

Convergence of repeated quantum nondemolition measurements and wave-function collapse

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Motivated by recent experiments on quantum trapped fields, we give a rigorous proof that repeated indirect quantum nondemolition (QND) measurements converge to the collapse of the wave function as predicted by the postulates of quantum mechanics for direct measurements. We also relate the rate of convergence toward the collapsed wave function to the relative entropy of each indirect measurement, a result which makes contact with information theory.

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Wave function collapse is a basic axiom of quantum direct measurement à la Von Neumann [1]. A quantum nondemolition (QND) measurement [2] is one for which the collapsed state is an eigenstate of the free evolution. Repeating the measurement on the collapsed state yields identical results since this state is preserved by the evolution. Indirect measurements [3] consists in letting the quantum system under study be entangled with another quantum system, called the probe, and in implementing a direct measurement on the probe. Since the system and the probe are entangled, one gains information. Repeating the process of entanglement and measurement increases statistically the information one gets on the system.

Developing experimental and theoretical expertise on quantum measurement processes is mandatory for developing quantum state manipulation. It was early realized [4,5] that modeling quantum measurements require systems with infinitely many degrees of freedom, e.g., as in the phenomenological stochastic models of Ref. [6]. The need to describe quantum jumps and randomness inherent to repeated measurements lead to the concept of quantum trajectories [7,8]. In parallel, tools of open quantum systems, specifically those of quantum stochastic calculus [9], have been adapted to the description of quantum continual measurements [10] and quantum feedback [11]. In most of these stochastic models, the driving noises, often classical or quantum Brownian motions, are linked to the degrees of freedom of the measurement apparatus. Although bearing similarities with these frameworks, our proof of the wave collapse in series of quantum nondemolition (QND) measurements is based on a purely quantum description of the repeated probe-system interactions.

Experiments on repeated indirect quantum nondemolition measurements have recently been performed, in particular in quantum optics. As an example, let us look at Ref. [12] whose setup is the following. The tested quantum system is a resonant electromagnetic cavity selecting photons of given frequency. It is probed by sending Rydberg atoms through it, one after the other. During the atom-photon interaction, each atom behaves as a two-state system modeled by a spin one-half [13]. The

atoms are prepared with their effective spins pointing in the $0x$ direction [14]. The experimental protocol ensures that the atom effective spin rotates around the $0z$ axis by an angle proportional to the number of photons \hat{n}_{ph} in the cavity, say $\hat{n}_{\text{ph}}\theta$ with θ a fixed angle. After interaction, the atom-photon system is entangled, but the cavity state gets unchanged if it is initially an eigenstate of the free photon Hamiltonian. The effective atom spin is then measured along a direction perpendicular to $0z$ but at angle ϕ with respect to $0x$. The output of the spin measure is \pm with probabilities $p_+(\phi|\hat{n}_{\text{ph}}) = \cos^2[(\hat{n}_{\text{ph}}\theta - \phi)/2]$ and $p_-(\phi|\hat{n}_{\text{ph}}) = \sin^2[(\hat{n}_{\text{ph}}\theta - \phi)/2]$, if there are \hat{n}_{ph} photons in the cavity. If the initial photon distribution is $q_0(\hat{n}_{\text{ph}})$, the probability to measure an effective spin \pm is $\sum_{\hat{n}_{\text{ph}}} q_0(\hat{n}_{\text{ph}}) p_{\pm}(\phi|\hat{n}_{\text{ph}})$. No direct measurement on the cavity is done. The experimental aim is to reconstruct the initial photon distribution by accumulating informations from the repeated atom effective spin measurements. The photon distribution is recalculated after each atom measurement using Bayes law [15]. Figure 1 shows experimental data for the evolution of reconstructed photon distributions. For each realization, they converge, experimentally and numerically [12], to peaked distributions whose centers depend on the realization. This is the collapse.

