Critical exponent of a quantum-noise-driven phase transition: The open-system Dicke model

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The quantum phase transition of the Dicke model has been observed recently in a system formed by motional excitations of a laser-driven Bose-Einstein condensate coupled to an optical cavity [Baumann *et al*., Nature (London) **464**, 1301 (2010)]. The cavity-based system is intrinsically open: photons leak out of the cavity where they are detected. Even at zero temperature, the continuous weak measurement of the photon number leads to an irreversible dynamics toward a steady state. In the framework of a generalized Bogoliubov theory, we show that the steady state exhibits a dynamical quantum phase transition. We find that the critical point and the mean field are only slightly modified with respect to the phase transition in the ground state. However, the critical exponents of the singular quantum correlations are significantly different in the two cases. There is also a drastic modification of the atom-field entanglement, since the divergence of the logarithmic negativity of the ground state at the critical point is suppressed and a finite entanglement is found in the steady state.

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I. INTRODUCTION

Experiments with ultracold atomic gases in optical fields laid down a new path to discover strongly correlated many-body quantum systems. In particular, the high degree of control over the interaction parameters allows for using atomic systems as quantum simulators of generic theoretical models [2]. Central to these efforts lies the possibility of observing quantum phase transitions (QPT). At effectively zero temperature ($T = 0$), by tuning an external field, the system can be scanned through a quantum critical point which separates regions with different symmetries in the ground state. One celebrated example is the QPT from a superfluid to a Mott insulator in the Bose-Hubbard model [3] that was realized with a gas of ultracold atoms in an optical lattice [4]. Additional quantum phases appear in this system when a dipole-dipole interaction is present [5].

A fundamental question is how quantum phase transitions are influenced by nonequilibrium conditions. The ordinary way to prepare a stationary system out of equilibrium at $T = 0$ can be illustrated by a BEC in a rotating trap. It undergoes the vortex formation QPT above a critical angular velocity [6]. External driving can impose that only a certain subset of states in the Hilbert space, those having a given moment of inertia in the previous example, be populated. A similar effect has been described for a spin chain in ring geometry: it can manifest criticality while being confined into the subspace of energy current carrying states [7]. In both examples the system is effectively Hamiltonian.

One can go beyond the effectively Hamiltonian systems by adding external nonequilibrium noise on critical states. It was shown that the $1/f$ noise, ubiquitous in electronic circuits, preserves the quantum phase transition in the steady state of a system, moreover, it gives a knob to tune the critical exponent by the noise strength [8]. This is in sharp contrast with the well-known effect of thermal fluctuations that destroy quantum critical correlations. In a more general level, *reservoir engineering* is a route toward designing specific noise sources in a dissipation process which leads to pure many-body states in the dynamical steady state. An example is a lattice gas immersed in a BEC of another species of atoms [9], which serves as a zero-temperature reservoir of Bogoliubov excitations. The resulting dissipative Bose-Hubbard model exhibits a dynamical phase transition between a pure superfluid state and a thermal-like mixed state as the on-site interaction is increased [10]. Note that this method for the preparation of strongly correlated quantum states makes dissipation a resource for quantum simulation [11] and universal quantum computation [12].

In this paper we will consider the bare electromagnetic vacuum at $T = 0$ as a reservoir and its effect on a Dicke-type Hamiltonian system, which is known to produce a singularity of the ground state [13]. Placed into a dissipative environment, the system evolves irreversibly into a steady state which is a dynamical equilibrium between driving and damping. The intrinsic noise accompanying the dissipation process is in accordance with the dissipation-fluctuation theorem. Even in this very natural case of nonequilibrium, at a certain value of the parameters, the correlation functions diverge by power law. That is, the loss does not destroy quantum criticality. But what is the relation of the criticality expected in the steady state to that of the ground state in the closed Hamiltonian system?

Our specific example is the self-organization phase transition of laser-driven atoms in an optical resonator [14–19]. The laser impinges on the atoms from a direction perpendicular to the resonator axis (see Fig. 1). Below a critical value of the pump intensity, the spatial distribution of the atoms is homogeneous along the axis and the mean cavity photon number is zero, since the photons scattered by the atoms into the cavity interfere destructively. Above a threshold pump power, there appears a wavelength-periodic modulation of the distribution, from which laser photons can be Bragg scattered into the resonator. Spontaneous symmetry breaking takes place between two possible solutions for the cavity field phase and the atomic distribution. This is a nonequilibrium phase transition, which has an experimentally accessible $T = 0$ limit if the atomic cloud is represented by a Bose-Einstein condensate. The phase diagram has been experimentally mapped by Baumann *et al*. [1].

