Revivals of zitterbewegung of a bound localized Dirac particle

Elvira Romera

Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, Fuentenueva s/n, 18071 Granada, Spain and Departamento de Física Atómica, Molecular y Nuclear, Universidad de Granada, Fuentenueva s/n, 18071 Granada, Spain

(Received 3 March 2011; published 8 November 2011)

In this paper a bound localized Dirac particle is shown to exhibit a revival of the zitterbewegung (ZB) oscillation amplitude. These revivals go beyond the known quasiclassical regenerations in which the ZB oscillation amplitude is decreasing from period to period. This phenomenon is studied in a Dirac oscillator and it is shown that it is possible to set up wave packets in which there is a regeneration of the initial ZB amplitude.

DOI: 10.1103/PhysRevA.84.052102 PACS number(s): 03.65.Pm, 05.30.Fk, 03.65.Ge

I. INTRODUCTION

In the context of relativistic quantum mechanics there is a surprising phenomenon that was introduced by Schrödinger in 1930 as zitterbewegung (ZB) [1]. He showed that there is a rapid trembling motion of a Dirac particle around its otherwise rectilinear average trajectory that is due to the interference between negative- and positive-energy eigenvalues. There have been many theoretical studies of ZB, but no direct observation due to the fact that the predicted frequency and amplitude are impossible to measure experimentally at present. Lock showed that ZB has a transient character for a free localized Dirac particle, pointing out that the ZB effect for a localized wave packet in an external field depends on the eigenvalues of the Hamiltonian [2]. Nowadays, there is an intense interest in the ZB of electrons in semiconductors (see the review of Zawadzki and Rusin [3] and references therein). Recently, ZB has been studied in graphene [3–11] where it was related to electric conductivity. In particular, revivals and ZB were studied in the electric current in monolayer graphene in a perpendicular magnetic field [10]. Gerritsma *et al.* [12] simulated experimentally the electron ZB by means of trapped ions and laser excitations by adjusting experimentally some parameters of the Dirac equation.

On the other hand the quantum revival of wave packets is an interference quantum phenomenon related to the relativistic and nonrelativistic temporal evolution of wave packets. Quantum revivals have been investigated theoretically in, for example, atomic, molecular, and nonlinear systems [13–19] and observed experimentally in many different quantum systems, such as Rydberg atoms and molecules, and Bose-Einstein condensates [16,20].

In what follows it is shown that there is a revival of the ZB oscillation amplitude when a bound Dirac electron is considered. A Dirac oscillator has been chosen to analyze this behavior because it is exactly soluble and is a model that has applications in several branches of physics (see Ref. [21] and references therein). In this work it is demonstrated that besides the ZB and quasiclassical oscillations studied previously by other authors [22], there exists a revival or regeneration of the ZB oscillation amplitude.

To describe quantum revivals, let us consider an initial wave packet that is a superposition of eigenstates localized around some energy level E_{n_0}. It is appropriate to expand the energy around n_0 if $|n-n_0|/n_0 \ll 1$,

$$E_n \approx E_{n_0} + E'_{n_0}(n-n_0) + \frac{E''_{n_0}}{2}(n-n_0)^2 + \cdots, \qquad (1)$$

and each term in the series defines an important time scale $T_{\rm cl} = \frac{2\pi\hbar}{|E'(n_0)|}$, $T_R = \frac{2\pi\hbar}{|E''(n_0)|/2}$, where $T_{\rm cl}$ is associated with the classical periodic motion of the wave packet and T_R is the revival time (the validity of this expansion has been demonstrated in Refs. [14,23,24]). The wave packet initially evolves quasiclassically with period $T_{\rm cl}$ and then spreads and collapses; at later times, around T_R, the wave packet regenerates and reaches approximately its initial shape. For times that are rational fractions of T_R, the wave packets split into clones of themselves [16,25]. After the revival time a new cycle starts with quasiclassical behavior, collapses, fractional revivals, and revivals. Revivals are usually analyzed using the autocorrelation function $A(t)$, which is the overlap between the initial and the time-evolving wave packet. An alternative approach in terms of uncertainty entropic relations has been proposed [26].