Let us abstract and generalize the previous situation. At initial time, the system is in state $|\varphi_0\rangle \equiv |\varphi\rangle$. It interacts during time Δt with a probe initially in state $|\psi\rangle$, so that the pair (probe + system) evolves into $U(|\psi\rangle \otimes |\varphi\rangle)$, where U is some unitary operator acting on the Hilbert space $\mathcal{H}_{\text{probe}} \otimes \mathcal{H}_{\text{sys}}$. After Δt , the system-probe interaction can be neglected. A perfect measurement à la Von Neumann is then performed on the probe. This means that there is an orthonormal basis $|i\rangle$, $i \in I$, of $\mathcal{H}_{\text{probe}}$ such that, after the measurement, the (probe + system)-state is proportional to $(|i\rangle\langle i| \otimes \text{Id})U(|\psi\rangle \otimes |\varphi\rangle)$ with probability $\|(|i\rangle\langle i| \otimes \text{Id})U(|\psi\rangle \otimes |\varphi\rangle)\|^2$. The vanishing of this probability for a certain state $|i\rangle$ means that the probe cannot be found in state $|i\rangle$, so we can (and shall) simply forget about that possibility.

We make the following assumption, related to nondemolition, on the evolution operator U : there is an orthonormal basis $|\alpha\rangle$, $\alpha \in A$, of \mathcal{H}_{sys} and a collection of operators U_α acting on $\mathcal{H}_{\text{probe}}$ such that, for each α ,

$$U(|\psi\rangle \otimes |\alpha\rangle) = (U_\alpha|\psi\rangle) \otimes |\alpha\rangle. \quad (1)$$

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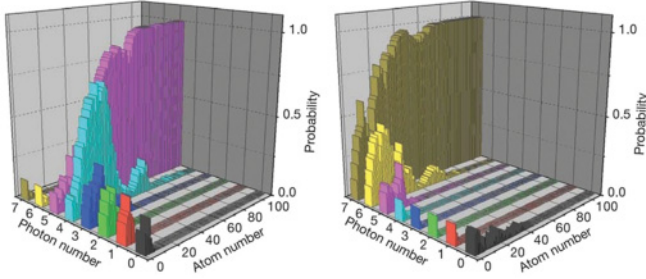


FIG. 1. (Color online) Two experimental samples [16] of reconstructed photon distributions as functions of the number of indirect measurements (i.e., the number of atoms traversing the cavity) according to Ref. [12]. The collapse of the photon distribution to a realization-dependent sharply defined number is clearly visible.

The operators U_α are automatically unitary. If the probe is found in state $|i\rangle$ after the measurement (probability $\sum_{\alpha \in A} |i|U_\alpha|\psi\rangle|^2|\langle\alpha|\varphi\rangle|^2$), the pair (probe + system) is again in a tensor product state $|i\rangle \otimes |\varphi_1\rangle$, where

$$|\varphi_1\rangle = \frac{\sum_{\alpha \in A} \langle i|U_\alpha|\psi\rangle \langle\alpha|\varphi\rangle |\alpha\rangle}{\left(\sum_{\alpha \in A} |\langle i|U_\alpha|\psi\rangle|^2 |\langle\alpha|\varphi\rangle|^2\right)^{1/2}}. \quad (2)$$

It is clear that the motivating experiment fulfills this property if $|\alpha\rangle$ is the occupation number basis [17].

The physics of this hypothesis is that the final aim is to measure an observable on the system for whom the states $|\alpha\rangle$ are eigenstates. As we shall see, this (direct) measurement can be (indirectly) achieved by repeated measurements on successive probes. So, one presents another probe to the system in state $|\varphi_1\rangle$, let them interact, and, after interaction, measures the probe to get $|\varphi_2\rangle$ and so on. Notice that, in general, at each step one could change the probe initial state, the observable measured on the probe (this is indeed what happens in the motivating example), and even the type of probes: the only thing one has to keep fixed is the basis $|\alpha\rangle$ for which property of Eq. (1) holds. Most of the following discussion can be extended to the general setting [18] but to keep notation simple, we concentrate on the case when $|\psi\rangle$ and the basis $|i\rangle$ are the same for all probes.

We start with a summary of our results:

(i) If a series of repeated indirect measurements is conducted, the state of the system will stabilize over time and go to a limit. Carrying identical independent experiments again, the system state will stabilize over time again but possibly with different limits.

(ii) Under a physically meaningful nondegeneracy condition, the only possible limits for the state of the system are the pointer states $|\alpha\rangle$, and the probability to end in state $|\alpha\rangle$ starting from state $|\varphi\rangle$ is $|\langle\alpha|\varphi\rangle|^2$. Hence, the outcome of a large number of repeated indirect measurements satisfying the condition of Eq. (1) obeys the standard rules of quantum mechanics direct measurements.

(iii) Under the same nondegeneracy condition, the measurements on the probes allow to infer the limit pointer state for each independent experiment.

