Time evolution of entangled biatomic states in a cavity

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We study the time evolution of entangled states of a pair of identical atoms, considered in the harmonic approximation, coupled to an environment represented by an infinite set of free oscillators, with the whole system confined within a spherical cavity of radius R. Taking the center-of-mass and the relative-position coordinates, and using the dressed-state approach, we present the time evolution of some quantities measuring the entanglement for both limits of a very large and a small cavity; the chosen examples are simple and illustrate these very distinct behaviors.

Introduction. Interacting or noninteracting parts of a quantum system can share entangled states that hold quantum correlations [1,2]. A recent review covering all relevant aspects of the subject is in Ref. [3]. Entanglement is a purely quantum phenomenon due to the attribution of physical meaning to superposed states, a concept with no correspondence in classical physics. Entanglement means that individual parts of a quantum system are not independent of each other, even if they do not interact, and their quantum properties are described by their common wave function. In particular, entanglement properties of bipartite systems have been largely investigated in recent years. Within this context we consider here a simple biatomic system in which each atom is modeled by a harmonic oscillator. In this case, studies have been performed for a noninteracting bipartite system, with different approaches, for instance in Refs. [4,5] and when an interaction between the oscillators is taken into account [6–14].

In this Brief Report we study the time evolution of a superposition of two biatomic states of identical atoms which interact indirectly via the coupling with the harmonic modes of a field force representing the environment. We consider the two atoms in the harmonic approximation and assume that the whole system resides inside a spherical cavity of radius R. Our basic objects will be *dressed* states, corresponding to the atoms dressed by the field. The biatomic system will be consistently described by the pair consisting of the "center-of-mass" and the "relative-position" oscillators, a procedure already employed in the literature, for instance in Refs. [6,7]. In our case, these oscillators will be appropriately dressed by the field. We will consider the entangled state formed by the superposition of two kinds of states: one state in which the center-of-mass oscillator is in its first excited level and the relative-position oscillator is in the ground state; this state is superposed with another state in which the oscillators have their roles reversed. We will be concerned by the system at zero temperature, that is, all the field modes are in their ground states. Actually, it has been shown in [15,16] that thermal effects on dressed oscillators are important only for high temperatures, they are negligible for room temperatures. This means that we can make the approximation of taking the time evolution of dressed states as established for zero temperature [4]. Thus, our results apply in fact for temperatures up to the order of room temperatures.

The model. We start from a slightly modified version of the Hamiltonian used in Refs. [6,7], describing two atoms A and B in the harmonic approximation, coupled to an environment, the whole system being contained in a perfectly reflecting cavity of radius R [17],

$$H = \frac{1}{2}\left[p_A^2 + \omega_A^2 q_A^2 + p_B^2 + \omega_B^2 q_B^2 + \sum_{k=1}^{N} \left(p_k^2 + \omega_k^2 q_k^2 \right) \right] - \sum_{k=1}^{N} \sqrt{2}(\eta_A q_A + \eta_B q_B)\omega_k q_k, \quad (1)$$

where the limit $N \rightarrow \infty$ will be understood. Since the atoms are identical, we write $\omega_A = \omega_B \equiv \omega_0$ and $\eta_A = \eta_B \equiv \eta = 2\sqrt{g\,\Delta\omega/\pi}$, where g is a constant with dimension of frequency measuring the strength of the coupling and $\Delta\omega = \pi c/R$ is the interval between two neighboring frequencies of the field [18].

For identical atoms we define new coordinates q_+ (center of mass) and q_- (relative position), such that

$$q_+ = \frac{1}{\sqrt{2}}(q_A + q_B), \quad q_- = \frac{1}{\sqrt{2}}(q_A - q_B), \quad (2)$$

and corresponding formulas for momenta. Then in terms of $p_\pm$, $q_\pm$, the Hamiltonian is written as $H = H_- + H_+$, where $H_- = \frac{1}{2}[p_-^2 + \omega_0^2 q_-^2]$ and

$$H_+ = \frac{1}{2}\left[p_+^2 + \omega_0^2 q_+^2 + \sum_{k=1}^{N} \left(p_k^2 + \omega_k^2 q_k^2 - \eta q_+ \omega_k q_k \right) \right]. \quad (3)$$

We see that the center-of-mass oscillator q_+ couples to the field while the relative-position one q_- oscillates freely. The Hamiltonian H_+ describes a single oscillator linearly coupled to the field. Therefore, for the system ($q_+ \oplus$ field), composed by the center-of-mass oscillator coupled to the field, we can define the renormalized frequency $\bar{\omega}$ dressed coordinates and dressed states. Dressed coordinates and states are defined by transformations of the normal-mode coordinates and of the eigenstates of the diagonalized Hamiltonian, which depend on the normal-mode frequencies and on the renormalized oscillator frequency. An important aspect is that excited

dressed states decay with time while normal modes are stationary. Details of this formalism are presented in Refs. [4,16,18,19] and references therein.