II. REVIVALS OF ZITTERBEWEGUNG IN A DIRAC OSCILLATOR

An appropriate system to discuss revivals of ZB for bounded states is a 2 + 1 Dirac oscillator due to the fact that it is exactly soluble and allows us to study this phenomenon in a simple system. Thus we shall consider the Hamiltonian for a Dirac oscillator [27] with frequency ω,

$$H = c\alpha\cdot(\mathbf{p} - im\omega\beta\mathbf{r}) + \beta mc^2, \qquad (2)$$

where m is the rest mass of the Dirac particle (for example an electron), α and β are the Dirac matrices, and c is the speed of light. We shall introduce the complex coordinate as in Ref. [21], $z = x + iy$, and using the usual creation and annihilation operator notation in terms of z and $\bar{z}$,

$$a = \frac{1}{\sqrt{m\omega\hbar}}p_{\bar{z}} - \frac{i}{2}\sqrt{\frac{m\omega}{\hbar}}z,$$
$$a^\dagger = \frac{1}{\sqrt{m\omega\hbar}}p_z + \frac{i}{2}\sqrt{\frac{m\omega}{\hbar}}\bar{z},$$

the Hamiltonian reads

$$H = \begin{pmatrix} mc^2 & 2c\sqrt{m\omega\hbar}a^\dagger \\ 2c\sqrt{m\omega\hbar}a & -mc^2 \end{pmatrix}. \qquad (3)$$

It is not difficult to show that the energy eigenfunctions are given by

$$|\phi_n^{\pm}\rangle = \begin{pmatrix} \pm\sqrt{\frac{1}{2}\pm\xi_n}|n\rangle \\ \mp\sqrt{\frac{1}{2}\mp\xi_n}|n-1\rangle \end{pmatrix}, \quad (4)$$

with

$$\xi_n = \frac{1}{2\sqrt{1+\frac{4\hbar\omega n}{mc^2}}} \quad (5)$$

and $n = 0,1,\ldots,$ and the energy spectrum is, in turn,

$$E_n^{\pm} = \pm mc^2\sqrt{1+\frac{4\hbar\omega n}{mc^2}}. \quad (6)$$

A superposition state is constructed that consists of two wave packets as the initial particle wave packet,

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|\Psi_-\rangle + |\Psi_+\rangle), \quad (7)$$

where the above wave packets are defined as the linear combination

$$|\Psi_+\rangle = \sum_n c_n^+|\phi_n^+\rangle, \quad |\Psi_-\rangle = \sum_n c_n^-|\phi_n^-\rangle, \quad (8)$$

each of them centered around a given eigenvalue $E_{n_0}^+$ and $E_{n_0}^-$, respectively, with coefficients distributed in Gaussian form ($c_n^+ = c_n^- = c_n$) as

$$c_n = \sqrt{\frac{1}{\pi\sqrt{\sigma}}}e^{-(n-n_0)^2/2\sigma}. \quad (9)$$

We can write the temporal evolution of the initial wave packet as

$$|\Psi_0(t)\rangle = \frac{1}{\sqrt{2}}\sum_n(c_n^+|\phi_n^+\rangle e^{iE_n^+t/\hbar} + c_n^-|\phi_n^-\rangle e^{iE_n^-t/\hbar}), \quad (10)$$

taking into account that

$$|\Psi_{\pm}(t)\rangle = \sum_n(c_n^{\pm}|\phi_n^{\pm}\rangle e^{iE_n^{\pm}t/\hbar}. \quad (11)$$

The series expansion in Eq. (1) should be interpreted in the context of the temporal evolution of the wave packet. If we replace the value of E_n by the expansion in Eq. (1) in Eq. (11) we can see that each term in the exponential (except the first) defines an important characteristic time scale. The first term is unimportant because it is an overall phase. The following two terms define the classical periodicity and the revival time, respectively [14,23,24].