(iv) The rate of convergence to one of the pointer states is governed by the relative entropy of certain probability measures in classical probe space. The order of magnitude of the probability that, while the repeated measurements are conducted, the state of the system comes close to a pointer state but ends up finally in another one can be computed explicitly.

The tools to prove these statements come from the classical theory of random processes: strong law of large numbers, martingale convergence theorem, large deviations. A proof of the wave function collapse using the martingale convergence theorem appeared in Ref. [19]. These works are based on nonlinear stochastic extensions of the Schrödinger equation [20], whereas our results are pure consequences of quantum mechanics (with measurements on probes) [21] and are closer in spirit to quantum trajectory approaches [7,8] and to experiments.

We now turn to the proofs. One can rephrase Eq. (2) by saying that, for each $\alpha \in A$,

$$\langle\alpha|\varphi_1\rangle = \frac{\langle i|U_\alpha|\psi\rangle \langle\alpha|\varphi_0\rangle}{\left(\sum_{\alpha \in A} |\langle i|U_\alpha|\psi\rangle|^2 |\langle\alpha|\varphi_0\rangle|^2\right)^{1/2}},$$

if the probe is found in state $|i\rangle$. Thus, a crucial consequence of Eq. (1) is that there are no interference terms for different α , so that taking the modulus squared does not lead to (much) loss of information. We set $p(i|\alpha) \equiv |\langle i|U_\alpha|\psi\rangle|^2$, and $q_0(\alpha) \equiv |\langle\alpha|\varphi_0\rangle|^2$, $q_1(\alpha) \equiv |\langle\alpha|\varphi_1\rangle|^2$, $q_2(\alpha) \equiv |\langle\alpha|\varphi_2\rangle|^2$ and so on. Observe that after measuring the n th probe, one has, for each $\alpha \in A$,

$$q_{n+1}(\alpha) = q_n(\alpha) \frac{p(i|\alpha)}{\sum_{\beta \in A} q_n(\beta) p(i|\beta)}, \quad (3)$$

with probability $\pi_n(i) \equiv \sum_{\beta \in A} q_n(\beta) p(i|\beta)$.

This is a random recursion relation that is of Markovian type: to compute the possible values of $q_{n+1}(\alpha)$ and their respective probabilities, all one needs to know are the $q_n(\beta)$. Each probe measurement leads to a choice among the probe states $|i\rangle$ such that $\pi_n(i) \neq 0$. The question to be settled is the long time behavior of the resulting random sequences $q_n(\alpha)$.

Observe that the $q_n(\alpha)$ and the $p(i|\alpha)$ are ≥ 0 . Moreover, $\sum_{\alpha \in A} q_n(\alpha) = 1$ and $\sum_{i \in I} p(i|\alpha) = 1$ for each $\alpha \in A$. It follows that $\sum_{i \in I} \pi_n(i) = 1$ as it should be. A crucial question is the following: having observed the random sequences $q_m(\beta)$ for $m = 0, \dots, n$ and all β in A , what is the average value of $q_{n+1}(\alpha)$? From Eq. (3), it is immediate that this (conditional) average, which we denote by $\mathbb{E}[q_{n+1}(\alpha)|q_0, \dots, q_n]$, is

$$\begin{aligned} \mathbb{E}[q_{n+1}(\alpha)|q_0, \dots, q_n] &= \sum_{i, \pi_n(i) \neq 0} q_n(\alpha) \frac{p(i|\alpha)}{\pi_n(i)} \pi_n(i) \\ &= \sum_{i, \pi_n(i) \neq 0} q_n(\alpha) p(i|\alpha). \end{aligned}$$

Now, $\pi_n(i) = \sum_{\beta \in A} q_n(\beta) p(i|\beta)$, and for this to vanish, the product $q_n(\beta) p(i|\beta)$ has to vanish for all $\beta \in A$, and in particular for $\beta = \alpha$, so that $\sum_{i, \pi_n(i) \neq 0} q_n(\alpha) p(i|\alpha) = \sum_{i \in I} q_n(\alpha) p(i|\alpha) = q_n(\alpha)$. Hence, we find that

$$\mathbb{E}[q_{n+1}(\alpha)|q_0, \dots, q_n] = q_n(\alpha). \quad (4)$$

In the theory of random processes, such a property defines the concept of martingale: the sequence q_0, q_1, \dots is a

martingale, because if one knows it up to time n (i.e., if one knows q_0, \dots, q_n), its expectation at time $n+1$ is its value at time n (i.e., q_n). To connect quantum measures to conditional expectations is not so surprising because both rely on orthogonal projections in Hilbert spaces.

