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Dispersion of nonlinear group velocity determines shortest envelope solitons

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We demonstrate that a generalized nonlinear Schrödinger equation (NSE), which includes dispersion of the intensity-dependent group velocity, allows for exact solitary solutions. In the limit of a long pulse duration, these solutions naturally converge to a fundamental soliton of the standard NSE. In particular, the peak pulse intensity times squared pulse duration is constant. For short durations, this scaling gets violated and a cusp of the envelope may be formed. The limiting singular solution determines then the shortest possible pulse duration and the largest possible peak power. We obtain these parameters explicitly in terms of the parameters of the generalized NSE.

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I. INTRODUCTION

Optical solitons are waves localized either in space or time that are formed as a result of the interplay between nonlinearity and dispersion [1,2]. A typical soliton, e.g., in an optical fiber, is often described by its complex envelope ψ which satisfies the slowly varying envelope approximation (SVEA). The SVEA assumes that the pulse spectrum is concentrated around a well-defined carrier frequency ω_0. Then, the dispersion is represented by the Taylor expansions around ω_0. The dispersion of the linear response given by the frequency-dependent wave vector $\beta(\omega)$ is then encoded within a discrete set, $\beta_m = \beta^{(m)}(\omega_0)$, of expansion coefficients, where at least three lowest-order terms with $m = 0,1,2$ are taken into account. For unidirectional propagation, the complex pulse envelope $\psi(z,t)$ is normally described in a comoving frame, $\psi = \psi(\zeta,\tau)$, with $\zeta = z$ and $\tau = t - \beta_1 z$, where β_1 is the inverse group velocity and τ is referred to as retarded time. In the case of an instantaneous cubic nonlinearity, the envelope is governed by the nonlinear Schrödinger equation (NSE),

$$i\partial_\zeta\psi + \frac{\beta_2}{2}(i\partial_\tau)^2\psi + \gamma|\psi|^2\psi = 0, \tag{1}$$

where β_2 is the group velocity dispersion (GVD) and the parameter γ is determined by the linear bulk dispersion, fiber geometry (effective fiber area), and nonlinear susceptibility of the third order [3]. The envelope is usually scaled so that $|\psi|^2$ represents the pulse power. For the focusing ($\gamma > 0$) nonlinearity, bright solitary solutions appear in the domain of negative dispersion ($\beta_2 < 0$).

The model given by the NSE does not impose any restrictions on soliton duration: a twofold decrease of the duration simply means a fourfold increase of the peak power. It is the SVEA that lacks precision for shorter pulse durations and broader pulse spectra. That is why Eq. (1) is commonly replaced by a more accurate, generalized NSE for short pulses. In what follows, we are primarily interested in solitary solutions of generalized pulse-propagation equations and therefore we deliberately exclude linear losses and Raman scattering.

The simplest generalization of Eq. (1) is to use a larger number of purely real dispersion parameters β_m. The term $\frac{1}{2}\beta_2(i\partial_\tau)^2\psi$ then must be replaced by the so-called dispersion operator,

$$\hat{\mathfrak{D}}\psi = \sum_{m=2}^{M_{\max}} \frac{\beta_m}{m!}(i\partial_\tau)^m\psi, \tag{2}$$

which covers the behavior of $\beta(\omega)$ in a larger frequency domain. Still, even an infinite number of parameters β_m does not guarantee the necessary convergence [4,5].

Another important generalization of the NSE is to introduce dispersion into the nonlinear term in Eq. (1). Such effective dispersion eliminates some of the deficiencies of the SVEA. It naturally appears in the description of short pulses, even in an ideal Kerr media with the nondispersive, instantaneous, nonlinear response, and in the absence of Raman scattering. For example, if the group velocity $1/\beta_1$ and the phase velocity ω_0/β_0 are similar, one can derive the following generalized NSE [6,7]:

$$i\partial_\zeta\psi + \hat{\mathfrak{D}}\psi + \gamma\left(1 + \omega_0^{-1} i\partial_\tau\right)|\psi|^2\psi = 0, \tag{3}$$

where the derivative of the nonlinear term describes the self-steepening effect and is a key factor in the extension of the envelope-based, generalized NSE toward the single-cycle regime.