Therefore, the corresponding classical period and the revival time for $|\Psi\rangle$ yield, straightforwardly,

$$T_{\mathrm{cl}} = \frac{\pi}{\omega}\sqrt{1+\frac{4\hbar\omega}{mc^2}n_0} \quad (12)$$

and

$$T_R = \frac{\pi mc^2}{\hbar\omega^2}\left(1+\frac{4\hbar\omega}{mc^2}n_0\right)^{3/2}. \quad (13)$$

To calculate the period of ZB the temporal evolution of the x and y components of the velocity are determined, which are given by $\langle v_j\rangle = \langle i[H,r_j]/\hbar\rangle$ $(j = x,y)$, where σ_x and σ_y are the Pauli matrices. For the wave packet $|\Psi_0\rangle$, after some algebra and taking into account that $|n\rangle$ is an orthonormal set, the temporal evolution for the velocities is given by

$$\langle v_x\rangle = 2\sum_{n=0}^{\infty} c_n c_{n+1}\{\eta_n\cos[(E_n+E_{n+1})t/\hbar] - \nu_n\cos[(E_n-E_{n+1})t/\hbar]\}, \quad (14)$$

$$\langle v_y\rangle = 0, \quad (15)$$

where $\eta_n = \gamma_n\gamma_{n+1} + \delta_n\delta_{n+1}$ and $\nu_n = \gamma_n\delta_{n+1} + \gamma_n\delta_{n+1}$, with $\gamma_n = \sqrt{\frac{1}{2}+\xi_n}$ and $\delta_n = \sqrt{\frac{1}{2}-\xi_n}$ for $n = 0,1,2,\ldots$. Several types of oscillatory motion emerge for the velocity evolution. The first term in the v_x temporal evolution is weighted by $\cos[(E_n+E_{n+1})t/\hbar]$, which is responsible for the ZB oscillatory motion. The ZB period is estimated using Eq. (1), which enables us to write $E_n + E_{n+1} \approx 2E_{n_0}$ [10] and then

$$T_{\mathrm{ZB}} = \pi\hbar/|E_{n_0}| = \frac{\pi\hbar}{mc^2\sqrt{1+\frac{4\hbar\omega}{mc^2}n}}. \quad (16)$$

The second term in the v_x temporal evolution is weighted by $\cos[(E_n-E_{n+1})t/\hbar]$, which lets us extract different periodicities in the velocity temporal evolution. Using Eq. (1) again, we obtain other oscillatory scales $E_n - E_{n+1} \approx E'_{n_0}(n-n_0) + E''_{n_0}(n-n_0)^2 + \cdots$, which are given by T_{cl} and T_R.

The velocity behavior is clearly illustrated in Fig. 1. The value $\langle v_x\rangle$ is numerically computed as a function of time for the temporal evolution of the initial wave packet $|\Psi_0\rangle$ with $n_0 = 30$ and $\sigma = 3.0$ and for an oscillator frequency $\omega = 10^3$ a.u. (Throughout, the results are generated in atomic units

