsf{P}^Q$ and $\mathsf{P}^P$ as its Cartesian margins. However, this problem can be overcome by replacing the sharp position and momentum observables with a pair of smeared ones $(\mu*\mathsf{P}^Q,\nu*\mathsf{P}^P)$. In this case, the observables have a joint observable exactly when there exists a positive trace one operator $T$ such that $\mu(X) = \mathrm{tr}[T\mathsf{P}^Q(-X)]$ and $\nu(Y) = \mathrm{tr}[T\mathsf{P}^P(-Y)]$ [22], in which case we denote $\mu = \mu^T$ and $\nu = \nu^T$. One joint observable is then always given by the *covariant phase space observable* $\mathsf{G}^T:\mathcal{B}(\mathbb{R}^2)\to\mathcal{L}(\mathcal{H})$ generated by $T$, that is,

$$\mathsf{G}^T(Z) = \frac{1}{2\pi}\int_Z W_{qp}TW_{qp}^*\,dqdp,\quad Z\in\mathcal{B}(\mathbb{R}^2), \quad (9)$$

where $W_{qp} = e^{i\frac{qp}{2}}e^{-iqP}e^{ipQ}$ are the Weyl operators. The structure and properties of covariant phase space observables have been studied extensively and are well understood [23,24] (see also Refs. [25,26]). For practical purposes it should be noted that any $\mathsf{G}^T$ can in principle be measured by using a quantum optical eight-port homodyne detector, provided that one can prepare a single mode electromagnetic field in a state $T$ (for a detailed analysis, see Ref. [27]).

The simplest example of a covariant phase space observable is $\mathsf{G}^{|0\rangle}$, the observable generated by the ground state of the harmonic oscillator, or the vacuum state, $|0\rangle$. The corresponding probability density is then the Husimi $Q$ function of the state and is widely used in quantum optics. For the present purpose, however, the important feature of

this observable is that the margins agree with position and momentum at the level of first moments, i.e., $\mu^{|0\rangle} * \mathsf{P}^Q[1] = Q$ and $\nu^{|0\rangle} * \mathsf{P}^P[1] = P$. Since the determination of the weak value requires only averaging, it is possible to use this single observable as a pointer for the scheme. One only needs to make sure that all of the conditions of Propositions 1 and 2 for the probe state are satisfied. This can be accomplished by choosing the state to be the vacuum which in the coordinate representation is given by $\phi_0(x) = (\frac{1}{\pi})^{1/4} e^{-\frac{x^2}{2}}$. We also use the same notation $f^\lambda$ for the two-dimensional pointer function, $f^\lambda(x,y) = (\lambda^{-1}x, \lambda^{-1}y)$. This then results in the measurement scheme $\mathcal{M}^{|0\rangle} = \langle \mathcal{K}, |0\rangle, \Phi^\lambda, \mathsf{G}^{|0\rangle}, f^\lambda \rangle$ whose associated instrument we denote by $\mathcal{I}^{|0\rangle}$. It follows that the marginal instruments $\mathcal{I}_1^{|0\rangle}(X) = \mathcal{I}^{|0\rangle}(X \times \mathbb{R})$ and $\mathcal{I}_2^{|0\rangle}(Y) = \mathcal{I}^{|0\rangle}(\mathbb{R} \times Y)$ can be expressed in the dual form as

$$\mathcal{I}_1^{|0\rangle}(X)^*(B) = \int \mu^{|0\rangle}[\lambda(X - x)]\, \mathcal{I}^\lambda(dx)^*(B)$$

$$\mathcal{I}_2^{|0\rangle}(Y)^*(B) = \int \nu^{|0\rangle}[\lambda(Y - y)] \mathcal{J}^\lambda(dy)^*(B),$$

for all $B \in \mathcal{L}(\mathcal{H})$, where it is understood that

$$\mathcal{I}^\lambda(dx)^*(B) = V_\psi^* K_x^*(B \otimes I) K_x V_\psi \, dx,$$

$$\mathcal{J}^\lambda(dy)^*(B) = |\sqrt{\lambda}\phi_0(\lambda y)|^2 \, V_\psi^* L_y^*(B \otimes I) L_y V_\psi \, dy,$$

where now $K_x = \sqrt{\lambda}\phi_0\{-[\lambda(A - x)]\}$ and we have used the fact that $\hat{\phi}_0 = \phi_0$. The calculations for the averages and the limits are now identical to the previous one. Therefore, we can again summarize these considerations in the form of a proposition.