Time evolution of entangled biatomic states. We shall describe the pair of atoms by the pair center-of-mass and relative-position oscillators. Consider the product states $|\Gamma_{10}^{(+-)}\rangle \equiv |\Gamma_1^{(+)}\rangle \otimes |\Gamma_0^{(-)}\rangle$ and $|\Gamma_{01}^{(+-)}\rangle \equiv |\Gamma_0^{(+)}\rangle \otimes |\Gamma_1^{(-)}\rangle$ in which the dressed center-of-mass oscillator is in the first excited level, while the relative-position oscillator is in its ground state, and vice versa, respectively. The dressed first-excited state of the center-of-mass oscillator evolves in time according to [4,18]

$$|\Gamma_1^{(+)}(t)\rangle = \sum_\nu f_{+\nu}(t)|\Gamma_1^\nu(0)\rangle, \quad (4)$$

with $\sum_\nu |f_{+\nu}(t)|^2 = 1$, where the label ν runs over the center-of-mass oscillator $(+)$ and the field modes $(\{k\})$. The quantity $f_{+\nu}(t)$ thus represents the probability amplitude that the excitation is at the νth dressed oscillator at time t. On the other hand, the first excited state of the relative-position oscillator is stationary. This implies that the time evolution of the state $|\Gamma_{10}^{(+-)}\rangle$ is governed by Eq. (4), while the state $|\Gamma_{01}^{(+-)}\rangle$ remains stationary.

We now consider at $t = 0$ the family of states

$$|\Psi^{AB}(0)\rangle = \sqrt{\xi}\,|\Gamma_{10}^{(+-)}(0)\rangle + \sqrt{1-\xi}\,e^{i\phi}|\Gamma_{01}^{(+-)}(0)\rangle \quad (5)$$

which belongs to the Hilbert space $\mathcal{H}_{+,\{k\}} \otimes \mathcal{H}_-$, representing states of the system of the two atoms coupled to the environmental field. At the instant t, the density matrix corresponding to Eq. (5) is written as

$$\begin{aligned}
\varrho(t) = |\Psi^{AB}(t)\rangle\langle\Psi^{AB}(t)| &= \xi|\Gamma_{10}^{(+-)}(t)\rangle\langle\Gamma_{10}^{(+-)}(t)| \\
&+ (1-\xi)|\Gamma_{01}^{(+-)}\rangle\langle\Gamma_{01}^{(+-)}| \\
&+ \sqrt{\xi(1-\xi)}\,e^{i\phi}|\Gamma_{01}^{(+-)}\rangle\langle\Gamma_{10}^{(+-)}(t)| \\
&+ \sqrt{\xi(1-\xi)}\,e^{-i\phi}|\Gamma_{10}^{(+-)}(t)\rangle\langle\Gamma_{01}^{(+-)}|.
\end{aligned} \quad (6)$$

We adopt a more explicit notation and write $|\Gamma_{10}^{(+-)}(t)\rangle \equiv |1_+(t),0_-;0,0,\ldots\rangle$ and $|\Gamma_{01}^{(+-)}\rangle \equiv |0_+,1_-;0,0,\ldots\rangle$. To analyze how the two-atom state evolves in time, we consider the reduced density matrix obtained by taking the trace over the field modes $\rho(t) = \sum_{k_i}\langle k_1,k_2,\ldots|\varrho(t)|k_1,k_2,\ldots\rangle$. Using Eq. (4),

$$\begin{aligned}
|1_+(t),0_-;0,0,\ldots\rangle &= f_{++}(t)|1_+(0),0_-;0,0,\ldots\rangle \\
&+ \sum_i f_{+i}(t)|0_+,0_-;0,\ldots,0,1_i,0,\ldots\rangle,
\end{aligned}$$

