Acceleration of adiabatic quantum dynamics in electromagnetic fields

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(Received 17 August 2011; published 31 October 2011)

We show a method to accelerate quantum adiabatic dynamics of wave functions under electromagnetic field (EMF) by developing the preceding theory [Masuda and Nakamura, Proc. R. Soc. London Ser. A **466**, 1135 (2010)]. Treating the orbital dynamics of a charged particle in EMF, we derive the driving field which accelerates quantum adiabatic dynamics in order to obtain the final adiabatic states in any desired short time. The scheme is consolidated by describing a way to overcome possible singularities in both the additional phase and driving potential due to nodes proper to wave functions under EMF. As explicit examples, we exhibit the fast forward of adiabatic squeezing and transport of excited Landau states with nonzero angular momentum, obtaining the result consistent with the transitionless quantum driving applied to the orbital dynamics in EMF.

DOI: 10.1103/PhysRevA.84.043434 PACS number(s): 32.80.Qk, 03.65.Ta, 37.90.+j

I. INTRODUCTION

The technology to manipulate tiny objects is rapidly evolving, and nowadays we can control even individual atoms [1]. Various methods to control quantum states have been reported in Bose Einstein condensates (BEC) [2–5] in quantum computing with use of spin states [6] and in many other fields of applied physics. It would be important to consider the acceleration of such manipulations of quantum states for manufacturing purposes and for innovation of technologies. Earlier we proposed [7] the acceleration of quantum dynamics with use of the additional phase of wave functions. We can accelerate a given quantum dynamics and exactly obtain a target state in any desired short time, where the target state is defined as the final state in a given standard dynamics. This kind of acceleration is called fast forward of quantum dynamics.

One of the most important application of the theory of fast forward is the acceleration of quantum adiabatic dynamics presented in our latest work [8], which will hereafter be referred as our preceding work. The adiabatic process occurs when the external parameter of the Hamiltonian of the system is adiabatically changed. Quantum adiabatic theorem [9–11] states that, if the system is initially in an eigenstate of the instantaneous Hamiltonian, it remains so during the adiabatic process [12–19]. The rate of change in the parameter of the Hamiltonian with respect to time is infinitesimal, so that it takes infinite time to reach the final state in the adiabatic process. However, by using our theory [8], the target states (final adiabatic states) are available in any desired short time. The infinitesimally slow change in the adiabatic dynamics is compensated by the infinitely fast forward.

On the other hand, electromagnetic field (EMF) is often used to control quantum states, for example, in manifestation of the quantum Hall effect [20] and manipulation of BEC [3]. The acceleration of the adiabatic dynamics in EMF is far from being trivial, and therefore it is highly desirable to extend our theory of the fast forward to systems under EMF. In this paper we extend the theory of our preceding work [8] to the system in EMF, and derive a driving field to generate the target adiabatic state exactly aside from a spatially uniform phase like dynamical and adiabatic phases [12].

In Sec. II we explain the method of the standard fast forwarding under EMF. Section III is devoted to the fast forward of the adiabatic dynamics with EMF. Section IV is concerned with a technical procedure of removing nontrivial singularities in both the additional phase and driving potential due to nodes proper to wave functions under EMF. Examples for the fast forward of excited Landau states with nonzero angular momentum are given in Sec.V. Summary and discussion are given in Sec. VI. The Appendix explains a way to apply the abstract scheme of transitionless quantum driving to the orbital dynamics of a charged particle in EMF.

II. STANDARD FAST FORWARD

Before embarking on the fast forward of adiabatic dynamics, we shall find the driving field which accelerates the (nonadiabatic) standard dynamics of wave function under the electromagnetic field (EMF) and drives wave function in any desired short time from an initial state to the target state defined as the final state of the standard dynamics. The Hamiltonian for the electronic system with EMF is written as $H_0 = \frac{1}{2m_0}(\mathbf{p} + \frac{e}{c}\mathbf{A}_0)^2 + V_0$. The electric and magnetic fields are related to the vector potential $\mathbf{A}_0$ and scalar potential V_0 as $\mathbf{E}_0 = -\frac{d\mathbf{A}_0}{dt} - \nabla V_0$ and $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$, respectively. For simplicity of notation, we shall put $\frac{e}{c}$ to be 1 hereafter. The Schrödinger equation is represented as

$$i\hbar\frac{\partial \Psi_0}{\partial t} = -\frac{\hbar^2}{2m_0}\nabla^2\Psi_0 - \frac{i\hbar}{2m_0}(\nabla\cdot\mathbf{A}_0)\Psi_0 - \frac{i\hbar}{m_0}\mathbf{A}_0\cdot\nabla\Psi_0 + \frac{\mathbf{A}_0^2}{2m_0}\Psi_0 + V_0\Psi_0. \quad (1)$$

Ψ_0 is a known function of space $\mathbf{x}$ and time t and is called a standard state. For any long time T called a standard final time, we choose $\Psi_0(t = T)$ as a target state that we are going

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to generate. Let $\Psi_\alpha(\mathbf{x},t)$ be a virtually fast-forwarded state of $\Psi_0(\mathbf{x},t)$ defined by

$$|\Psi_\alpha(t)\rangle = |\Psi_0(\alpha t)\rangle, \quad (2)$$

where α (>1) is a time-independent magnification factor of the fast forward.