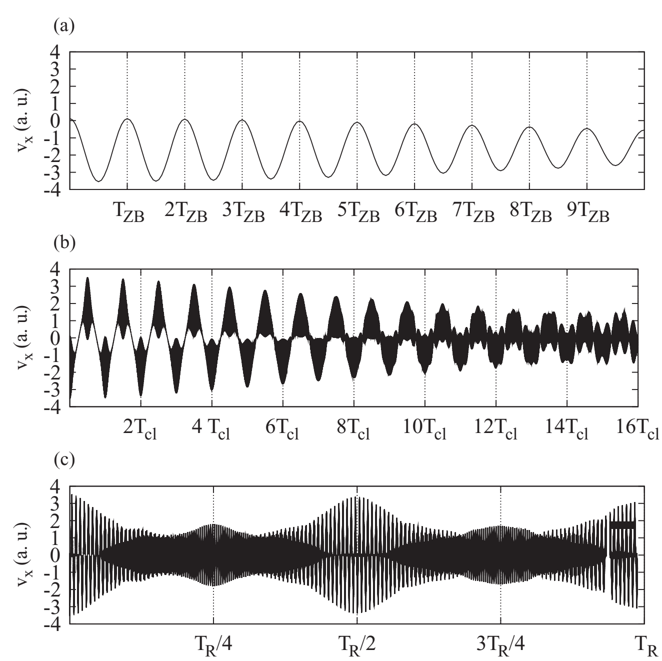

FIG. 1. Time dependence of $\langle v_x\rangle$ for the initial wave packet with $\sigma = 3$, $n_0 = 30$, and oscillator frequency $\omega = 10^3$, for which $T_{\mathrm{ZB}} = 6.15\times10^{-5}$, $T_{\mathrm{cl}} = 8.54\times10^{-3}$, and $T_R = 1.19$ (all in a.u.). The vertical dotted lines stand for (a) T_{ZB} periods, (b) T_{cl} periods, and (c) T_R periods.

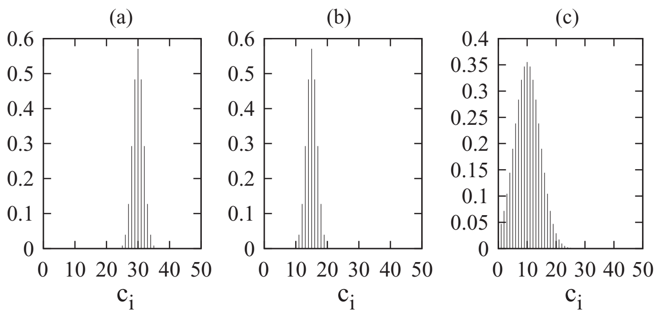

FIG. 2. Coefficients [Eq. (9)] for (a) $n_0 = 30, \sigma = 3$, (b) $n_0 = 15$, $\sigma = 3$, and (c) $n_0 = 10, \sigma = 20$.

$m = \hbar = e = 1$.) The initial wave packet [see Eqs. (7)–(9)] have been constructed, with the level population given in Fig. 2(a). We observe in Fig. 1(a) that there is an oscillatory behavior for the T_{ZB} time scale. For greater time scales we can see in Fig. 1(b) that quasiclassical oscillations appear to be enveloping the ZB oscillations whose amplitude is decreasing from period to period. This behavior was previously observed in Ref. [22]. Finally, in Fig. 1(c) we can clearly see a new time-scale oscillation T_R, which is enveloping the previous oscillations, and it is apparent that for $t = mT_R/2$ (for $m = 1,2,\ldots$) there is a revival of the ZB oscillation amplitude [Fig. 1(c)] and the quasiclassical oscillations.

This phenomenon is illustrated with another example. In Fig. 3 an initial localized wave packet with a different value of the parameter $n_0 = 15$ is considered [that is, with the level population given in Fig. 2(b)]. Then $T_{\rm cl}$ and T_R are smaller as ω is smaller and we observe that T_{ZB} is somewhat lower than in the preceding case, as expected from Eqs. (12), (13), and