*Proposition 3* Let $\mathcal{I}^{|0\rangle}$ be the instrument defined by the measurement scheme $\mathcal{M}^{|0\rangle} = \langle \mathcal{K}, |0\rangle, \Phi^\lambda, \mathsf{G}^{|0\rangle}, f^\lambda \rangle$ and let $\mathcal{I}_1^{|0\rangle}$ and $\mathcal{I}_2^{|0\rangle}$ be its marginal instruments. Then

$$\lim_{\lambda \to 0} \int x \, \frac{\langle \varphi | \mathcal{I}_1^{|0\rangle}(dx)^*[\mathsf{F}(Y)]\varphi \rangle}{\langle \varphi | \mathcal{I}_1^{|0\rangle}(\mathbb{R})^*[\mathsf{F}(Y)]\varphi \rangle} = \mathrm{Re}\, \frac{\langle \varphi | \mathsf{F}(Y)\mathsf{E}[1]\varphi \rangle}{\langle \varphi | \mathsf{F}(Y)\varphi \rangle} \tag{10}$$

and

$$\lim_{\lambda \to 0} \int y \, \frac{\langle \varphi | \mathcal{I}_2^{|0\rangle}(dy)^*[\mathsf{F}(Y)]\varphi \rangle}{\langle \varphi | \mathcal{I}_2^{|0\rangle}(\mathbb{R})^*[\mathsf{F}(Y)]\varphi \rangle} = \mathrm{Im}\, \frac{\langle \varphi | \mathsf{F}(Y)\mathsf{E}[1]\varphi \rangle}{\langle \varphi | \mathsf{F}(Y)\varphi \rangle} \tag{11}$$

for all $\varphi \in \widetilde{\mathcal{D}}(x, \mathsf{E})$.

## VI. STATE RECONSTRUCTION METHODS

### A. Reconstructing the wavefunction via weak measurements

In Ref. [3], Lundeen *et al.* reported an experiment where they claim to have measured the pointwise values of the wavefunction of a photon using weak measurements. Though the localization of photons is at best problematic, their state determination method deserves attention. To avoid the question of photon localization, we consider a spin-$\frac{1}{2}$ particle. In the original experiment the part of the inner degree of freedom was played by the polarization of the photon.

Consider the position of the particle in, say, the $z$ direction, so that the spacial part of the Hilbert space can be taken to be $\mathcal{H} = L^2(\mathbb{R})$. Now as a "probe" we take the spin-degree of freedom, $\mathcal{K} = \mathbb{C}^2$, with the initial state $|+\rangle$. In order to measure the pointwise value of the wavefunction, we divide the position space into disjoint intervals $(I_i)_{i \in \mathbb{N}}$ with the assumption that the intervals are of equal length and the center is labeled by $x_i$. For each interval we will then perform a weak measurement of the two-valued observable $1 \mapsto Q_i = \mathsf{P}^Q(I_i), 0 \mapsto I - Q_i = \mathsf{P}^Q(\mathbb{R} \setminus I_i)$, thus scanning the whole position space. In order to accomplish the weak measurement, the position and spin of the particle are coupled via the unitary transformation

$$\Phi^\alpha(\rho \otimes \sigma) = e^{-i\alpha Q_i \otimes \sigma_y} \rho \otimes \sigma e^{i\alpha Q_i \otimes \sigma_y}.$$

In particular, an initial vector state $\varphi \otimes |+\rangle$ evolves into a superposition

$$\Psi^\alpha = Q_i\varphi \otimes (\cos\alpha|+\rangle + \sin\alpha|-\rangle) + (I - Q_i)\varphi \otimes |+\rangle. \tag{12}$$

As the pointer observable we choose either $\sigma_x$ or $\sigma_y$ or, more precisely, their two-valued spectral measures $\mathsf{P}^{\sigma_j}$, $j = x, y$. We have thus arrived at the measurement schemes $\mathcal{M}_j^i = \langle \mathcal{K}, |+\rangle, \Phi^\alpha, \mathsf{P}^{\sigma_j} \rangle$, $i \in \mathbb{N}$, $j = x, y$.