In general, the magnification factor can be time dependent. Hereafter α is assumed to be time dependent, $\alpha = \alpha(t)$. In this case, the virtually fast-forwarded state is defined as

$$|\Psi_\alpha(t)\rangle = |\Psi_0(\Lambda(t))\rangle, \quad (3)$$

where

$$\Lambda(t) = \int_0^t \alpha(t')dt'. \quad (4)$$

Since the generation of Ψ_α requires an anomalous mass reduction, we cannot generate Ψ_α as it stands [7]. But we can obtain the target state by considering a fast-forwarded state $\Psi_{FF} = \Psi_{FF}(\mathbf{x},t)$ which differs from Ψ_α by an additional space-dependent phase, $f = f(\mathbf{x},t)$, as

$$\Psi_{FF}(t) = e^{if}\Psi_\alpha(t) = e^{if}\Psi_0(\Lambda(t)), \quad (5)$$

where $f = f(\mathbf{x},t)$ is a real function of space $\mathbf{x}$ and time t and is called the additional phase. Schrödinger equation for fast-forwarded state Ψ_{FF} is supposed to be given as

$$i\hbar\frac{\partial\Psi_{FF}}{\partial t} = -\frac{\hbar^2}{2m_0}\nabla^2\Psi_{FF} - \frac{i\hbar}{2m_0}(\nabla\cdot\mathbf{A}_{FF})\Psi_{FF} - \frac{i\hbar}{m_0}\mathbf{A}_{FF}\cdot\nabla\Psi_{FF} + \frac{\mathbf{A}_{FF}^2}{2m_0}\Psi_{FF} + V_{FF}\Psi_{FF}. \quad (6)$$

$V_{FF} = V_{FF}(\mathbf{x},t)$ and $\mathbf{A}_{FF} = \mathbf{A}_{FF}(\mathbf{x},t)$ are called a driving scalar potential and a driving vector potential, respectively. Driving EMF is related with V_{FF} and $\mathbf{A}_{FF}$ as

$$\mathbf{E}_{FF} = -\frac{d\mathbf{A}_{FF}}{dt} - \nabla V_{FF}, \quad (7a)$$

$$\mathbf{B}_{FF} = \nabla\times\mathbf{A}_{FF}. \quad (7b)$$

If we appropriately tune the time dependence of α (the detail will be shown later), the additional phase can vanish at the final time of the fast-forward T_F, and we can obtain the exact target state

$$\Psi_{FF}(T_F) = \Psi_0(T), \quad (8)$$

where T_F is the final time of the fast forward defined by

$$T = \int_0^{T_F}\alpha(t)dt. \quad (9)$$

Substituting Eqs. (1), (4), and (5) into Eq. (6) and taking its real and imaginary parts, we obtain a pair of equations

$$\nabla\cdot\left(\nabla f - \frac{\alpha\mathbf{A}_0 - \mathbf{A}_{FF}}{\hbar}\right) + 2\,\mathrm{Re}[\nabla\Psi_0/\Psi_0] \times\left(\nabla f - \frac{\alpha\mathbf{A}_0 - \mathbf{A}_{FF}}{\hbar}\right) - (\alpha-1)\mathrm{Im}[\nabla^2\Psi_0/\Psi_0] = 0 \quad (10)$$

and

$$\frac{V_{FF}}{\hbar} = -\frac{\partial f}{\partial t} - (\alpha-1)\frac{\hbar}{2m_0}\mathrm{Re}[\nabla^2\Psi_0/\Psi_0] - \frac{\hbar}{m_0}\left(\nabla f - \frac{\alpha\mathbf{A}_0 - \mathbf{A}_{FF}}{\hbar}\right)\cdot\mathrm{Im}[\nabla\Psi_0/\Psi_0] - \frac{\hbar}{2m_0}(\nabla f)^2 + \frac{\hbar}{2m_0}\frac{\alpha\mathbf{A}_0^2 - \mathbf{A}_{FF}^2}{\hbar^2} - \frac{\hbar}{m_0}\frac{\mathbf{A}_{FF}}{\hbar}\cdot\nabla f + \alpha\frac{V_0}{\hbar}, \quad (11)$$

where $f(\mathbf{x},t)$, $\Psi_0(\mathbf{x},\Lambda(t))$, $\alpha(t)$, $\mathbf{A}_0(\mathbf{x},\Lambda(t))$, $\mathbf{A}_{FF}(\mathbf{x},t)$, and $V_{FF}(\mathbf{x},t)$ are abbreviated by f, Ψ_0, α, $\mathbf{A}_0$, $\mathbf{A}_{FF}$, and V_{FF}, respectively. The same abbreviation will be used throughout in this section. We can take the driving scalar potential from Eq. (11) and the additional phase f which is a solution of Eq. (10).