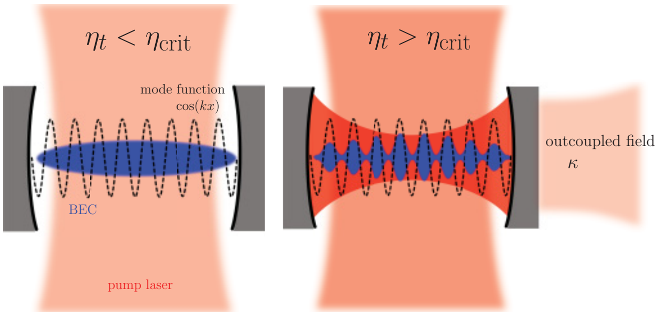

FIG. 1. (Color online) Self-organization phase transition of a BEC in a cavity. Below a threshold in the transverse driving field (left) the condensate is quasihomogeneous, and there are no photons inside the cavity. Above threshold (right), a standing matter wave of period λ appears that scatters photons into the cavity.

II. OPEN-SYSTEM DESCRIPTION

Consider the dispersive coupling of a one-dimensional matter wave field $\Psi(x)$ to a single cavity mode a in the transverse pump geometry shown in Fig. 1. The dispersive limit appears when the laser pump is far detuned from the atomic resonance ($\Delta_A = \omega - \omega_A$ exceeds the atomic linewidth γ by orders of magnitude). In the frame rotating at the pump frequency ω, the many-particle Hamiltonian reads

$$H/\hbar = -\Delta_C\, a^\dagger a + \int_0^L \Psi^\dagger(x)\Big[-\frac{\hbar}{2m}\frac{d^2}{dx^2} + U_0\, a^\dagger a\, \cos^2(kx) + i\eta_t\, \cos kx (a^\dagger - a)\Big]\Psi(x)dx. \tag{1}$$

The detuning $\Delta_C = \omega - \omega_C$ defines the effective photon energy in the cavity. Atom-atom s-wave collisions are neglected; the length of the condensate along the cavity axis is L. The atom-light interaction originates from coherent photon scattering. The absorption of a cavity photon and stimulated emission back into the cavity gives rise to the term proportional to $U_0 = g^2/\Delta_A$. The coherent redistribution of photons between the pumping laser and the cavity mode results in an effective pump with amplitude $\eta_t = \Omega g/\Delta_A$, where the dipole coupling of the atoms to the cavity mode and to the pump laser are characterized by single-photon Rabi frequency g and Rabi frequency Ω, respectively. Note that this term describes the external driving of the system, and the explicit time dependence in the optical frequency range, due to the laser field, has been eliminated by the transformation into the rotating frame. The remaining frequencies are in the kHz range of the recoil frequency $\omega_R = \hbar k^2/2m$.

The critical behavior can be described in a subspace spanned by two motional modes, i.e.,

$$\Psi(x) = \frac{1}{\sqrt{L}}c_0 + \sqrt{\frac{2}{L}}c_1\, \cos kx, \tag{2a}$$

with the bosonic annihilation operators c_0 and c_1. With the closed subspace constraint

$$c_0^\dagger c_0 + c_1^\dagger c_1 = N \tag{2b}$$

imposed, the Hamiltonian of the system formally reduces to that of the Dicke model [20]. Originally, it was introduced to describe the dipole coupling of N two-level atoms to a single quantized field mode [21]. It has been known for a long time that the Dicke model can exhibit a thermodynamic phase transition at finite temperature [22], and a quantum phase transition at zero temperature [13] between an unexcited normal phase and a superradiant phase, where both the atoms and the mode are macroscopically excited. There is a renewed interest in studying the zero-temperature properties of this system with particular respect to critical entanglement [23], finite-size scaling [24,25], and quantum chaos [26]. Dimer *et al.* proposed a realization of the Dicke model with photon loss by means of multilevel atoms coupled to a ring cavity mode via Raman transitions [27]. Another collective spin model exhibiting dynamical QPT, the Lipkin-Meshkov-Glick model, was constructed in cavity QED systems [28].

We consider a single dissipation channel which is the photon leakage through one of the mirrors. The corresponding dissipation process can be modeled by a Heisenberg-Langevin equation for the field amplitude a, which includes a loss term with rate κ and a Gaussian noise operator $\xi(t)$,

$$\frac{d}{dt}a = -i[a\,,H] - \kappa a + \xi. \tag{3}$$

The effect of continuous weak measurement of the photon number is described by the same equation. The noise operator ξ has zero mean and its only nonvanishing correlation is $\langle \xi(t)\xi^\dagger(t')\rangle = 2\kappa\delta(t-t')$ at $T=0$. For finite temperature, other correlations would appear proportional with the thermal photon number. The given second-order correlation expresses the fluctuation-dissipation theorem. The noise operator can be seen as a necessary source for maintaining the commutation relation and general algebraic properties during the time evolution.

This equation applies to the decay of a single uncoupled harmonic oscillator. In the present case, however, the cavity mode interacts with the matter wave field. As was shown in [29], when considering the relaxation of a coupled system, the interacting Hamiltonian has to be diagonalized first, and then the energy-conserving terms in the interaction with the environment can be separated and retained in a Markovian approximation. The actual decay process is photon loss through one of the cavity mirrors which involves the exchange of energy quanta between the system and the reservoir in the optical frequency range. This high-frequency component, which stems from the laser pump, has been eliminated formally from the system dynamics by going into a frame rotating at the pump frequency. The modulation of the system energies due to coupling between the cavity and the matter wave field is in the range between kHz and MHz (determined by the recoil frequency ω_R and the detuning Δ_C, this latter is in the range of κ). Having so many orders of magnitude difference, the original decay process of the photon mode is practically unaffected and Eq. (3) holds for the interacting system. But one must keep in mind that the original problem is intrinsically time dependent and thus there can be an energy current from the laser into the reservoir through the system.