After the first measurement we measure the momentum $\mathsf{P}^P$ of the system and postselect the values that lie in the small interval $J_\epsilon = (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. The conditional probabilities are then given by

$$\{\pm 1\} \mapsto \frac{\langle \Psi^\alpha | \mathsf{P}^P(J_\epsilon) \otimes \mathsf{P}^{\sigma_j}(\{\pm 1\}) \Psi^\alpha \rangle}{\langle \Psi^\alpha | \mathsf{P}^P(J_\epsilon) \otimes I \Psi^\alpha \rangle}. \tag{13}$$

Now if we calculate the conditional average, and perform additional scaling by the factor $2 \sin\alpha$ [which corresponds to using a pointer function $f(x) = \frac{x}{2\sin\alpha}$], then in the weak limit $\alpha \to 0$ we arrive at the two values

$$\xi_i = \mathrm{Re}\langle \varphi | \mathsf{P}^P(J_\epsilon) Q_i \varphi \rangle, \tag{14}$$

$$\eta_i = \mathrm{Im}\langle \varphi | \mathsf{P}^P(J_\epsilon) Q_i \varphi \rangle, \tag{15}$$

where $\xi_i$ and $\eta_i$ refer to the measurements of $\sigma_x$ and $\sigma_y$, respectively, with the fixed position interval $I_i$. The claim of Ref. [3] is that Eqs. (14) and (15) give the real and imaginary parts of the wavefunction at the point $x_i$, that is,

$$\varphi(x_i) \simeq \mathrm{constant} \langle \varphi | \mathsf{P}^P(J_\epsilon) Q_i \varphi \rangle. \tag{16}$$

This is the more precise meaning of Eqs. (2) and (3) in Ref. [3]. It is now straightforward to show that for a sufficiently regular $\varphi$, such as a compactly supported $C^\infty$ function with $\hat{\varphi}(0) \neq 0$, Eq. (16) is true when the lengths of the intervals approach zero.

The proposed method has obvious limitations as a method of state determination. Even after scanning through all the disjoint intervals $I_i$, the method can succeed only for those state vectors $\varphi$ for which $\mathsf{P}^P(J_\epsilon)\varphi \neq 0$. Indeed, if the momentum of the system is localized outside the vicinity of the origin, then $\mathsf{P}^P(J_\epsilon)\varphi = 0$ and no information can be obtained from the measurement. As an example, consider the superposition $\hat{\varphi} = \frac{1}{\sqrt{2}}[\chi_{(-3/2,-1/2)} + \chi_{(1/2,3/2)}]$, where $\chi_I$ denotes the characteristic function of the interval $I$. That is, $\hat{\varphi}$ is a kind of two-slit state in momentum space. The corresponding position representation is then $\varphi(x) = \frac{2}{\sqrt{\pi}x} \cos(x) \sin(x/2)$. For this state one clearly has $\mathsf{P}^P(J_\epsilon)\varphi = 0$ provided that $\epsilon < \frac{1}{2}$ and the state could not be determined. Thus, this method is far from being generally valid. Moreover, if it happens for instance that $\mathsf{P}^P(J_\epsilon)\varphi = \varphi$, then it is known from the basic Fourier theory that all the component vectors $Q_i\varphi$, $i \in \mathbb{N}$, are nonzero and

one has to scan through all the intervals $I_i$ in order to determine the state. Now a simple example is given by $\hat{\varphi} = \frac{1}{\sqrt{\epsilon}}\chi_{J_\epsilon}$, that is, a state which is well localized around the origin in momentum space. The position representation gives $\varphi(x) = \sqrt{\frac{2}{\epsilon\pi}}\frac{\sin(\frac{\epsilon}{2}p)}{p}$. If we choose, for instance, $\epsilon = \frac{1}{4}$, we obtain

$$\langle\varphi|\mathsf{P}^Q((-10,10))\varphi\rangle = \int_{-10}^{10} |\varphi(x)|^2\,dx \simeq 0.67.$$

This shows that even after scanning the position space interval $(-10,10)$, only "approximately $\frac{2}{3}$ of the state" is reconstructed.