If the group and phase velocities differ, one faces a more complex dispersion in the nonlinear term, i.e., it becomes nonlocal in time. Equation (3) is then replaced with [5,8,9]

$$i\partial_\zeta\psi + \hat{\mathfrak{D}}\psi + \gamma\frac{n(\omega_0)}{\omega_0}\frac{\omega_0 + i\partial_\tau}{n(\omega_0 + i\partial_\tau)}|\psi|^2\psi = 0, \tag{4}$$

where $n(\omega) = c\beta(\omega)/\omega$ is the refractive index. If dispersion of the nonlinear susceptibility and the effective fiber area cannot be ignored, an even more complicated operator appears in the nonlinear term in (4). Such a nonlocal, nonlinear term has either (i) to be evaluated in the frequency domain or (ii) to be approximated in the spirit of the Taylor expansion, analogous to Eq. (2).

In what follows, we take the second point of view and expand the nonlocal term in Eq. (4) up to the second order.

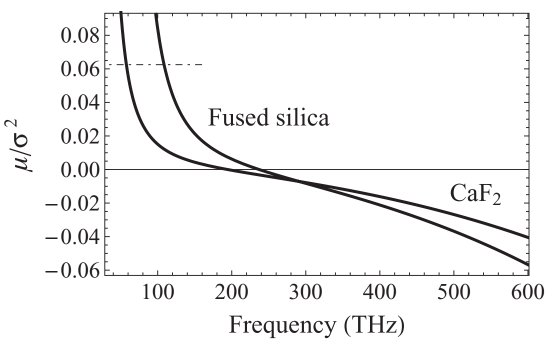

FIG. 1. Dimensionless parameter μ/σ^2 calculated in accordance with Eq. (6) vs carrier frequency $\omega_0/(2\pi)$ for exemplary material dispersions. Refractive indexes are taken from the handbook [10]. The dash-dotted line indicates the parameter value we used for solitary solutions shown in Fig. 5.

After proper rescaling of the space, time, and field variables, we obtain the following model equation:

$$i\partial_z\psi + \frac{1}{2}\partial_t^2\psi + \left(1 + i\sigma\partial_t - \frac{\mu}{2}\partial_t^2\right)|\psi|^2\psi = 0, \qquad (5)$$

where, from now on, z, t, and ψ refer to rescaled variables and σ and μ are dimensionless parameters. The higher-order linear dispersion is neglected. The self-steepening term $\sigma(i\partial_t)|\psi|^2\psi$ may be understood as an intensity-dependent contribution to the group velocity. In this venue, the next term $\frac{1}{2}\mu(i\partial_t)^2|\psi|^2\psi$ describes temporal dispersion of such a contribution.

Equation (5) is invariant with respect to the following rescaling of variables:

$$z \to \Lambda^2 t, \quad t \to \Lambda t, \quad \psi \to \psi/\Lambda,$$
$$\sigma \to \Lambda\sigma, \quad \mu \to \Lambda^2\mu,$$

such that the numerical values of μ and σ depend on normalization. System properties are determined by a single invariant parameter,

$$\frac{\mu}{\sigma^2} = \frac{f(\omega_0)f''(\omega_0)}{[f'(\omega_0)]^2}, \quad \text{with} \quad f(\omega) = \frac{\omega}{n(\omega)}. \qquad (6)$$

Exemplary curves indicating how μ/σ^2 depends on the carrier frequency ω_0 are shown in Fig. 1.