III. STEADY STATE VS GROUND STATE

Taking the $N \to \infty$ limit, the mean-field approach, which can be adopted both in the lossy and lossless cases of the Dicke-model, gives an adequate approximation of the true steady state and the ground state, respectively. The ground state of the Hamiltonian (1) in the subspace (2a) has been calculated previously [20]. The calculation of the steady state in the mean-field approximation is presented in Appendix A. In contrast to the calculation of the ground state, the steady state is obtained from the Heisenberg equations of motion including the damping and the Langevin noise [c.f. the last two terms in Eq. (3)]. It is interesting to note that, although formally one can regain the equations of motion of the closed system by setting κ to zero, the steady-state solution does not tend to the ground state in the $\kappa \to 0$ limit. One can understand this by noting that the time required to reach the steady state diverges with κ^{-1}, and consequently, the open system description is ill defined in this limit.

Figure 2 presents a comparison of the quantum critical behavior in these two cases. The operators are split into their steady-state expectation values and quantum fluctuations,

$$a(t) = \sqrt{N}\alpha_0 + \delta a(t), \qquad c_k(t) = e^{-i\mu t}[\sqrt{N}\gamma_k + \delta c_k(t)], \tag{4}$$

where α_0 corresponds to a coherent state in the cavity, and γ_0,γ_1 are the condensate wave-function components. The exponential factor describes the time evolution of the condensate wave function. The chemical potential μ is the lower eigenvalue of the Gross-Pitaevskii equation that corresponds to Eqs. (A1b) and (A1c) coupling the motional modes. We can choose $\gamma_0 = \sqrt{1-\beta_0^2}$ and $\gamma_1 = \beta_0$, where $0 \leqslant \beta_0 \leqslant 1$. The mean-field amplitudes $|\alpha_0|^2,\beta_0^2$ are plotted by thick lines (right scale) against the pumping strength $y = \sqrt{2N}\eta_t$ normalized to the critical value y_c. The critical point separates two phases with different symmetries. In the normal phase $\alpha_0 = \beta_0 = 0$, and in the superradiant phase there are two mean-field solutions corresponding to $\pm\alpha_0$ and $\gamma_1 = \pm\beta_0$. The critical point y_c depends on κ [see Eq. (A3) in Appendix A], however, the mean-field solutions α_0 and β for the steady state and for the ground state, expressed as a function of y/y_c, overlap. The perfect overlap is not a general property, but there is always a smooth connection so that they coincide for $\kappa \to 0$. This is because the mean-field solution is dominated by the identical Hamiltonian part of the dynamics.

In contrast, the correlation functions of quantum fluctuations may signify different kinds of second-order phase transitions, because the quantum noise source in the open system is dissipation (or the backaction of the measurement), which is completely missing from the closed system. For the matter field we consider the excitation mode δb, which is orthogonal to the condensate wave function. In the normal phase $\delta b = \delta c_1$, while in the superradiant phase $\delta b = -\beta_0\delta c_0 + \sqrt{1-\beta_0^2}\delta c_1$. In Fig. 2, we compare the number of incoherent photons $\langle \delta a^\dagger \delta a \rangle$ [panel (a), thin red lines] and the *condensate depletion* $\langle \delta b^\dagger \delta b \rangle$ [panel (b) thin blue lines] in the steady state (solid lines) and in the ground state (dashed lines). There is a divergence for both cases, however, the exponent is -0.5 for the ground state (usual mean-field exponent), whereas it is -1 for the steady state (see ln-ln scale in Fig. 3). This difference is independent of κ and is related to the different physical origins of the divergence.