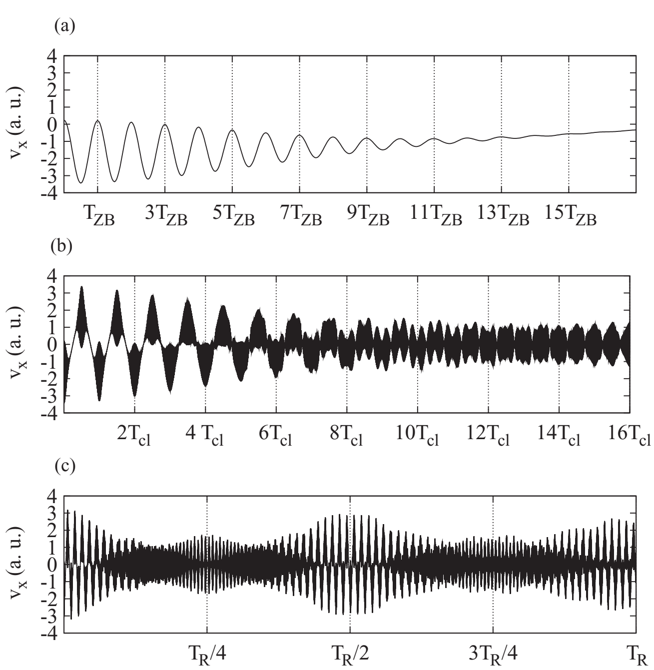

FIG. 3. Time dependence of $\langle v_x \rangle$ for the initial wave packet with $\sigma = 3$, $n_0 = 15$, and oscillator frequency $\omega = 10^3$, for which $T_{ZB} = 8.16 \times 10^{-5}$, $T_{\rm cl} = 6.4 \times 10^{-3}$, and $T_R = 0.5$ (all in a.u.). The vertical dotted lines stand for (a) T_{ZB} periods, (b) $T_{\rm cl}$ periods, and (c) T_R periods.

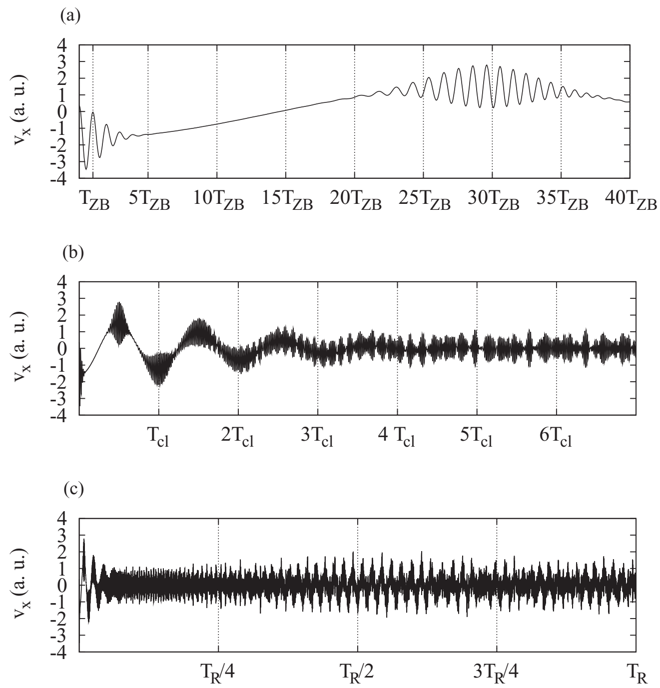

FIG. 4. Time dependence of $\langle v_x \rangle$ for the initial wave packet with $\sigma = 20$, $n_0 = 10$, and oscillator frequency $\omega = 10^3$, for which $T_{ZB} = 9.46 \times 10^{-5}$, $T_{\rm cl} = 5.55 \times 10^{-3}$, and $T_R = 0.33$ (all in a.u.). The vertical dotted lines stand for (a) T_{ZB} periods, (b) $T_{\rm cl}$ periods, and (c) T_R periods.

(16), respectively. Again, a revival of the ZB amplitude can be observed [Fig. 3(c)].

Moreover, it should be stressed that the appearance of revivals of the ZB oscillation amplitude depends on the shape of the initial wave packet, i.e., we have to work with a localized wave packet. If we consider a broader wave packet around lower energies $\pm E_{n_0}$, with $n_0 = 10$ and $\sigma = 20$ (Fig. 4) [see level population in Fig. 2(c)], we observe in Fig. 4(a) that there is an oscillatory behavior similar to Figs. 1 and 3 for the first quasiclassical periods where the ZB oscillation amplitude is greater when the $|\langle v_x \rangle|$ is greater. Next, however, we can observe a quasiclassical modulation that disappears in three classical periods [Fig. 4(b)] and we will have ZB, but there is no regeneration of the initial ZB amplitude (≈ 3.8 a.u.). For much longer times the revival or quasiclassical behavior never appears (it has been checked it from 0 to $10T_R$).