A. Additional phase and driving field

In order to determine the driving field, we should first calculate the additional phase in Eq. (10). Here we derive a general solution of Eq. (10) from the continuity equation for Ψ_0 and Ψ_{FF}. With use of Eq. (1), we have a continuity equation for Ψ_0,

$$\frac{\partial}{\partial t}|\Psi_0|^2 = \frac{\hbar}{m_0}\nabla\cdot\left(\mathrm{Im}[\Psi_0\nabla\Psi_0^*] - \frac{\mathbf{A}_0}{\hbar}|\Psi_0|^2\right), \quad (12)$$

and by using Eqs. (5) and (6), the continuity equation for Ψ_{FF} is

$$\frac{\partial}{\partial t}|\Psi_{FF}|^2 = \frac{\hbar}{m_0}\nabla\cdot\left(-\nabla f|\Psi_0|^2 + \mathrm{Im}[\Psi_0\nabla\Psi_0^*] - \frac{\mathbf{A}_{FF}}{\hbar}|\Psi_0|^2\right), \quad (13)$$

where $\Psi_{FF}(\mathbf{x},t)$ is abbreviated by Ψ_{FF}. From Eq. (5), which is the definition of Ψ_{FF}, we have a relation between time derivatives of Ψ_0 and Ψ_{FF} as

$$\frac{\partial}{\partial t}|\Psi_{FF}|^2 = \alpha\frac{\partial}{\partial t}|\Psi_0|^2. \quad (14)$$

Combining Eqs. (12), (13), and (14), we have the gradient of the additional phase

$$\nabla f(\mathbf{x},t) = (\alpha-1)\mathrm{Im}[\nabla\Psi_0/\Psi_0] + \left(\alpha\frac{\mathbf{A}_0}{\hbar} - \frac{\mathbf{A}_{FF}}{\hbar}\right). \quad (15)$$

Noting the equivalence of gauges for $\mathbf{A}_0$ and $\mathbf{A}_{FF}$ due to the initial condition $[\mathbf{A}_0(t=0) = \mathbf{A}_{FF}(t=0)]$, we can take the gradient of the additional phase and $\mathbf{A}_{FF}$ as

$$\nabla f = (\alpha-1)\mathrm{Im}[\nabla\Psi_0/\Psi_0], \quad (16)$$

$$\mathbf{A}_{FF} = \alpha\mathbf{A}_0. \quad (17)$$

It is easily confirmed that Eqs. (16) and (17) satisfy Eq. (10). Equation (17) implies that MF should be magnified by α times, that is,

$$\mathbf{B}_{FF}(\mathbf{x},t) = \alpha(t)\mathbf{B}_0(\mathbf{x},\Lambda(t)), \quad (18)$$

where $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$ and $\mathbf{B}_{FF} = \nabla \times \mathbf{A}_{FF}$. When the standard state Ψ_0 is expressed in terms of its real amplitude $\tilde{\Psi}_0$ and phase η as

$$\Psi_0(\mathbf{x},t) = \tilde{\Psi}_0(\mathbf{x},t)e^{i\eta(\mathbf{x},t)}, \tag{19}$$

Eq. (16) leads to the expression for the additional phase and its gradient as

$$\nabla f(\mathbf{x},t) = (\alpha - 1)\nabla\eta(\mathbf{x},\Lambda(t)), \tag{20a}$$
$$f(\mathbf{x},t) = (\alpha - 1)\eta(\mathbf{x},\Lambda(t)). \tag{20b}$$

In Eq. (20b) a space-independent constant term was neglected.

Substitution of Eq. (17) into Eq. (11) gives

$$\begin{aligned}\frac{V_{FF}}{\hbar} &= -\frac{\partial f}{\partial t} - (\alpha-1)\frac{\hbar}{2m_0}\mathrm{Re}[\nabla^2\Psi_0/\Psi_0] \\ &\quad -\frac{\hbar}{m_0}\nabla f\cdot \mathrm{Im}[\nabla\Psi_0/\Psi_0] - \frac{\hbar}{2m_0}(\nabla f)^2 \\ &\quad -\frac{\hbar}{m_0}\frac{\alpha\mathbf{A}_0}{\hbar}\cdot\nabla f - \frac{\hbar}{2m_0}\alpha(\alpha-1)\frac{\mathbf{A}_0^2}{\hbar^2} + \alpha\frac{V_0}{\hbar}. \end{aligned}\tag{21}$$

Using Eq. (19) in Eq. (1) we have

$$-\frac{\hbar}{2m_0}\mathrm{Re}\left[\frac{\nabla^2\Psi_0}{\Psi_0}\right] = -\frac{\partial\eta}{\partial t} - \frac{\hbar}{m_0}\frac{\mathbf{A}_0}{\hbar}\cdot\nabla\eta - \frac{\hbar}{2m_0}\frac{\mathbf{A}_0^2}{\hbar^2} - \frac{V_0}{\hbar}. \tag{22}$$

Substituting Eqs. (20) and (22) into Eq. (21) we obtain the driving scalar potential as

$$\begin{aligned}\frac{V_{FF}}{\hbar} &= -\frac{d\alpha}{dt}\eta - 2(\alpha-1)\frac{\partial\eta}{\partial t} \\ &\quad -\frac{\hbar}{2m_0}(\alpha^2-1)\left(\nabla\eta + \frac{\mathbf{A}_0}{\hbar}\right)^2 + \frac{V_0}{\hbar}.\end{aligned}\tag{23}$$

Therefore, once we have Ψ_0 and $\mathbf{A}_0$, the driving field can be obtained from Eq. (7) with use of Eqs. (17) and (23). By applying $\mathbf{E}_{FF}$ and $\mathbf{B}_{FF}$ against the initial standard state, we can generate the target state in any short time T_F related to the standard final time T through Eq. (9).