and we obtain

$$\rho(t) = \begin{pmatrix} a(t) & 0 & 0 & 0 \\ 0 & b(t) & d(t) & 0 \\ 0 & d^*(t) & c(t) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (7)$$

where

$$a(t) \equiv (\rho)_{0_+0_-}^{0_+0_-} = \xi[1 - |f_{++}(t)|^2], \quad (8)$$

$$b(t) \equiv (\rho)_{0_+1_-}^{0_+1_-} = 1 - \xi, \quad (9)$$

$$c(t) \equiv (\rho)_{1_+0_-}^{1_+0_-} = \xi|f_{++}(t)|^2, \quad (10)$$

$$d(t) \equiv (\rho)_{0_+1_-}^{1_+0_-} = \sqrt{\xi(1-\xi)}\,e^{i\phi}f_{++}^*(t), \quad (11)$$

$$d^*(t) \equiv (\rho)_{1_+0_-}^{0_+1_-} = \sqrt{\xi(1-\xi)}\,e^{-i\phi}f_{++}(t). \quad (12)$$

As it should, $\text{Tr}\rho(t) = a(t) + b(t) + c(t) = 1$. The degree of impurity of the state (7) is given by $D = 1 - \text{Tr}\rho^2(t)$; then it follows that

$$D(\xi;t) = 2\xi(1 - |f_{++}(t)|^2)(1 - \xi + \xi|f_{++}(t)|^2). \quad (13)$$

To find the time dependence of the above quantities, we need to evaluate the function $f_{++}(t)$ which governs the behavior of the system. There are two significantly different situations, depending on the size of the cavity. We shall analyze the limit of a very large cavity $(R \to \infty)$ and the case of a small cavity. In any case, independently of emission frequency, we must have $0 < |f_{++}(t)|^2 \leqslant 1$.

Large cavity: for a very large cavity, we have [20]

$$f_{++}(t) = e^{-gt}\left[\cos\kappa t - \frac{g}{\kappa}\sin\kappa t\right] + iG(t;\bar{\omega},g), \quad (14)$$

where $\kappa^2 = \bar{\omega}^2 - g^2$ and the function G is given by

$$G(t;\bar{\omega},g) = -\frac{4g}{\pi}\int_0^\infty dy\,\frac{y^2\sin yt}{(y^2 - \bar{\omega}^2)^2 + 4g^2y^2}. \quad (15)$$

We will consider $\kappa^2 > 0$, which includes the weak coupling regime $g^2 \ll \bar{\omega}^2$.

Small cavity: for a finite cavity, the spectrum of eigenfrequencies is discrete, and the continuum language used in the case of large cavity is not valid. However, for a very small cavity, with a radius R much smaller than the coherence length $\pi c/g$ $(R \ll \pi c/g)$, we can obtain $f_{++}(t)$ to a good approximation as [20]

$$\begin{aligned}
f_{++}(t) \approx \left(1 + \frac{2}{3}\pi\delta\right)^{-1} &\left\{\exp\left[-i\bar{\omega}\left(1 - \frac{\pi\delta}{2}\right)t\right]\right. \\
&\left.+ \sum_{k=1}^\infty \frac{4\delta}{\pi k^2}\exp\left[-i\frac{g}{\delta}\left(k + \frac{2\delta}{\pi k}\right)t\right]\right\}, \quad (16)
\end{aligned}$$

where $\delta = gR/\pi c \ll 1$ is a dimensionless parameter, characterizing the smallness of the cavity. These two extreme cases are illustrated in Fig. 1, where we plot the degree of impurity as a function of the time.

Entanglement. Let us now examine the degree of entanglement of the states described by the density matrix (7) and how it evolves in time. Let us initially calculate the concurrence [21] associated with the density matrix ρ. The "spin-flipped" state is $\tilde{\rho} = (\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2)$, where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the Pauli matrix. Taking the basis $\{|0_+,0_-\rangle,|0_+,1_-\rangle,|1_+,0_-\rangle,|1_+,1_-\rangle\}$ we find

$$\rho\tilde{\rho} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2b(t)c(t) & 2b(t)d(t) & 0 \\ 0 & 2c(t)d^*(t) & 2b(t)c(t) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (17)$$

where we have used that $b(t)c(t) = |d(t)|^2$. The concurrence of a mixed bipartite state ρ is given by [21]