In this work, we are primarily interested in the influence of the μ term on the localized solitary solutions of Eq. (5). For simplicity's sake, such solutions will be first obtained for $\sigma = 0$. Physically, this corresponds to a hypothetic medium in which at some ω_0 the phase velocity is two times larger than the group velocity [8]. Then the solitary solutions will be obtained for $\sigma \neq 0$ and realistic values of μ/σ^2. Positive values of μ are of special interest here because of the interval of carrier frequencies in which $\mu > 0$ appears to be close to the domain of negative dispersion where solitons exist. For completeness, the case $\mu < 0$ (and $\beta_2 < 0$) will be briefly mentioned at the end of the paper.

Without loss of generality, we assume that $|\psi|^2$ takes its maximum value at $t = 0$, which is referred to as peak power P_0. Localized solutions must decay at large $|t|$. Solitons, in particular, are expected to decay exponentially, $\psi \sim \exp(-|t|/t_0)$ for $t \to \pm\infty$. The quantity t_0 is the pulse duration. This parameter and the peak power are mutually related, as it will be shown below. In particular, we will parametrize solitons by t_0 and look for the corresponding pulse shape and P_0.

The classical NSE (1) is recovered from Eq. (5) for $\sigma = \mu = 0$. The so-called fundamental soliton solution is given by

$$\psi = \frac{1/t_0}{\cosh t/t_0} e^{iz/(2t_0^2)}. \qquad (7)$$

For this solution,

$$P_0 t_0^2 = 1. \qquad (8)$$

When $\sigma \neq 0$ but $\mu = 0$, Eq. (5) corresponds to the generalized NSE (3) in which all $\beta_{m\geqslant 3}$ are set to zero. Solitary solutions of Eq. (3) were recently found by adopting a universal Lax pair technique [11]. The shape of a direct generalization of the fundamental soliton solution reads

$$|\psi|^2 = \frac{2/t_0^2}{1 + \sqrt{1 + (\sigma/t_0)^2}\cosh(2t/t_0)}. \qquad (9)$$

It reduces to (7) for $\sigma \ll t_0$. Furthermore,

$$P_0 t_0^2 = \frac{2}{1 + \sqrt{1 + (\sigma/t_0)^2}} < 1, \qquad (10)$$

such that for the same pulse duration t_0, the resulting peak power P_0 appears to be smaller than that of the fundamental soliton (8). Note that as long as Eq. (5) is valid, both (7) and (9) formally allow for an arbitrarily short pulse duration and an arbitrarily high peak power. We recall that the underlying generalized NSE (3) goes beyond the SVEA and applies even to few-cycle pulses [6]. Strictly speaking, $\psi(z,t)$ should then be replaced by an analytic signal for the electric field [12].

In what follows, we demonstrate that Eq. (5) allows for exact solitary solutions, even for $\mu \neq 0$. With an increase of the pulse duration, these solutions receive $1/\cosh$ shape and are indistinguishable from the fundamental NSE soliton (7). A principal new feature is observed for short durations: the value of $P_0 t_0^2$ increases above unity and a cusp of the envelope $|\psi|$ may develop for $\mu > 0$. This limiting singular solution determines the shortest-possible pulse duration and the highest-possible peak power. We explicitly obtain these pulse characteristics in terms of μ and σ.

II. DERIVATION

To derive solitary solutions, we apply the following ansatz:

$$\psi(z,t) = A(t)e^{iz/(2t_0^2)}, \qquad (11)$$

as suggested by the fundamental soliton solution (7). By inserting (11) into (5), we obtain an equation for the complex soliton amplitude $A(t)$,

$$A'' - \frac{A}{t_0^2} + 2|A|^2A + 2i\sigma(|A|^2A)' - \mu(|A|^2A)'' = 0, \qquad (12)$$

where derivatives with respect to t are denoted by a prime. Far from the soliton center, Eq. (12) can be linearized. Then we immediately see that $A \sim \exp(-|t|/t_0)$, in accordance with the definition of t_0. All solitary solutions of Eq. (12) asymptotically behave like the fundamental soliton (7). However, the peak power is generally different from $1/t_0^2$.