III. FAST FORWARD OF ADIABATIC DYNAMICS

So far we presented the fast forward of the standard dynamics in EMF which enables us to generate the target state in any desired short time. Now we shall investigate the fast forward of adiabatic dynamics of wave function under EMF in a manner as employed in the preceding work [8]. Here we cannot directly apply the issue in the previous section, and it should be noted: (i) the adiabatic states are merely energy eigenstates of the instantaneous Hamiltonian and are not suitable as a standard state to be accelerated and (ii) since we shall use an infinitely large α to compensate the infinitesimally slow dynamics, the expressions for f in Eq. (20) and V_{FF} in Eq. (23) should diverge as they stand. Hence we first need to regularize the adiabatic state so that it should satisfy the time-dependent Schrödinger equation with an adiabatically time-changing parameter.

Let us consider Ψ_0 under $\mathbf{E}_0$ and $\mathbf{B}_0$ corresponding to the vector potential $\mathbf{A}_0 = \mathbf{A}_0(\mathbf{x},R(t))$ and the scalar potential $V_0 = V_0(\mathbf{x},R(t))$ which adiabatically change through a parameter $R = R(t)$ defined by

$$R(t) = R_0 + \varepsilon t. \tag{24}$$

The constant value ε is the rate of adiabatic change in $R(t)$ with respect to time and is infinitesimally small, that is,

$$\frac{dR(t)}{dt} = \varepsilon, \tag{25}$$
$$\varepsilon \ll 1. \tag{26}$$

The Hamiltonian of the system is represented as

$$H_0 = \frac{[\mathbf{p} + \mathbf{A}_0(\mathbf{x},R(t))]^2}{2m_0} + V_0[\mathbf{x},R(t)], \tag{27}$$

and Schrödinger equation for Ψ_0 is given as

$$\begin{aligned} i\hbar\frac{\partial\Psi_0}{\partial t} &= -\frac{\hbar^2}{2m_0}\nabla^2\Psi_0 - \frac{i\hbar}{2m_0}(\nabla\cdot\mathbf{A}_0)\Psi_0 - \frac{i\hbar}{m_0}\mathbf{A}_0\cdot\nabla\Psi_0 \\ &\quad + \frac{\mathbf{A}_0^2}{2m_0}\Psi_0 + V_0\Psi_0. \end{aligned}\tag{28}$$

If a system is in the nth energy eigenstate at the initial time, the adiabatic theorem guarantees that, in the limit $\varepsilon \to 0$, Ψ_0 remains in the nth energy eigenstate of the instantaneous Hamiltonian throughout the time evolution. Then Ψ_0 is written as

$$\Psi_0(\mathbf{x},t,R(t)) = \phi_n(\mathbf{x},R(t))e^{-\frac{i}{\hbar}\int_0^t E_n(R(t'))dt'}e^{i\Gamma(t)}, \tag{29}$$

where $E_n = E_n(R)$ and $\phi_n = \phi_n(\mathbf{x},R)$ are the nth energy eigenvalue and eigenstate corresponding to the parameter R, respectively, and $\Gamma = \Gamma(t)$ is the adiabatic phase given by

$$\Gamma(t) = i\int_0^t\int_{-\infty}^{\infty} d\mathbf{x}dt\,\phi_n^*\frac{d}{dt}\phi_n, \tag{30}$$

which is independent of space coordinates. ϕ_n fulfills

$$\frac{\partial\phi_n}{\partial t} = 0, \tag{31}$$

$$\begin{aligned} &-\frac{\hbar^2}{2m_0}\nabla^2\phi_n - \frac{i\hbar}{2m_0}(\nabla\cdot\mathbf{A}_0)\phi_n - \frac{i\hbar}{m_0}\mathbf{A}_0\cdot\nabla\phi_n \\ &+\frac{\mathbf{A}_0^2}{2m_0}\phi_n + V_0\phi_n = E_n\phi_n. \end{aligned}\tag{32}$$

The second and the third factors of the right-hand side of Eq. (29) are space-independent dynamical and adiabatic phase factors, respectively, which we will not intend to realize in the fast forwarding. The adiabatic dynamics in the limit $\varepsilon \to 0$ takes infinitely long time until we obtain an aimed adiabatic state (target state).

For the fast forward of the adiabatic dynamics, we should first choose an appropriate standard state and Hamiltonian. The original adiabatic state is not appropriate as the standard state. Quantum dynamics in Eq. (28) with small but finite ε inevitably induces nonadiabatic transition, but Ψ_0 in Eq. (29) ignores such transition. To overcome this difficulty, we regularize the standard state and Hamiltonian corresponding to the adiabatic dynamics [8], so that the following two conditions are satisfied.