The martingale at hand has a peculiar property: it is bounded [every $q_n(\alpha) \geq 0$ and $\sum_{\alpha \in A} q_n(\alpha) = 1$]. We can then quote a special case of the martingale convergence theorem (see any modern textbook on probability theory, e.g., Ref. [24], for a precise statement): *A random sequence q_0, q_1, \dots which is a bounded martingale converges almost surely and in \mathbb{L}^1 . The limit, a random variable q_∞ , is such that its expectation satisfies $\mathbb{E}(q_\infty) = q_0$.*

This is a deep theorem and there is no intuitive argument that we know to explain it [22]. But in our case its meaning is simple. The statement of almost sure convergence is precisely the mathematical formulation of (i). The statement of \mathbb{L}^1 convergence is a simple consequence of the Lebesgue dominated convergence theorem, because our martingale is bounded. The statement on the expectation of the limit random variable yields the second part of (ii) once we have given an independent argument to show that the possible limits are the pointer states.

To get this, we observe that the convergence of $q_n(\alpha)$ leads to the convergence of $\pi_n(i) = \sum_{\beta \in A} q_n(\beta) p(i|\beta)$. If i is such that $\pi_\infty(i) \neq 0$ then, for n large enough, $\pi_n(i) > \pi_\infty(i)/2 > 0$ which implies that, with probability 1, the n th probe will be found in state i for arbitrarily large values of n . This allows to take the large n limit in Eq. (3) for this value of i . Hence,

$$q_\infty(\alpha) = q_\infty(\alpha) \frac{p(i|\alpha)}{\sum_{\beta \in A} q_\infty(\beta) p(i|\beta)},$$

for any $i \in I$ such that $\pi_\infty(i) \neq 0$. Only the α for which $q_\infty(\alpha) \neq 0$ yield a nontrivial equation, so we can restrict to these α . Then, we can simplify to get $p(i|\alpha) = \pi_\infty(i)$ for any i such that $\pi_\infty(i) \neq 0$. If $q_\infty(\alpha) \neq 0$, $\pi_\infty(i) = 0$ implies $p(i|\alpha) = 0$, so that $p(i|\alpha) = \pi_\infty(i)$ is actually valid for any i . The right-hand side may depend on i but it does not depend on α . So, the same holds for the left-hand side: this means that the evolution operator U and the probe measurement act in a degenerate way on the corresponding kets $|\alpha\rangle$. In such a degenerate situation, we cannot expect to measure them individually, just as in a standard quantum measure of a system observable we cannot separate the $|\alpha\rangle$ having the same eigenvalue [23]. So, we assume that for any $\alpha, \beta \in A$ there is some $i \in I$ such that $p(i|\alpha) \neq p(i|\beta)$, and we get that $q_\infty(\alpha) = \delta_{\alpha, \gamma}$ for some γ , i.e., the only possible values for $q_\infty(\alpha)$ are 0 or 1. The equality $\mathbb{E}(q_\infty(\alpha)) = q_0(\alpha)$ then implies that $q_\infty(\alpha)$ takes value 1 with probability $q_0(\alpha) = \langle |\alpha\rangle | \varphi \rangle^2$ and 0 with probability $1 - q_0(\alpha)$ as expected in a perfect measurement of a nondegenerate system observable with the $|\alpha\rangle$ as eigenstates.

The proofs of statements (iii) and (iv) use the same tools. We start by determining the rate of convergence to the limiting system state. This turns out to depend on this limiting state and this is also the clue to statement (iii). What we have proved so far implies that at some time, say n_0 , one of the components, say $q_{n_0}(\gamma)$, will be large, i.e., close to 1, so that all other

components will be small. We can then replace Eq. (3) by an approximate linear recursion relation, namely, for $\alpha \neq \gamma$,

$$q_{n+1}(\alpha) = q_n(\alpha) \frac{p(i|\alpha)}{p(i|\gamma)}, \quad (5)$$

with probability $p(i|\gamma)$ (if nonzero). The proof given above shows again that this random recursion relation defines a martingale. There is a subtle point, however: this martingale is not bounded anymore and the martingale convergence theorem does not apply. However, we can rely on a simpler tool. Defining $l_n \equiv \ln q_n$, we get, for $\alpha \neq \gamma$,