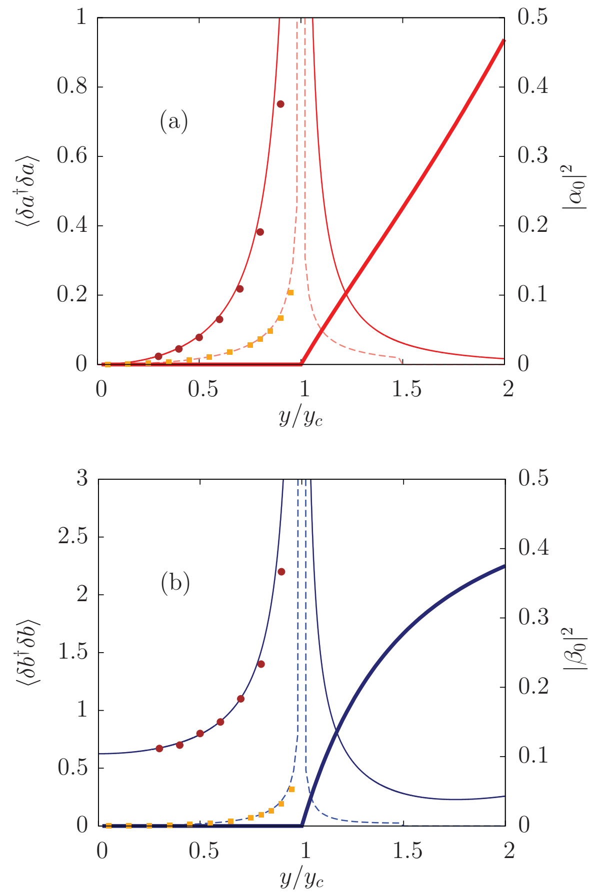

FIG. 2. (Color online) Comparison of the criticality in the ground state of the closed, and in the steady state of the open Dicke models. The mean fields and the incoherent excitation numbers of the photon (a) and the atom (b) fields are plotted as a function of the coupling normalized to its critical value y_c. With this scaling, the mean-field amplitudes (thick solid lines) coincide in the two states. There is a marked difference between the excitation numbers in the steady state (thin solid lines) and in the ground state (thin dashed lines). The points show the numerical results for finite atom numbers, which we obtained using a general quantum simulation framework for cavity QED systems [30]. The steady state (circles) and the ground state (squares) are calculated for $N = 200$ and 400, respectively. The parameters are $\Delta_C = -2$, $U_0 = 0$, and $\kappa = 2\,\omega_R$.

On the one hand, the ground state is a two-mode squeezed state with a squeezing parameter which tends to infinity at the critical point. This results in a singularity of the entanglement between the cavity and the atomic subsystems. On the other hand, the steady state is driven by quantum noise associated with dissipation (or measurement), which heats up the quasinormal mode population infinitely where the imaginary part of its eigenvalue vanishes. The steady state is a mixed state having a regular entanglement at the critical point, reflected by the logarithmic negativity $E_{\mathcal{N}}$ in Fig. 4.

Some final remarks are appropriate about experimental detection. Direct photon counting can discriminate between the ground state and steady state. Because the mean field vanishes

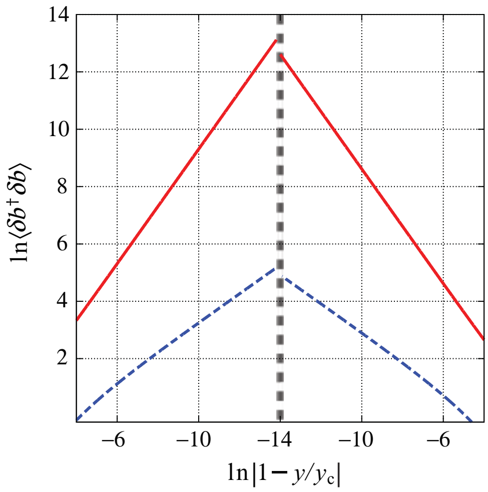

FIG. 3. (Color online) Critical exponents for the phase transition in the ground state and in the steady state. The scaling of the condensate depletion with the relative deviation from the critical point is shown in a ln-ln scale on both sides of the critical point. At the center $|1-y/y_c| = e^{-14}$. We compare the averages in the steady state (solid red lines) for $\kappa = 2\omega_R$ and in the ground state of the system (dashed blue lines) without dissipation ($\kappa = 0$). The corresponding exponents are -1.00 and -0.50, respectively. Other parameters are the same as in Fig. 2.

below threshold ($y < y_c$), any detected photon corresponds to the fluctuations $\langle\delta a^\dagger\delta a\rangle$. As shown in Fig. 2(a) by solid and dashed lines, the photon number of the steady state is above that of the ground state. Therefore a measurement yielding a photon number above the ground-state level would indicate that the system has evolved from the initially prepared ground state toward the steady state due to the quantum noise penetrating into the cavity. For the finite duration of the experiment the actual steady state is not necessarily reached, especially close to the critical point where the damping of the quasinormal modes tends to vanish ("critical slowing down").

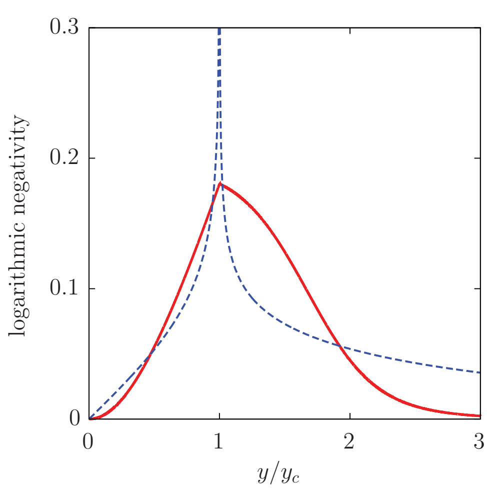

FIG. 4. (Color online) Entanglement at the critical point. Logarithmic negativity [defined in Appendix C, in Eq. (C3)] as a function of the transverse pumping strength y for the steady state (solid red lines, $\kappa = 2\,\omega_R$) and for the ground state (dashed blue lines, $\kappa = 0$). Parameters: $\delta_C = -2$.