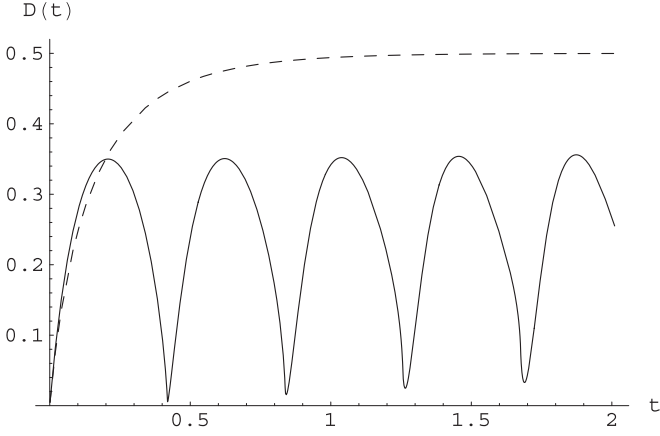$$C_\rho = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \quad (18)$$

FIG. 1. Degree of impurity as function of time [Eq. (13)] taking $\bar{\omega} = 1.5$ and $g = 1.0$ (in arbitrary units) for states with $\xi = 0.5$: for a very large cavity (dashed line) and for a small cavity with $\delta = 0.1$ (full line).

where the λ_i are the square roots of the eigenvalues of the non-Hermitian matrix $\rho\tilde{\rho}$, written in decreasing order. In our case, there is only one nonvanishing eigenvalue of $\rho\tilde{\rho}$, given by $\lambda(t) = 4b(t)c(t)$. We then find the concurrence as

$$C_\rho(t) = 2\sqrt{b(t)c(t)} = 2\sqrt{\xi(1-\xi)}\,|f_{++}(t)|. \quad (19)$$

Another measure of entanglement is the negativity [22,23], which can be defined by

$$\mathcal{N}_\rho = \|\rho^{T_-}\|_1 - 1, \quad (20)$$

where ρ^{T_-} is the partial transpose of the bipartite mixed state ρ and $\|\cdot\|_1$ denotes the trace norm. The trace norm of an operator O is defined by $\|O\|_1 = \text{Tr}\sqrt{O\,O^\dagger}$ which reduces, for Hermitian operators, to the sum of the absolute values of its eigenvalues. It can be easily shown that, for Hermitian operators,

$$\|O\|_1 = \sum_j |\lambda_j| = \sum_{\lambda_j > 0} \lambda_j - \sum_{\lambda_j < 0} \lambda_j = 1 + 2\sum_{\lambda_j < 0} |\lambda_j|,$$

so that the negativity is given by the absolute value of the sum of the negative eigenvalues of ρ^{T_-}, that is,

$$\mathcal{N}_\rho = 2\left|\sum_{\lambda_j < 0} \lambda_j\right|. \quad (21)$$

The partial transpose of ρ, ρ^{T_-} is given by

$$\rho^{T_-}(t) = \begin{pmatrix} a(t) & 0 & 0 & d(t) \\ 0 & b(t) & 0 & 0 \\ 0 & 0 & c(t) & 0 \\ d^*(t) & 0 & 0 & 0 \end{pmatrix}, \quad (22)$$

whose eigenvalues are $\lambda_1(t) = b(t)$, $\lambda_2(t) = c(t)$, $\lambda_3(t) = [a(t) + \sqrt{a^2(t) + 4|d(t)|^2}]/2$ and $\lambda_4(t) = [a(t) - \sqrt{a^2(t) + 4|d(t)|^2}]/2$. We thus obtain the negativity as

$$\mathcal{N}_\rho(t) = 2|\lambda_4(t)| = \sqrt{a^2(t) + 4|d(t)|^2} - a(t)$$
$$= \sqrt{\xi^2 + (4\xi - 6\xi^2)|f_{++}(t)|^2 + \xi^2|f_{++}(t)|^4}$$
$$- \xi + \xi|f_{++}(t)|^2. \quad (23)$$
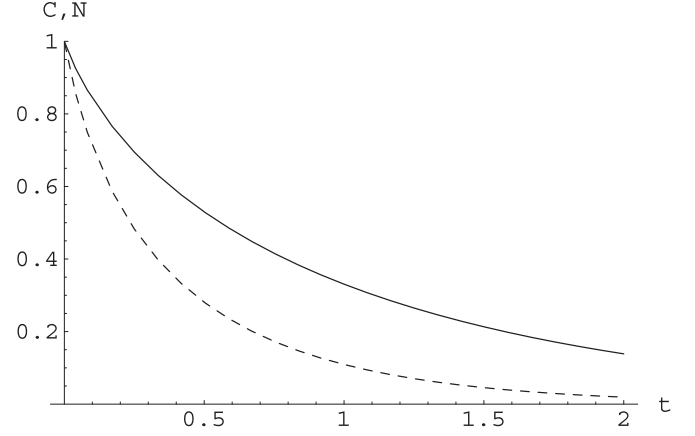