$$l_{n+1}(\alpha) = l_n(\alpha) + \ln \frac{p(i|\alpha)}{p(i|\gamma)}, \quad (6)$$

with probability $p(i|\gamma)$ (if nonzero). So, $l_n(\alpha) - l_0(\alpha)$ is the sum of n independent identically distributed random variables with mean $-S(\gamma|\alpha) \equiv \sum_i p(i|\gamma) \ln p(i|\alpha)/p(i|\gamma)$. Remember that for each β , the collection $p(i|\beta)$, $i \in I$, defines a probability on I , and $S(\gamma|\alpha)$ is nothing but the relative entropy of $p(i|\gamma)$ with respect to $p(i|\alpha)$, a quantity that is always nonnegative and, in fact, strictly positive under the nondegeneracy assumption. The law of large numbers yields $l_n(\alpha) \sim -nS(\gamma|\alpha) \rightarrow -\infty$, so that $q_n(\alpha)$ converges exponentially to 0 with rate $S(\gamma|\alpha)$. Hence, as soon as one of the components, say $q(\gamma)$, has become reasonably close to one, with high probability the state of the system will converge to $|\gamma\rangle$. In this situation, each measurement on the probe leads to a gain of information on the system state, which in average is given for each component $\alpha \neq \gamma$ by the relative entropy $S(\gamma|\alpha)$.

By the strong law of large numbers, the previous discussion also implies that if the limit state is $|\gamma\rangle$, the frequency of measurements leading to probe state $|i\rangle$ will converge to $p(i|\gamma)$. By the nondegeneracy hypothesis, this fixes the limit pointer state unambiguously. This proves statement (iii). In practice, an histogram of all n_i/n , the fraction of probes measured in state $|i\rangle$ in a single series of a large number n of repeated measurements, for $i \in I$, will be close to $p(i|\gamma)$ for a single $|\gamma\rangle$, allowing to identify $|\gamma\rangle$. Then, conducting many independent homogeneous series (starting each experiment with the same system state) allows to reconstruct the probabilities q_0 . Hence, the homogeneous repeated measurement scheme is fully equivalent to an ideal Von Neumann measurement.

To finish the discussion, note that by the martingale property, knowing the results of probe measurements up to time n_0 , the probability to end in pointer state $|\gamma\rangle$ is exactly $q_{n_0}(\gamma)$, which is close to 1. The quantity $1 - q_{n_0}(\gamma)$ is the probability to end in another pointer state. It is also the order of magnitude of the probability that the above discussion breaks down. This occurs precisely when the random evolution invalidates the linear approximation. If this happens, it will be likely to happen quickly after n_0 because, if for a long time after n_0 the $q_n(\alpha)$ remain small, the law of large numbers implies that they are very likely to decrease exponentially so that escaping away from the pointer state $|\gamma\rangle$ will get harder and harder. Take some $\varepsilon > 0$ such that if $1 - \varepsilon < q_n(\gamma)$ the linear approximation is good to describe the transition from time n to time $n+1$. Suppose that during a random evolution this

condition on $q_n(\gamma)$ remains valid for $n_0 \leq n \leq n_1$. By standard large deviation theory (Cramer's theorem) if $n_1 - n_0$ is large, the probability that, for a given α , $q_{n_1}(\alpha)$ is of order ε (instead of being of order $\varepsilon \exp[-(n_1 - n_0)S(\gamma|\alpha)]$) is estimated crudely as $\sim \lambda_*^{n_1 - n_0}$ for a certain $\lambda_* < 1$ which is the minimum over $s > 0$ of the function $\lambda(s) = \sum_i p(i|\gamma) \left(\frac{p(i|\alpha)}{p(i|\gamma)}\right)^s$.

Finally, we emphasize that the (infinite) series of indirect experiments may be viewed as building a measurement

apparatus [18]. Indeed, the reading of the asymptotic behavior of the frequencies of the probe measurement outcomes allows us to register the limit pointer state.

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- [21] Point (iii) does not seem to have been addressed in Ref. [19], which only proves the collapse without explaining how to identify the limit pointer state in a given realization. Point (iv) shows that the convergence rate depends on the limit pointer state, a fact which contrasts with the usual estimates [19].
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