IV. DISCUSSION

In this paper we have adapted the famous Dicke model to an intrinsically *nonequilibrium* setting and pointed out distinctive features of this experimentally accessible driven-damped open system. The underlying Dicke problem, a closed, conservative system, has been a subject of intensive research for many decades. It remained an intriguing question, however, what happens with the critical point under nonequilibrium effects? While in classical physics the extension from equilibrium to nonequilibrium systems has been extensively studied, this step has not been made in quantum theory. On the other hand, the ongoing experimental work will significantly shape the research on quantum phase transitions, too. In particular, we believe that the cavity-QED-based systems given in several laboratories worldwide raise relevant new aspects for classifying quantum critical phenomena. In this paper we revealed nonequilibrium critical effects in the case of the simplest possible environment. It consists of a single well-defined dissipation channel (photon leakage out of the cavity), which is equivalent with the backaction of a weak quantum measurement on the photon number observable (continuous photodetection). This innocent looking intrusion in the system drastically modifies the critical exponent of the singularity at the phase transition point.

Note added. Recently, we became aware of similar results by Öztop *et al.* [34].

ACKNOWLEDGMENTS

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APPENDIX A: MEAN-FIELD SOLUTION OF THE TWO-MODE MODEL

We solve the steady state of the system within a mean-field approach, which is consistent with the assumptions that (i) there is a macroscopically populated BEC wave function, and (ii) the state of the cavity field is close to a coherent state. After restricting the atomic dynamics to the spatial modes, Eqs. (2) (see also Ref. [20]), we proceed by obtaining the equations of motion of the operators a, c_0, and c_1. The first equation is the Heisenberg-Langevin equation given by Eq. (3), while the Heisenberg equations of motion for the operators c_i follow directly from the Hamiltonian dynamics provided by Eq. (1).

$$\dot{a} = \left[i\left(\delta_C - \frac{u}{N}c_1^\dagger c_1\right) - \kappa\right]a + \frac{y}{2\sqrt{N}}(c_0^\dagger c_1 + c_1^\dagger c_0) + \xi, \tag{A1a}$$

$$\dot{c}_0 = i\left[\frac{\omega_R}{2} + \frac{u}{2N}a^\dagger a\right]c_0 + \frac{y}{2\sqrt{N}}(a^\dagger - a)c_1, \tag{A1b}$$

$$\dot{c}_1 = -i\left[\frac{\omega_R}{2} + \frac{u}{2N}a^\dagger a\right]c_1 + \frac{y}{2\sqrt{N}}(a^\dagger - a)c_0, \tag{A1c}$$

where we introduce the parameters $\delta_C = \Delta_C - 2u$, $\omega_R = \hbar k^2/(2m)$, $u = NU_0/4$, and $y = \sqrt{2N}\eta_t$. Note that in the thermodynamic limit $N \to \infty$ and $U_0,\eta_t \to 0$, while u and y are kept constant, and they can be expressed with the atom density. The noise operator ξ has zero mean and its only nonvanishing correlation is $\langle \xi(t)\xi^\dagger(t')\rangle = 2\kappa\delta(t-t')$, with 2κ being the photon loss rate [31,32].

By using the decomposition of the operators to mean part and fluctuations, given by Eq. (4), in the equations of motion (A1) and neglecting fluctuations we arrive at the mean-field equations determining α_0, γ_0, and γ_1. Since the BEC wave function is normalized to unity, we can choose $\gamma_0 = \sqrt{1-\beta_0^2}$ and $\gamma_1 = \beta_0$, where $0 \leqslant \beta_0 \leqslant 1$. By choosing β_0 positive, we select one of the two mean-field solutions, which also fixes the phase of the cavity field. For the steady state, we obtain

$$\left[i\left(\delta_C - u\beta_0^2\right) - \kappa\right]\alpha_0 = -y\beta_0\sqrt{1-\beta_0^2}, \tag{A2a}$$

$$(\omega_R + u|\alpha_0|^2)\beta_0 = -y\,\mathrm{Im}(\alpha_0)\frac{1-2\beta_0^2}{\sqrt{1-\beta_0^2}}, \tag{A2b}$$

and the chemical potential $\mu = -\frac{1}{2}(\omega_R + u|\alpha_0|^2)/(1-2\beta_0^2)$.