FIG. 2. Concurrence (full line) and negativity (dashed line) as functions of time for states with $\xi = 0.5$, $\bar{\omega} = 1.5$, and $g = 1.0$ (in arbitrary units) in a very large cavity.

Notice that, for pure bipartite states, the concurrence and the negativity, as defined above, are equal to each other [23]. However, for mixed bipartite states one has $\mathcal{N}_\rho \leqslant C_\rho$ [24]. These features are illustrated in Fig. 2 for a very large cavity and in Fig. 3 for a small one, where we plot the concurrence and the negativity for some of the states (7), as a function of time.

Conclusions. Our study of how an entangled non-Gaussian biatomic state contained in a cavity evolves in time leads to, as an overall conclusion, that the behavior of the system is very contrasting in the cases of a very large cavity (free space) or of a small cavity. This is essentially due to the behavior of the quantity $|f_{++}(t)|^2$, which governs the time evolution of the system in each case. This quantity (the probability that the center-of-mass oscillator remain excited at the first level at time t), goes monotonically to *zero* as $t \to \infty$ for a very large cavity, while it oscillates indefinitely with a strictly positive minimum for a small cavity. These rather distinct behaviors are exhibited by the degree of impurity as illustrated in Fig. 1.

With regard to entanglement, we have examined the concurrence and the negativity as measures of entanglement; these are plotted in Figs. 2 and 3 for a very large and a small cavity, respectively. In a very large cavity, these quantities decrease monotonically to zero for large times and do not
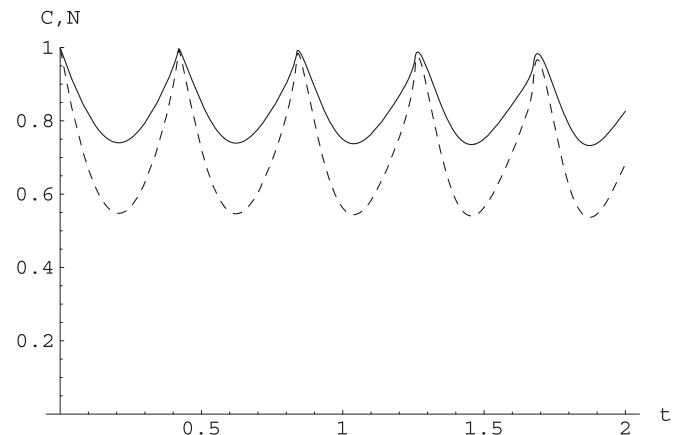

FIG. 3. Same as in Fig. 2, but for a small cavity with $\delta = 0.1$.

show any sudden death. In contrast, for a small cavity both present an oscillatory behavior as time evolves and never vanish, having finite values for any arbitrarily long elapsed time. This can be related to former results in another context, obtained using dressed states, for decay and stability of excited atoms in very large or small cavities, for both zero or finite temperature [16,19,25,26].

Nevertheless, it is worthy mentioning that, in some cases, a finite asymptotic entanglement persists for long times in free space, that is in very large cavities, as pointed out in Refs. [8,10,12–14]. However, in most cases, this phenomenon exists when the environment is a thermal bath and for initial states that are two-mode squeezed (Gaussian) states, with a high degree of squeezing. These situations are different from the case of the zero-temperature bath and non-Gaussian initial states we have considered in this note, for which no asymptotic entanglement for large cavities exists.

In our case, for large cavities the entanglement vanishes asymptotically. On the other hand, we have a *permanent*, oscillating entanglement for small cavities at all times. In the case of a bath as we have adopted here, the existence of the entanglement for all times in a sufficiently small cavity, in contrast to the fast decay for long times in the very large cavity (free space), favors small cavities as candidates to engender phenomena that require entanglement as a resource. The discussion presented here may be extended to other systems relevant to quantum computation.

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