In contrast to the simplest NSE case, Eq. (12) does not allow for real valued solutions for $\sigma \neq 0$. Therefore, we introduce both the amplitude and the phase,

$$A(t) = a(t)e^{i\Phi(t)},$$

where, in accordance with our notations,

$$a(0) = \sqrt{P_0}, \quad a'(0) = 0.$$

We now rewrite Eq. (12) in the form

$$[(a - \mu a^3)e^{i\Phi}]''e^{-i\Phi} - \frac{a}{t_0^2} + 2a^3 + 2i\sigma(a^3e^{i\Phi})'e^{-i\Phi} = 0. \tag{13}$$

The imaginary part of this equation is

$$(a - \mu a^3)\Phi'' + 2(a - \mu a^3)'\Phi' + 6\sigma a^2 a' = 0.$$

It can be integrated once

$$\Phi' = -\frac{\sigma}{2}\frac{3 - 2\mu a^2}{(1 - \mu a^2)^2}a^2, \tag{14}$$

where we apply a natural restriction: $\Phi' \to 0$ as $a \to 0$. Without loss of generality, one can assume that the phase $\Phi(t \to -\infty) = 0$, then, in general, $\Phi(t \to +\infty) \neq 0$. The latter value can be found from Eq. (14) after the shape function $a(t)$ is determined.

By using Eq. (14), the real part of Eq. (13) can be transformed to the form

$$(a - \mu a^3)'' - \frac{a}{t_0^2} + 2a^3 + \frac{\sigma^2}{4}\frac{4(1 - \mu a^2)^2 - 1}{(1 - \mu a^2)^3}a^5 = 0.$$

We multiply the latter equation by $(1 - 3\mu a^2)a'$, integrate it once, and obtain

$$(1 - 3\mu a^2)^2 a'^2 - \left(1 - \frac{3}{2}\mu a^2\right)\frac{a^2}{t_0^2} + (1 - 2\mu a^2)a^4 + \frac{\sigma^2}{4}\frac{(1 - 2\mu a^2)^2}{(1 - \mu a^2)^2}a^6 = C, \tag{15}$$

where C is an integration constant. For a solitary solution with finite energy, both $a(t), a'(t) \to 0$ at $t \to \pm\infty$, and therefore, $C = 0$. Formally, all solutions of Eq. (15) can now be found in quadratures.

A noticeable peculiarity of Eq. (15) is that the factor ahead of a'^2 vanishes if $\mu > 0$ and $a = 1/\sqrt{3\mu}$. This is why the singular soliton may appear and impose restrictions on the duration and power of the physically meaningful, nonsingular solutions. We will now see how it happens in the limit $P_0 \to 1/(3\mu)$.

III. SOLITARY SOLUTIONS

We now consider Eq. (15) for $C = 0$ in more detail and describe localized solutions for $a(t)$, which is our main goal here.

A. The case $\mu = 0$

The fundamental soliton is recovered from Eq. (15) when $\mu = 0$. Then Eq. (15) is simplified to the form

$$a'^2 - \frac{a^2}{t_0^2} + a^4 + \frac{\sigma^2}{4}a^6 = 0, \tag{16}$$

and is solved by introducing $b = 1/a^2 - t_0^2/2$. Then one defines $\Phi(t)$ from Eq. (14), inserts $A = a\exp(i\Phi)$ into Eq. (11), and finally obtains the solitary solutions (7) and (9). The soliton peak power is determined from the relation

$$\frac{\sigma^2}{4}P_0^2 + P_0 - \frac{1}{t_0^2} = 0,$$

which leads directly to Eq. (10).