Note that the coherent field amplitude α_0 is complex, while β_0 is real. The solution $\alpha_0 = \beta_0 = 0$ always satisfies these equations, and it corresponds to the normal phase in which the condensate is homogeneous ($\gamma_0 = 1$) and there is no photon inside the cavity. Above the pumping threshold,

$$y_c^2 = -\omega_R\left(\delta_C^2 + \kappa^2\right)/\delta_C, \tag{A3}$$

the solution bifurcates, and the normal phase loses stability. The stable solution becomes

$$\beta_0^2 = \frac{\delta_C}{u}\left(1 - \sqrt{1 - \frac{u}{\delta_C}\frac{y^2 - y_c^2}{y^2 + u\omega_R}}\right), \tag{A4}$$

which corresponds to the superradiant phase, where the condensate is modulated ($\beta_0 > 0$) and the cavity field is finite ($|\alpha_0|^2 > 0$). For $u = 0$, the expression in Eq. (A4) needs to be reformulated as $\beta_0^2 = (y^2 - y_c^2)/2y^2$, and accordingly the mean-field amplitudes coincide both for the steady state and for the ground state ($\kappa = 0$), if expressed as a function of y/y_c, leaving κ the only role of shifting the critical pumping strength y_c.

The critical behavior is unaffected by u, and the parameters u and y can be tuned independently, thus for simplicity we set $u = 0$.

APPENDIX B: FLUCTUATIONS IN THE STEADY STATE

To go beyond mean field one has to keep the operator-valued fluctuations δa and δc_i in Eqs. (A1). We consider quantum fluctuations up to linear order. Note, that the zeroth-order term vanishes due to the mean-field equations, and we arrive at a set of linear, stochastic differential equations for the fluctuations. There are two types of fluctuations in the atom field ($\delta c_0,\delta c_1$). The zero mode fluctuations, $\delta c = \sqrt{1-\beta_0^2}\delta c_0 + \beta_0\delta c_1$, give rise to a phase diffusion of the condensate. The dynamics of the zero mode decouples from that of the other types of fluctuations. We are interested in the dynamics of the nonzero mode $\delta b = -\beta_0\delta c_0 + \sqrt{1-\beta_0^2}\delta c_1$, which describe the condensate depletion $\delta N = \langle \delta b^\dagger \delta b\rangle$. The coupled equations of motion read

$$\frac{d}{dt}\delta a = \left[i\left(\delta_C - u\beta_0^2\right) - \kappa\right]\delta a + \xi + \left[\frac{y}{2}\left(1-2\beta_0^2\right) - iu\alpha_0\beta_0\sqrt{1-\beta_0^2}\right](\delta b^\dagger + \delta b), \tag{B1a}$$

$$\frac{d}{dt}\delta b = -i\,\frac{\omega_R + u|\alpha_0|^2}{1-2\beta_0^2}\,\delta b + \frac{y}{2}\left(1-2\beta_0^2\right)(\delta a^\dagger - \delta a) - iu\beta_0\sqrt{1-\beta_0^2}\,(\alpha_0\delta a^\dagger + \alpha_0^*\delta a), \tag{B1b}$$

We solve Eqs. (B1a) and (B1b) by calculating the normal mode excitations of the system. Arranging the fluctuations in the vector $\underline{\hat{R}} = [\delta a,\delta a^\dagger,\delta b,\delta b^\dagger]$, Eqs. (B1a) and (B1b) are written in the compact form

$$\frac{\partial}{\partial t}\underline{\hat{R}} = \mathbf{M}\underline{\hat{R}} + \underline{\hat{\xi}}, \tag{B2}$$

where $\mathbf{M}$ is the linear stability matrix of the mean-field solution, and the driving term $\underline{\hat{\xi}} = [\hat{\xi},\hat{\xi}^\dagger,0,0]$ includes the quantum noise of the cavity field. The matrix $\mathbf{M}$ is non-normal, therefore it has different left and right eigenvectors $\underline{l}^{(k)}$ and $\underline{r}^{(k)}$, which form a biorthogonal system, i.e., their scalar product $(\underline{l}^{(k)},\underline{r}^{(l)}) = \delta_{k,l}$. The quasinormal modes defined by $\hat{\rho}_k = (\underline{l}^{(k)},\underline{\hat{R}})$ are decoupled from each other, and evolve as

$$\hat{\rho}_k(t) = e^{\lambda_k t}\hat{\rho}_k(0) + \int_0^t e^{\lambda_k(t-t')}\hat{Q}_k(t')dt'\,. \tag{B3}$$

Generally, the noise enters into all quasinormal modes via the projection $\hat{Q}_k = (\underline{l}^{(k)},\underline{\hat{\xi}})$. Since $\underline{\hat{R}}$ contains the fluctuation operators twice (the operators and their Hermitian adjoint), the operators $\hat{\rho}_k$ also form adjoint pairs $\rho_+,\rho_+^\dagger$ with