(1) A regularized standard Hamiltonian and state used for the fast forward should agree with H_0 and Ψ_0 except for space-independent phase, respectively, in the limit $\varepsilon \to 0$.

(2) The regularized standard state should satisfy the time-dependent Schrödinger equation corresponding to the regularized standard Hamiltonian up to $O(\varepsilon)$ with small but finite ε.

Hereafter $\Psi_0^{(\mathrm{reg})}$ and $H_0^{(\mathrm{reg})}$ denote the regularized standard state and Hamiltonian, respectively, which fulfill the conditions 1 and 2. In the fast forward we take the limit $\varepsilon \to 0$, $\alpha \to \infty$, and $\alpha\varepsilon \sim 1$. Applying this regularization procedure in advance, the adiabatic dynamics $\phi_n(R(0)) \to \phi_n(R(T))$ can be accelerated and the target state $\phi_n(R(T))$ is realized in any desired short time, where T is a standard final time which is taken to be $O(1/\varepsilon)$.

A. Regularization of standard state

Let us regularize the standard state so that it can be fast forwarded with infinitely large magnification factor. Let us consider a regularized Hamiltonian $H_0^{(\mathrm{reg})}$,

$$H_0^{(\mathrm{reg})} = \frac{(\mathbf{p} + \mathbf{A}^{(\mathrm{reg})})^2}{2m_0} + V_0^{(\mathrm{reg})}. \tag{33}$$

The scalar potential $V_0^{(\mathrm{reg})}$ in the regularized Hamiltonian is given as

$$V_0^{(\mathrm{reg})}(\mathbf{x},t) = V_0(\mathbf{x},R(t)) + \varepsilon \tilde{V}(\mathbf{x},t). \tag{34}$$

$\tilde{V}$ is a real function of $\mathbf{x}$ and t to be determined *a posteriori*, which is introduced to incorporate the effect of nonadiabatic transitions. On the other hand, for the vector potential we put $\mathbf{A}^{(\mathrm{reg})}(\mathbf{x},t) = \mathbf{A}_0(\mathbf{x},t)$. It is obvious that $H_0^{(\mathrm{reg})}$ agrees with H_0 in the limit $\varepsilon \to 0$, that is,

$$\lim_{\varepsilon\to 0} H_0^{(\mathrm{reg})}(\mathbf{x},t) = H_0(\mathbf{x},R(t)). \tag{35}$$

The standard state in the adiabatic dynamics should fulfill Schrödinger equation up to $O(\varepsilon)$. We suppose that a regularized standard state is given by

$$\Psi_0^{(\mathrm{reg})} = \phi_n e^{-\frac{i}{\hbar}\int_0^t E_n(R(t'))dt'} e^{i\varepsilon\theta}, \tag{36}$$

where $\theta = \theta(\mathbf{x},t)$ is real, and $\phi_n = \phi_n(\mathbf{x},R(t))$ and $E_n = E_n(R(t))$ are the nth energy eigenstate and eigenvalue of the original Hamiltonian H_0, respectively. ϕ_n satisfies the instantaneous eigenvalue problem in Eq. (32). The Schrödinger equation for regularized standard system is represented as

$$\begin{aligned} i\hbar\frac{\partial \Psi_0^{(\mathrm{reg})}}{\partial t} &= -\frac{\hbar^2}{2m_0}\nabla^2\Psi_0^{(\mathrm{reg})} - \frac{i\hbar}{2m_0}(\nabla\cdot\mathbf{A}_0)\Psi_0^{(\mathrm{reg})} \\ &\quad -\frac{i\hbar}{m_0}\mathbf{A}_0\cdot\nabla\Psi_0^{(\mathrm{reg})} + \frac{\mathbf{A}_0^2}{2m_0}\Psi_0^{(\mathrm{reg})} \\ &\quad + V_0\Psi_0^{(\mathrm{reg})} + \varepsilon\tilde{V}\Psi_0^{(\mathrm{reg})}. \end{aligned} \tag{37}$$

Substituting Eq. (36) into Eq. (37) and eliminating the equation of $O(1)$ with use of Eq. (32), we find the equation for $O(\varepsilon)$:

$$\begin{aligned} i\hbar\frac{\partial\phi_n}{\partial R} - \hbar\frac{d\theta}{dt}\phi_n &= -\frac{\hbar^2}{2m_0}[2i\nabla\theta\cdot\nabla\phi_n + i(\nabla^2\theta)\phi_n] \\ &\quad + \tilde{V}\phi_n + \frac{\hbar}{m_0}\mathbf{A}_0\cdot(\nabla\theta)\phi_n. \end{aligned} \tag{38}$$

Multiplying Eq. (38) by $\frac{i}{\hbar}\phi_n^*$ and taking the real and imaginary parts of the resultant equation, we have

$$\nabla^2\theta + 2\mathrm{Re}[\nabla\phi_n/\phi_n]\cdot\nabla\theta + \frac{2m_0}{\hbar}\mathrm{Re}\left[\frac{\partial\phi_n}{\partial R}\Big/\phi_n\right] = 0, \tag{39}$$

$$\begin{aligned} \frac{\tilde{V}}{\hbar} &= -\mathrm{Im}\left[\frac{\partial\phi_n}{\partial R}\Big/\phi_n\right] - \frac{\hbar}{m_0}\mathrm{Im}[\nabla\phi_n/\phi_n]\cdot\nabla\theta \\ &\quad - \frac{\hbar}{m_0}\left[\frac{\mathbf{A}_0}{\hbar}\cdot\nabla\theta\right]. \end{aligned} \tag{40}$$