Finally, in Fig. 5 the periods in terms of the parameter ω have been studied. We can see that $T_R > T_{\rm cl} > T_{ZB}$, T_{ZB} is almost constant for all ω, and $T_{\rm cl}$ and T_R increase when ω decreases. In addition, when ω is smaller the temporal scales move away form each other quickly. In fact, the revival of the ZB amplitude appears later. The revival of the ZB amplitude will disappear when $\omega = 0$ (which corresponds to Lock's result [2]). Note that in this limit case the ZB would be approximately 10^{-4} a.u. These results are an extension for a bound Dirac particle of the results found for massless quasiparticles in graphene in a perpendicular magnetic field [10].

It should be noted that the existence of revivals of the wave packet and, consequently, of the same initial quasiclassical behavior of $\langle v_x \rangle$ and ZB oscillation amplitude is due to (i) the way in which it has been constructed as a superposition

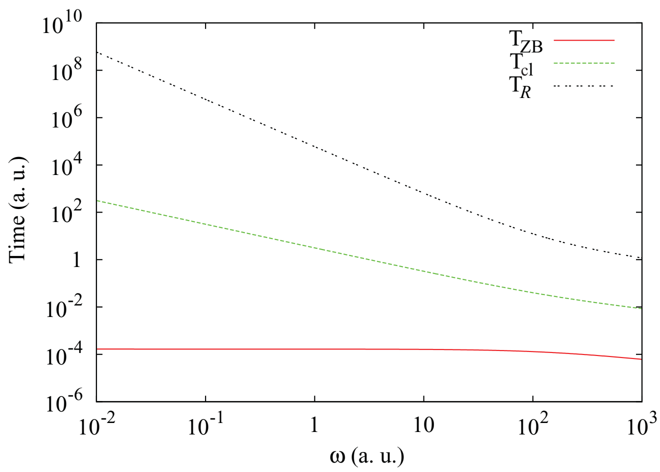

FIG. 5. (Color online) Temporal scales T_{ZB}, T_{cl}, and T_R vs ω (all in a.u.) for $n_0 = 30$.

of two wave packets localized around two given eigenvalues $E^+_{n_0}$ and $-E^-_{n_0}$ and (ii) the fact that the Dirac oscillator has electron-hole symmetry ($E^+_n = -E^-_n$), which is an essential property to obtain Eqs. (14) and (15).

Furthermore, a natural generalization of this result could be done as follows. If we consider a bound Dirac particle with a nonlinear spectrum $E^\pm_n$ in n and with electron-hole symmetry, we expect that the localized Dirac particle exhibits a revival of the ZB oscillation amplitude. Although it is an open problem to prove this assertion, it could be justified since one can always consider an initial wave packet as a superposition of two localized wave packets, with the coefficients centered around a mean value n_0 with $|n - n_0| \ll n_0$, and obtain an analogous behavior to Eq. (14) for the temporal evolution of the velocities. It should be remarked that the condition $E^+_n = -E^-_n$ is an essential point to have definite and visible temporal scales. If the initial wave packet is more localized and the n_0 value is higher, the revival of the ZB oscillation amplitude will be sharper due to the fact that the regeneration will be more accurate because the Taylor expansion is more accurate too.

III. CONCLUSION

In summary, we have studied the wave-packet dynamics for a Dirac oscillator demonstrating that for some particular election of the initial wave packet there is a regeneration or revival of the ZB oscillation amplitude apart from the quasi-classical modulation of ZB in which the oscillation amplitude is decreasing. These revivals appear to be associated with a nonlinearity in the relativistic eigenvalue spectrum. When the frequency of the oscillation is smaller, the regeneration appears at longer times. In the limit of frequency zero, that is, for a free Dirac particle, the regenerations disappear because in the case of the free Dirac particle the spectrum is continuous rather than discreet. We conjecture that this result may appear in any bound Dirac particle with electron-hole symmetry.

ACKNOWLEDGMENT

This work was supported by Project Nos. PYR-2010-24, FIS2008-01143, and FQM-165/0207.