The above discussion shows that the state reconstruction method via weak measurements clearly has its limitations when it comes to determining a completely unknown state. In general, some *a priori* knowledge concerning the state is needed in order to be reassured of the applicability of the method. However, it should be pointed out that if one is indeed interested in determining the actual pointwise values of the wavefunction, then an advantage of this method is that the data does not need to be processed; the value appears directly as the conditional average.

### B. Informationally complete sequential measurements

As an alternative and completely general method of state reconstruction, we present the more common approximate sequential measurement of position and momentum where the weak limit is not taken. Consider again the standard measurement scheme $\mathcal{M}^\lambda$ with the exception that we take the position $\mathsf{P}^Q$ to be the observable we wish to measure so that no dilation is needed and the transformation $\Phi^\lambda$ becomes the usual unitary transformation. The observable actually measured is thus a smeared position $\mu^\lambda * \mathsf{P}^Q$.

Suppose that after the measurement $\mathcal{M}^\lambda$, a momentum measurement is performed. This results in a sequential joint observable $\mathsf{G} : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$, which is in fact a covariant phase space observable whose generating operator is determined by the state of the probe system, $T = |\phi^\lambda\rangle\langle\phi^\lambda|$, where $\phi^\lambda(x) = \sqrt{\lambda}\,\overline{\phi(-\lambda x)}$. Moreover, the measurement of an arbitrary $\mathsf{G}^T$ can be realized as such a sequential measurement by choosing the probe state appropriately.

From the point of view of state reconstruction, the covariant phase space observables are extremely useful since a large class of them are known to be *informationally complete* [28], meaning that the initial state of the system is uniquely determined by the measurement outcome statistics of this observable. Indeed, if the generating operator satisfies $\mathrm{tr}[T W_{qp}] \neq 0$ for almost all $(q,p) \in \mathbb{R}^2$, the observable $\mathsf{G}^T$ is informationally complete [29]. In the sequential scheme, we immediately see that by choosing the probe state to be, for instance, a Gaussian, it is possible to determine the state of the object system with a *single* measurement scheme. We once again point out that a well-known quantum optical implementation of this measurement scheme is provided by an eight-port homodyne detector with a strong local oscillator.

In comparison to the reconstruction method via weak measurement, the phase space method has its advantages. The most important aspect is that the method is completely general. Indeed, an arbitrary state is uniquely determined by the statistics provided that the requirement of informational completeness is satisfied. The measured distribution itself is then, of course, a representation of the state and, at least in some cases, it can be used directly to calculate some other quantities which are of interest. However, one is typically interested in some other representation of the state, such as its matrix elements with respect to a given basis. In such a case, one often needs heavy processing of the data and this likely leads to errors. A common concern in these reconstruction algorithms is that there is no way of making sure that the reconstructed state, that is, its matrix representation, is positive.

## VII. CONCLUSIONS

We have generalized concepts and obtained new results concerning the weak value of a quantum observable. We have constructed a measurement scheme that generalizes the standard model to the case of an arbitrary real observable. We have then defined the weak value of an observable in this general context and have shown that the weak value can be obtained from sequential measurements. In particular, we have shown that a single measurement scheme with a phase space observable as the pointer can be used to obtain the weak value. We have also considered the determination of an unknown quantum state using both weak and approximate measurements. We have analyzed the state reconstruction method of Ref. [3] and discussed some of its shortcomings. As an alternative we have presented the well-known method of informationally complete phase space measurements.

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