From Eq. (39), θ turns out to be dependent on t only through $R(t)$. Therefore the minor term $d\theta/dt (= \varepsilon\frac{\partial\theta}{\partial R})$ was suppressed in Eq. (40). Equations (39) and (40) give θ and $\tilde{V}$, respectively. It is worth noting that θ is not explicitly affected by EMF.

B. Additional phase and driving field for fast forward of adiabatic dynamics

Putting $\phi_n = \tilde{\phi}_n e^{i\eta}$ with the real amplitude $\tilde{\phi}_n$ and phase η, the regularized standard state in Eq. (36) is rewritten as

$$\Psi_0^{(\mathrm{reg})} = \tilde{\phi}_n e^{i(\eta+\varepsilon\theta)} e^{-\frac{i}{\hbar}\int_0^t E_n dt}. \tag{41}$$

With use of $\Psi_0^{(\mathrm{reg})}$ in Eq. (41) instead of Ψ_0 in Eq. (10), we have

$$\begin{aligned} &\tilde{\phi}_n^2\nabla\cdot\left(\nabla f - \frac{\alpha\mathbf{A}_0 - \mathbf{A}_{\mathrm{FF}}}{\hbar}\right) + 2[\tilde{\phi}_n\nabla\tilde{\phi}_n]\left(\nabla f - \frac{\alpha\mathbf{A}_0 - \mathbf{A}_{\mathrm{FF}}}{\hbar}\right) \\ &\quad - (\alpha-1)\big[2\tilde{\phi}_n\nabla(\eta+\varepsilon\theta)\cdot\nabla\tilde{\phi}_n + \tilde{\phi}_n^2\nabla^2(\eta+\varepsilon\theta)\big] = 0, \end{aligned} \tag{42}$$

where $\tilde{\phi}_n(\mathbf{x},R(\Lambda(t)))$, $f(\mathbf{x},t)$, $\mathbf{A}_0(\mathbf{x},R(\Lambda(t)))$, $\mathbf{A}_{FF}(\mathbf{x},t)$, $\eta(\mathbf{x},R(\Lambda(t)))$, and $\theta(\mathbf{x},R(\Lambda(t)))$ are abbreviated by $\tilde{\phi}_n$, f, $\mathbf{A}_0$, $\mathbf{A}_{\mathrm{FF}}$, η, and θ, respectively, and the same abbreviations will be taken hereafter in this section.

Multiplying ϕ_n^* on both sides of Eq. (32) and taking its imaginary part with use of $\phi_n = \tilde{\phi}_n e^{i\eta}$, we have

$$\big[2\tilde{\phi}_n\nabla\eta\cdot\nabla\tilde{\phi}_n + \nabla^2\eta\tilde{\phi}_n^2\big] + \left[\nabla\cdot\frac{\mathbf{A}_0}{\hbar}\tilde{\phi}_n^2 + 2\tilde{\phi}_n\nabla\tilde{\phi}_n\cdot\frac{\mathbf{A}_0}{\hbar}\right] = 0. \tag{43}$$

By eliminating η between Eqs. (42) and (43), it follows:

$$\begin{aligned} &\tilde{\phi}_n^2\nabla\cdot\left(\nabla f - \frac{\mathbf{A}_0 - \mathbf{A}_{\mathrm{FF}}}{\hbar}\right) + 2[\tilde{\phi}_n\nabla\tilde{\phi}_n]\left(\nabla f - \frac{\mathbf{A}_0 - \mathbf{A}_{\mathrm{FF}}}{\hbar}\right) \\ &\quad - (\alpha-1)\varepsilon\big[2\tilde{\phi}_n\nabla\theta\cdot\nabla\tilde{\phi}_n + \tilde{\phi}_n^2\nabla^2\theta\big] = 0. \end{aligned} \tag{44}$$

We can easily confirm that

$$\nabla f - \frac{\mathbf{A}_0 - \mathbf{A}_{\mathrm{FF}}}{\hbar} = (\alpha-1)\varepsilon\nabla\theta \tag{45}$$

satisfies Eq. (44). Noting $\mathbf{A}_{\mathrm{FF}}(t=0) = \mathbf{A}_0(t=0)$, we have the vector potential $\mathbf{A}_{\mathrm{FF}}$ and gradient of the additional phase from Eq. (45) as

$$\mathbf{A}_{\mathrm{FF}}(t) = \mathbf{A}_0(\Lambda(t)), \tag{46}$$

$$\nabla f = (\alpha-1)\varepsilon\nabla\theta, \tag{47}$$

which should be compared with the result in Eqs. (17) and (20) in the case of the standard fast forward. It is noteworthy that we do not have to magnify the MF in the fast forward of adiabatic dynamics, while, in the fast forward of standard dynamics, we need to magnify the MF by α times as shown in Eq. (17). The result in Eq. (45) is also obtained from the continuity equation.