[1] E. Schrödinger, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl. **24**, 418 (1930).
[2] J. A. Lock, Am. J. Phys. **47**, 797 (1979).
[3] See W. Zawadzki and T. M. Rusin, J. Phys. Condens. Matter **23**, 143201 (2011), and references therein.
[4] T. M. Rusin and W. Zawadzki, Phys. Rev. B **76**, 195439 (2007); **78**, 125419 (2008); **80**, 045416 (2009).
[5] E. McCann and V. I. Falko, Phys. Rev. Lett. **96**, 086805 (2006).
[6] M. I. Katsnelson, Eur. Phys. J. B **51**, 157 (2006).
[7] G. M. Maksimova, V. Y. Demikhovskii, and E. V. Frolova, Phys. Rev. B **78**, 235321 (2008).
[8] J. C. Martinez, M. B. A. Jalil, and S. G. Tan, Appl. Phys. Lett. **97**, 062111 (2010).
[9] J. Schliemann, New J. Phys. **10**, 043024 (2008).
[10] Elvira Romera and F. de los Santos, Phys. Rev. B **80**, 165416 (2009).
[11] Y. X. Wang, Z. Yang, and S. J. Xiong, Europhys. Lett. **89**, 17007 (2010).
[12] R. Gerritsma, G. Kirchmair, F. Zahringer, E. Solano, R. Blatt, and C. F. Roos, Nature (London) **463**, 68 (2010).
[13] J. H. Eberly, N. B. Narozhny, and J. J. Sánchez-Mondragón, Phys. Rev. Lett. **44**, 1323 (1980).
[14] I. Sh. Averbukh and J. F. Perelman, Phys. Lett. A **139**, 449 (1989); Acta Phys. Pol. A **78**, 33 (1990).
[15] M. Mehring, K. Mueller, I. S. Averbukh, W. Merkel, and W. P. Schleich, Phys. Rev. Lett. **98**, 120502 (2007).
[16] R. W. Robinett, Phys. Rep. **392**, 1 (2004).
[17] V. V. Nesvizhevsky *et al.*, Nature (London) **415**, 297 (2002); V. V. Nesvizhevsky, A. Y. Voronin, R. Cubitt, and K. V. Protasov, Nature Phys. **6**, 114 (2010).
[18] V. V. Nesvizhevsky *et al.*, Eur. Phys. J. C **40**, 479 (2005); V. V. Nesvizhevsky, A. K. Petukhov, K. V. Protasov, and A. Y. Voronin, Phys. Rev. A **78**, 033616 (2008).
[19] P. Strange, Phys. Rev. Lett. **104**, 120403 (2010).
[20] G. Rempe, H. Walther, and N. Klein, Phys. Rev. Lett. **58**, 353 (1987); J. A. Yeazell, M. Mallalieu, and C. R. Stroud Jr., *ibid.* **64**, 2007 (1990); T. Baumert *et al.*, Chem. Phys. Lett. **191**, 639 (1992); M. J. J. Vrakking, D. M. Villeneuve, and A. Stolow, Phys. Rev. A **54**, R37 (1996); A. Rudenko *et al.*, Chem. Phys. **329**, 193 (2006).
[21] B. P. Mandal and S. Verma, Phys. Lett. A **374**, 1021 (2010).
[22] A. Bermudez, M. A. Martin-Delgado, and E. Solano, Phys. Rev. A **76**, 041801(R) (2007).
[23] J. Parker and C. R. Stroud Jr., Phys. Rev. Lett. **56**, 716 (1986).
[24] R. Bluhm and V. A. Kostelecky, Phys. Rev. A **50**, R4445 (1994); Phys. Lett. A **200**, 308 (1995); Phys. Rev. A **51**, 4767 (1995).
[25] D. L. Aronstein and C. R. Stroud Jr., Phys. Rev. A **55**, 4526 (1997); Laser Phys. **15**, 1496 (2005).
[26] Elvira Romera and F. de los Santos, Phys. Rev. Lett. **99**, 263601 (2007); Phys. Rev. A **78**, 013837 (2008).
[27] M. Moshinsky and A. Szczepaniak, J. Phys. A **22**, L817 (1989).