B. The case $\mu > 0$ and $\sigma = 0$

We start with the case $\mu > 0$ and put $\sigma = 0$ first. Equation (15) is then reduced to the following equation:

$$a'^2 \underbrace{- \frac{1 - \frac{3}{2}\mu a^2}{(1 - 3\mu a^2)^2}\frac{a^2}{t_0^2} + \frac{1 - 2\mu a^2}{(1 - 3\mu a^2)^2}a^4}_{U_{\text{eff}}(a)} = 0. \tag{17}$$

The last two terms in this equation can be considered as an effective potential $U_{\text{eff}}(a)$. Then the trajectory $a = a(t)$ defined by this dynamical system belongs to the region $U_{\text{eff}}(a) \leqslant 0$. The peak power is determined from the condition $U_{\text{eff}}(\sqrt{P_0}) = 0$, that is, from the equation

$$P_0 t_0^2 = \frac{1 - \frac{3}{2}\mu P_0}{1 - 2\mu P_0}. \tag{18}$$

We now consider the latter equation for a fixed $\mu > 0$ and various pulse durations t_0. For a temporally wide pulse with $t_0^2 \gg \mu$, we obtain $P_0 = 1/t_0^2$, as it should be for the fundamental soliton. With decreasing t_0, the peak power exceeds $1/t_0^2$. Equation (18) yields physically meaningful values for P_0 as long as

$$t_0 \geqslant t_0^{\min} = 3\sqrt{\frac{\mu}{2}}. \tag{19}$$

This provides the shortest pulse duration, and the largest peak power reads

$$P_0^{\max} = \frac{1}{3\mu}, \quad \left(P_0 t_0^2\right)_{\text{cusp}} = \frac{3}{2}. \tag{20}$$

The scaling law holds for the shortest soliton, but with a different constant; cf. Eqs. (8) and (10). In particular, the peak power is 50% larger than that for the fundamental soliton with the same duration.

These results can also be explained analyzing the effective potential in Eq. (17). For $t_0^2 \gg \mu > 0$, one can neglect all terms $\sim\mu a^2$ and obtain a standard double-well effective potential with the fundamental soliton solution (7). When t_0^2 decreases and approaches $9\mu/2$, the maximum value of a^2 approaches $1/(3\mu)$, such that the singular behavior of the effective potential can no longer be ignored. Representative plots of $U_{\text{eff}}(a)$ for t_0 slightly above and slightly below the critical value (19) are shown in Fig. 2. Clearly, nonsingular

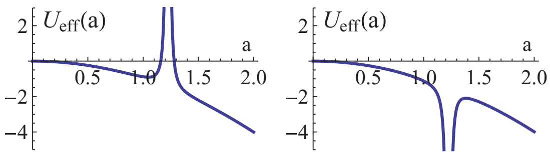

FIG. 2. (Color online) The effective potential in Eq. (17) for $\mu = 2/9$ such that $t_0^{\min} = 1$ and for t_0 slightly above and slightly below $t_0^{\min}$. Left: $t_0 = 1.01$. Right: $t_0 = 0.99$.

localized solutions of Eq. (17) do exist only in the first case. The smallest possible value of t_0 can also be derived by expanding the effective potential at the singularity point $a^2 = 1/(3\mu)$, namely,

$$U_{\text{eff}}(a)|_{a^2\to 1/(3\mu)} \to \frac{t_0^2 - 9\mu/2}{27\mu^2 t_0^2(1-3\mu a^2)^2} + O(1), \tag{21}$$

which exhibits the sign change in the singularity shown in Fig. 2 in accordance with Eq. (19).

A more accurate analysis of Eq. (17) shows that

$$a(t)|_{t_0\to t_0^{\min}} \to \frac{1}{\sqrt{3\mu}} \exp\left(-\frac{\sqrt{2}}{3\sqrt{\mu}}|t|\right), \tag{22}$$

where the latter expression is the limiting soliton shape. It has a cusp at $t = 0$, which actually prevents the existence of solitons with exactly this or shorter time durations.