By using $\Psi_0^{(\mathrm{reg})}$ and $V_0^{(\mathrm{reg})}$ instead of Ψ_0 and V_0, respectively, and noting Eqs. (32), (40), (46), (47), and (11) leads to the driving scalar potential as

$$\frac{V_{\mathrm{FF}}}{\hbar} = (\alpha-1)\frac{E_n}{\hbar} - \frac{d\alpha}{dt}\varepsilon\theta - \alpha^2\varepsilon^2\frac{\partial\theta}{\partial R} - \frac{\hbar}{2m_0}\alpha^2\varepsilon^2(\nabla\theta)^2 - \frac{\hbar}{m_0}\alpha\varepsilon\frac{\mathbf{A}_0}{\hbar}\cdot\nabla\theta - \alpha\varepsilon\,\mathrm{Im}\left[\frac{\partial\phi_n}{\partial R}\Big/\phi_n\right] - \alpha\varepsilon\frac{\hbar}{m_0}\mathrm{Im}\left[\frac{\nabla\phi_n}{\phi_n}\right]\cdot\nabla\theta + \frac{V_0}{\hbar}, \tag{48}$$

where we omitted a term of $O(\varepsilon)$. While the first term diverges with infinitely large α, it concerns only a spatially uniform phase of wave function, which we do not care about in the fast forward and can be omitted. Consequently, we have the driving scalar potential

$$\frac{V_{\mathrm{FF}}}{\hbar} = - \frac{d\alpha}{dt}\varepsilon\theta - \alpha^2\varepsilon^2\frac{\partial\theta}{\partial R} - \frac{\hbar}{2m_0}\alpha^2\varepsilon^2(\nabla\theta)^2 - \frac{\hbar}{m_0}\alpha\varepsilon\frac{\mathbf{A}_0}{\hbar}\cdot\nabla\theta - \alpha\varepsilon\,\mathrm{Im}\left[\frac{\partial\phi_n}{\partial R}\Big/\phi_n\right] - \alpha\varepsilon\frac{\hbar}{m_0}\mathrm{Im}\left[\frac{\nabla\phi_n}{\phi_n}\right]\cdot\nabla\theta + \frac{V_0}{\hbar}. \tag{49}$$

The driving field can be obtained from Eqs. (7), (46), and (49). Thus, for the regularized state, one can generate the fast-forwarded state Ψ_{FF} which agrees with ϕ_n at the initial and final time:

$$\Psi_{\mathrm{FF}}(t) = \phi_n(R(\Lambda(t)))e^{i\alpha(t)\varepsilon\theta(R(\Lambda(t)))}, \tag{50}$$

where $R(t) = R_0 + \varepsilon t$ and $\Lambda(t) = \int_0^t \alpha(t')dt'$. Here, because of $\alpha\varepsilon = O(1)$, $R(\Lambda(t))$ varies within a finite time range.

The idea of the acceleration of adiabatic dynamics was also proposed by Berry [21] and by Muga *et al.* [22]. The former presented an algorithm of the transitionless quantum driving (TLQD) by applying Kato's tool [10] used in proof of the adiabatic theorem. The latter showed the driving of wave function in a tunable harmonic trap with use of the invariants of motion [23] and inverse engineering techniques [24], finding a promising time dependence of the trapping frequency in the linear case [25,26] and the nonlinear case [22]. And recently Torrontegui *et al.* proposed the fast atomic transport without vibrational heating [27]. These approaches are concerned with a direct acceleration of the adiabatic states. Contrary to those works the present theory combines opposite ideas of the infinitely fast forward and infinitesimally slow adiabatic dynamics. Furthermore, with use of a suitable space-dependent additional phase, this theory enables us to accelerate adiabatic dynamics of wave function in any potential and EMF against any time dependence of the magnification factor (which begins from and ends at unity) of the fast forward.

IV. SOLUTIONS FOR θ

In the fast forward of adiabatic dynamics, we need to solve Eq. (39) to obtain the additional phase θ and thereby to determine the driving potential V_{FF} in Eq. (49). However, both Eqs. (39) and (49) seem to suggest the divergence of θ and V_{FF} at the nodes of wave function where $\phi_n = 0$. In fact, as ϕ_n is an eigenfunction of the Hamiltonian with the magnetic field, it is typically complex. Then the nodal set of a complex-valued wave function is of codimension two. That is, the nodal set of a two-dimensional complex wave function is made up of points; of a three-dimensional wave function, of curves, etc. Here, by solving Eq. (39) for prototype adiabatic dynamics like dilation and translation of wave function which may have nodes, we shall show that actually there appears no divergence in both θ and V_{FF}. In the case of general nodes, we shall suggest a way to remove such divergence as well.