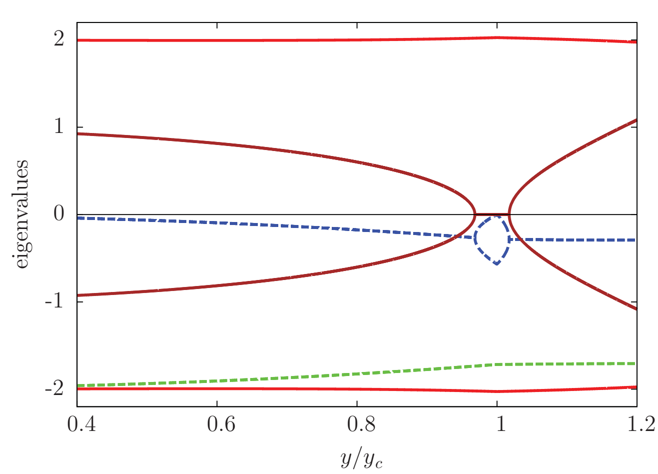

FIG. 5. (Color online) Spectrum of the linear stability matrix $\mathbf{M}$ vs the transverse pump strength y. Solid lines (dashed lines) correspond to the imaginary (real) part of the eigenvalues. The real parts are the same for a complex conjugate pair. Parameters: $\delta_C = -2$, $\kappa = 2\omega_R$.

eigenvalues λ_+, λ_+^* and $\rho_-, \rho_-^\dagger$ with eigenvalues λ_-, λ_-^*. Each pair corresponds to a quasinormal mode excitation of the system. Figure 5 shows the spectrum of the linear stability matrix **M** as a function of the pumping strength y. Solid lines correspond to the imaginary parts, and dashed lines to the real parts of the complex conjugate pairs of eigenvalues λ_k, λ_k^*. (The real parts are the same for each pair.) The conservative BEC and the lossy cavity modes form two dissipative quasinormal excitation modes; both are subjected to quantum noise and damping. One of them is a photonlike mode, with eigenvalues λ_+, λ_+^* starting from $-\kappa \pm i\delta_C$ at $y = 0$, with a slightly increasing frequency and decreasing decay rate as $y \to y_c$. The other quasinormal mode dominantly corresponds to the BEC mode. Its eigenvalues λ_-, λ_-^* are purely imaginary, $\pm i\omega_R$ at $y = 0$, however, with increasing y its decay rate increases and its frequency decreases down to zero. Interestingly, there is a finite interval where the imaginary part of λ_- vanishes. At the lower and upper limits of this interval, the matrix **M** becomes defective, i.e., it has only three independent eigenvectors and three eigenvalues with $\lambda_- = \lambda_-^*$ becoming a multiple eigenvalue. The critical point is reached, where the smallest decay rate becomes zero. At this point, the quantum noise is not balanced by damping, therefore the steady-state excitation numbers diverge.

The second-order correlations of the original fluctuation operators can be derived from the correlations $\langle \hat\rho_k(t)\hat\rho_l(t)\rangle$. In the regime of cavity cooling, where $\delta_C - u\beta_0^2 < 0$, the real parts of the eigenvalues λ_k are negative, thus the first term of Eq. (B3) dies out with time. The steady-state correlations are then obtained from the second term,

$$\langle \hat\rho_k(t)\hat\rho_l(t)\rangle \longrightarrow -\frac{2\kappa}{\lambda_k + \lambda_l} l_1^{(k)*} l_2^{(l)*}. \tag{B4}$$

This result is in contrast to the normal mode expectation values of zero-temperature systems. For such an equilibrium situation the expectation values are simply $\langle \hat\rho_k \hat\rho_l\rangle = 1$, provided that $\hat\rho_k$ is the annihilation, and $\hat\rho_l$ is the creation operator of the same normal mode, i.e., $0 \leqslant \mathrm{Im}(\lambda_l) = -\,\mathrm{Im}(\lambda_k)$.

The correlations of the original system operators can be calculated using their expansion with the quasinormal modes, $\underline{\hat R} = \sum_k \hat\rho_k \underline{r}^{(k)}$, that leads to

$$\langle \hat R_i \hat R_j\rangle = \sum_{k,l} \langle \hat\rho_k \hat\rho_l\rangle\, r_i^{(k)} r_j^{(l)}. \tag{B5}$$

For example, the condensate depletion is given by $\delta N = \langle \delta b^\dagger \delta b\rangle = \langle \hat R_4 \hat R_3\rangle$, while the number of incoherent cavity photons is expressed by $\langle \delta a^\dagger \delta a\rangle = \langle \hat R_2 \hat R_1\rangle$.

APPENDIX C: ATOM-FIELD ENTANGLEMENT

We quantify the entanglement between the BEC and cavity subsystems by calculating the logarithmic negativity from the steady-state correlation matrix. To this end, we introduce the quadrature operators $\delta x = (\delta a + \delta a^\dagger)/\sqrt{2}$, $\delta y = -i(\delta a - \delta a^\dagger)/\sqrt{2}$, $\delta X = (\delta b + \delta b^\dagger)/\sqrt{2}$, and $\delta Y = -i(\delta b - \delta b^\dagger)/\sqrt{2}$ and group them in the vector $\underline{u} = (\delta x, \delta y, \delta X, \delta Y)^T$. As these quadratures are Hermitian, one can construct a real correlation matrix

$$C_{ij} = \tfrac{1}{2}\langle u_i u_j + u_j u_i\rangle, \tag{C1}$$

which has the following block form:

$$\mathbf{C} = \begin{bmatrix} \mathbf{P} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{A} \end{bmatrix}, \tag{C2}$$

where **P** and **A** describe the correlations within the photon and atom fields, while **X** accounts for the cross correlations between the two. The logarithmic negativity can be expressed [33] by the symplectic invariants (det **P**, det **A**, det **X**) of the covariance matrix (C1) as

$$E_{\mathcal{N}} = \max(0, -\ln 2\tilde\nu_-), \tag{C3}$$

and

$$\tilde\nu_- = 2^{-1/2}\sqrt{\Sigma(\mathbf{C}) - \sqrt{\Sigma(\mathbf{C})^2 - 4\,\det \mathbf{C}}}, \tag{C4}$$

where $\Sigma = \det \mathbf{P} + \det \mathbf{A} - 2\det \mathbf{X}$. The state is separable, thus the entanglement is zero if $\tilde\nu_- \geqslant \frac{1}{2}$. The logarithmic negativity quantifies the amount by which this separability criterion is violated.

[1] K. Baumann, C. Guerlin, F. Brennecke, and T. Esslinger, Nature (London) **464**, 1301 (2010).
[2] R. Jördens *et al.*, Phys. Rev. Lett. **104**, 180401 (2010).
[3] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Phys. Rev. Lett. **81**, 3108 (1998).
[4] M. Greiner, O. Mandel, T. Esslinger, T. W. Hansch, and I. Bloch, Nature (London) **415**, 39 (2002).
[5] K. Góral, L. Santos, and M. Lewenstein, Phys. Rev. Lett. **88**, 170406 (2002).
[6] D. Dagnino, N. Barberan, M. Lewenstein, and J. Dalibard, Nat. Phys. **5**, 431 (2009).
[7] T. Antal, Z. Rácz, and L. Sasvári, Phys. Rev. Lett. **78**, 167 (1997).
[8] E. G. Dalla Torre, E. Demler, T. Giamarchi, and E. Altman, Nat. Phys. **6**, 806 (2010).
[9] S. Diehl, A. Micheli, A. Kantian, B. Kraus, H. P. Buchler, and P. Zoller, Nat. Phys. **4**, 878 (2008).
[10] S. Diehl, A. Tomadin, A. Micheli, R. Fazio, and P. Zoller, Phys. Rev. Lett. **105**, 015702 (2010).
[11] H. Weimer, M. Muller, I. Lesanovsky, P. Zoller, and H. P. Buchler, Nat. Phys. **6**, 382 (2010).
[12] F. Verstraete, M. M. Wolf, and J. Ignacio Cirac, Nat. Phys. **5**, 633 (2009).
[13] M. Hillery and L. D. Mlodinow, Phys. Rev. A **31**, 797 (1985).
[14] P. Domokos and H. Ritsch, Phys. Rev. Lett. **89**, 253003 (2002).
[15] J. K. Asbóth, P. Domokos, H. Ritsch, and A. Vukics, Phys. Rev. A **72**, 053417 (2005).
[16] A. T. Black, H. W. Chan, and V. Vuletić, Phys. Rev. Lett. **91**, 203001 (2003).
[17] J. Keeling, M. J. Bhaseen, and B. D. Simons, Phys. Rev. Lett. **105**, 043001 (2010).

[18] S. Gopalakrishnan, B. L. Lev, and P. M. Goldbart, Nat. Phys. **5**, 845 (2009).
[19] S. Fernandez-Vidal, G. De Chiara, J. Larson, and G. Morigi, Phys. Rev. A **81**, 043407 (2010).
[20] D. Nagy, G. Kónya, G. Szirmai, and P. Domokos, Phys. Rev. Lett. **104**, 130401 (2010).
[21] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
[22] K. Hepp and E. H. Lieb, Phys. Rev. A **8**, 2517 (1973).
[23] N. Lambert, C. Emary, and T. Brandes, Phys. Rev. Lett. **92**, 073602 (2004).
[24] J. Vidal and S. Dusuel, Europhys. Lett. **74**, 817 (2006).
[25] T. Liu, Y. Y. Zhang, Q. H. Chen, and K. L. Wang, Phys. Rev. A **80**, 023810 (2009).
[26] C. Emary and T. Brandes, Phys. Rev. Lett. **90**, 044101 (2003).
[27] F. Dimer, B. Estienne, A. S. Parkins, and H. J. Carmichael, Phys. Rev. A **75**, 013804 (2007).
[28] S. Morrison and A. S. Parkins, Phys. Rev. Lett. **100**, 040403 (2008).
[29] H. Zoubi, M. Orenstien, and A. Ron, Phys. Rev. A **67**, 063813 (2003).
[30] A. Vukics, Eur. Phys. J. D **44**, 585 (2007).
[31] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1990).
[32] C. W. Gardiner and M. J. Collett, Phys. Rev. A **31**, 3761 (1985).
[33] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **70**, 022318 (2004).
[34] B. Öztop, M. Bordyuh, Ö. E. Müstecaplioglu, and H. E. Türeci, e-print arXiv:1107.3108.