Numerical solutions of Eq. (17) for a fixed μ and several values of $t_0 \to t_0^{\min}$ are shown in Fig. 3. When decreasing t_0, the soliton shape approaches the uppermost limiting $a(t)$ given by the singular solution (22). The spectrum of the pulse spreads out in the vicinity of the singular solution. One may then wish to replace the envelope equation with a nonenvelope one, e.g., with the so-called short-pulse equation for the negative dispersion domain [13,14] (see also the earlier paper [15]). Remarkably, the cusp solutions reappear and still determine the shortest pulse duration [16,17].

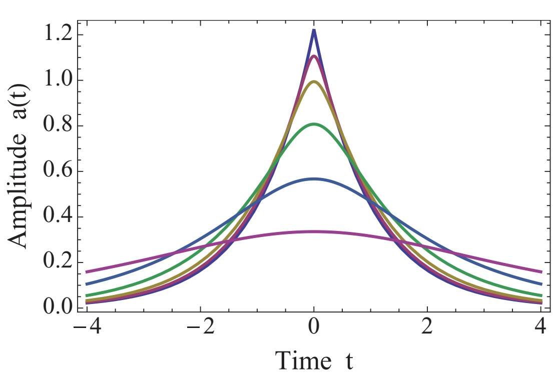

FIG. 3. (Color online) Illustrative solitary solutions of Eq. (17) for $\mu = 2/9$ such that $t_0^{\min} = 1$ and different values of $t_0 \geqslant t_0^{\min}$. From bottom to top, $t_0 = 3.0, 1.8, 1.3, 1.1, 1.03$. The uppermost line is given by the limiting singular solution (22).

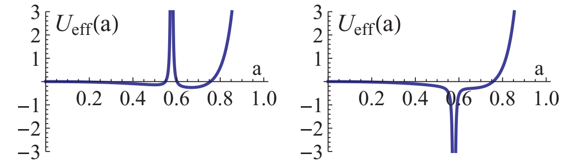

FIG. 4. (Color online) The effective potential in Eq. (23) for $\mu = 1$, $\sigma = 4$ such that $t_0^{\min} = 1.5$ and for t_0 slightly above and slightly below $t_0^{\min}$. Left: $t_0 = 1.52$. Right: $t_0 = 1.48$.

C. The case $\mu > 0$ and $\sigma \neq 0$

We now consider $a(t)$ for $\sigma \neq 0$. Again $C = 0$ in Eq. (15), which still has the form $a'^2 + U_{\text{eff}}(a) = 0$ but with a more complicated effective potential,

$$U_{\text{eff}}(a) = -\frac{1-\frac{3}{2}\mu a^2}{(1-3\mu a^2)^2}\frac{a^2}{t_0^2} + \frac{1-2\mu a^2}{(1-3\mu a^2)^2}a^4 + \frac{\sigma^2}{4}\frac{(1-2\mu a^2)^2}{(1-3\mu a^2)^2(1-\mu a^2)^2}a^6. \tag{23}$$

Analysis of the soliton behavior is now more cumbersome, but the basic features are similar to those described in the previous section. For given μ and σ, the solitary solutions exist as long as $t_0 \geqslant t_0^{\min}$. Two exemplary plots of $U_{\text{eff}}(a)$ for t_0 slightly above and slightly below the critical value are shown in Fig. 4. Evidently, a nonsingular soliton exists only in the first case. This critical value of t_0 can be found from the equation for the peak power [cf. Eq. (18)],

$$P_0 t_0^2\left[1 + \frac{\sigma^2}{4}\frac{1-2\mu P_0}{(1-\mu P_0)^2}P_0\right] = \frac{1-\frac{3}{2}\mu P_0}{1-2\mu P_0}. \tag{24}$$

As in the previous section, the fundamental soliton corresponds to the limiting case, $P_0 \to 1/t_0^2$ for $t_0^2 \gg \mu$. This gives us the possibility to trace the behavior of the peak power with decreasing t_0 for fixed μ and σ. Solitary solutions exist as long as $P_0 \leqslant P_0^{\max}$ and $t_0 \geqslant t_0^{\min}$ with

$$P_0^{\max} = \frac{1}{3\mu}, \quad t_0^{\min} = 3\sqrt{\frac{\mu/2}{1+\sigma^2/(16\mu)}}; \tag{25}$$

cf. Eqs. (19) and (20). In particular, Eq. (20) is replaced with

$$\left(P_0 t_0^2\right)_{\text{cusp}} = \frac{3/2}{1+\sigma^2/(16\mu)}. \tag{26}$$

Typical shapes of solitary solutions when t_0 is decreasing and approaching $t_0^{\min}$ are shown in Fig. 5. They look very similar to those shown in Fig. 3 and also evolve from the standard 1/cosh shape toward the limiting cusp solution. The only difference is that for the same t_0 and μ, the resulting peak power is smaller and decreases with the increase of σ.

D. The case $\mu < 0$

This case is of less interest, as conditions $\beta_2 < 0$ and $\mu < 0$ are compatible in a small domain of frequencies (if at all). Here, the behavior of solitary solutions of the generalized NSE (15) is qualitatively similar to that of the standard NSE (1). Namely, the effective potential in Eq. (15) is a regular function for all

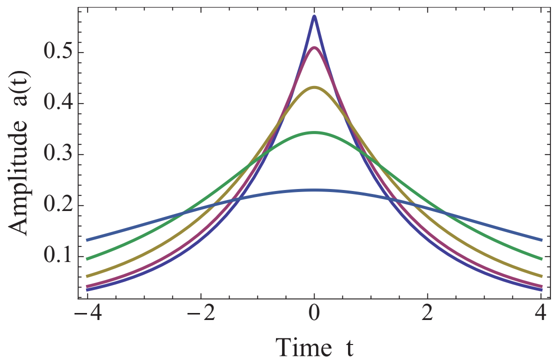

FIG. 5. (Color online) Illustrative solitary solutions of Eq. (15) for $C = 0$, $\mu = 1$, $\sigma = 4$ such that $t_0^{\min} = 3/2$ and different values of $t_0 \geqslant t_0^{\min}$. From bottom to top, $t_0 = 4.0$, 2.5, 1.9, 1.6, 1.501. The uppermost line is close to the limiting singular solution.

possible soliton shapes and the pulse duration parameter t_0 can be arbitrarily small. For extremely short solitons with $t_0 \to 0$, the peak power is determined by the relation

$$P_0 t_0^2 = \frac{3|\mu|}{2\sigma^2 + 4|\mu|} < 1.$$

In particular, the peak intensity is smaller than that of the fundamental soliton.

IV. CONCLUSIONS

We investigated how dispersion of the nonlinear term in the generalized NSE (5) affects fundamental solitons. The first-order dispersion ($\sigma \neq 0$ but $\mu = 0$) causes a reduction of the peak intensity of a soliton. A similar decrease of the peak intensity is observed if $\mu < 0$ is present. In both cases, the pulse duration parameter t_0 can take arbitrarily small values as long as the envelope equation is valid. New effects are expected for $\mu > 0$. This regime appears to be typical in the negative dispersion domain, where optical solitons are observed. In this case, the peak intensity is larger than that of the fundamental soliton. Soliton duration is bounded from below $t_0 > t_0^{\min}$, with the limiting value $t_0^{\min}$ being given by Eq. (26). The limiting soliton has a characteristic cusp profile. Such singular profiles have recently been found also for the nonenvelope pulse-propagation equations [16–18]. Our results suggest that cusp formation may be a universal mechanism responsible for the appearance of the limiting solitons. This cusp formation prevents the existence of solitons with exactly limiting or shorter durations.

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