A. Fast forward of adiabatic dilation

In this case, the position vector is scaled as

$$\mathbf{x}' = \lambda(R)\mathbf{x}, \tag{51}$$

where $\lambda(R)$ is a real scalar function of the adiabatic parameter $R = R(t)$ and stands for the spatial scaling. With use of the normalized eigenstate $\phi_n^{(0)}(\mathbf{x})$ for $\lambda[R(t=0)] = 1$, the adiabatically expanded or contracted state for $\lambda(R(t))(\neq 1)$ is given by

$$\phi_n(\mathbf{x}) = \lambda^{d/2}(R(t))\phi_n^{(0)}(\mathbf{x}'), \tag{52}$$

where d is the spatial dimension and the factor $\lambda^{d/2}$ is required to satisfy the normalization condition,

$$\int |\phi_n(\mathbf{x})|^2 d^d\mathbf{x} = \int \left|\phi_n^{(0)}(\mathbf{x}')\right|^2 d^d\mathbf{x}' = 1. \tag{53}$$

Then we see

$$\frac{\partial\phi_n(\mathbf{x})}{\partial R}\Big/\phi_n(\mathbf{x}) = \frac{d}{2}\frac{d\lambda}{dR}\Big/\lambda + \frac{\nabla_{\mathbf{x}'}\phi_n^{(0)}(\mathbf{x}')}{\phi_n^{(0)}(\mathbf{x}')}\cdot\frac{d\lambda}{dR}\mathbf{x} \tag{54}$$

and

$$\frac{\nabla_{\mathbf{x}}\phi_n(\mathbf{x})}{\phi_n(\mathbf{x})} = \lambda\frac{\nabla_{\mathbf{x}'}\phi_n^{(0)}(\mathbf{x}')}{\phi_n^{(0)}(\mathbf{x}')}. \tag{55}$$

Using Eqs. (54) and (55) in Eq. (39), we find the equation for θ:

$$\nabla_{\mathbf{x}}^2\theta + 2\left(\lambda\nabla_{\mathbf{x}}\theta + \frac{m_0}{\hbar}\frac{d\lambda}{dR}\mathbf{x}\right)\cdot\mathrm{Re}\left[\frac{\nabla_{\mathbf{x}'}\phi_n^{(0)}(\mathbf{x}')}{\phi_n^{(0)}(\mathbf{x}')}\right] + \frac{m_0 d}{\hbar}\frac{d\lambda}{dR}\Big/\lambda = 0. \tag{56}$$

Irrespective of the presence or absence of nodes of wave function $[\phi_n^{(0)}(\mathbf{x}')]$, Eq. (56) has the solution

$$\theta = -\frac{m_0}{\hbar}\frac{d\lambda}{dR}\frac{\mathbf{x}^2}{2\lambda}, \tag{57}$$

which makes the second term in Eq. (56) vanishing and guarantees the equality for the remaining terms.

On the other hand, two apparently divergent terms in Eq. (49) can now be rewritten as

$$-\alpha\varepsilon\,\mathrm{Im}\left[\frac{\partial\phi_n}{\partial R}\bigg/\phi_n\right]-\alpha\varepsilon\frac{\hbar}{m_0}\mathrm{Im}\left[\frac{\nabla_{\mathbf{x}}\phi_n}{\phi_n}\right]\cdot\nabla_{\mathbf{x}}\theta$$
$$=-\alpha\varepsilon\left(\frac{\hbar}{m_0}\lambda\nabla_{\mathbf{x}}\theta+\frac{d\lambda}{dR}\mathbf{x}\right)\cdot\mathrm{Im}\left[\frac{\nabla_{\mathbf{x}'}\phi_n^{(0)}(\mathbf{x}')}{\phi_n^{(0)}(\mathbf{x}')}\right]=0. \quad (58)$$

In obtaining the final equation in Eq. (58), we employed the solution in Eq. (57). Thus, even if $\frac{\nabla_{\mathbf{x}'}\phi_n^{(0)}(\mathbf{x}')}{\phi_n^{(0)}(\mathbf{x}')}$ would have singularities at nodes of wave function, both θ and V_{FF} are free from such singularity. A concrete example of the isotropic dilation in the case of $\lambda=(R(t))^{1/2}$ will be given in Sec. V.

While we have investigated the isotropic dilation, the result can be generalized to the case of anisotropic dilation defined by

$$x_j'=\lambda_j(R)x_j \quad (j=1,\ldots,d), \quad (59)$$

where $\{x_j'\}$ and $\{x_j\}$ are orthogonal space coordinates. Following Eq. (52), the general adiabatic state is given by

$$\phi_n(\{x_j\})=\prod_{j=1}^{d}\lambda_j^{1/2}\phi_n^{(0)}(\{x_j'\}) \quad (60)$$

with use of the eigenstate $\phi_n^{(0)}(\{x_j\})$ for $\lambda_j[R(t=0)]=1$ $(j=1,\ldots,d)$.

Applying the same procedure as in Eqs. (54)–(56), we obtain

$$\theta=-\frac{m_0}{\hbar}\sum_{j=1}^{d}\left(\frac{d\lambda_j}{dR}\bigg/\lambda_j\right)\frac{x_j^2}{2}, \quad (61)$$

which satisfies Eq. (39). Equation (61) reduces to Eq. (57) in the limit of isotropic dilation.

B. Fast forward of adiabatic translation

We then investigate the translation of wave function by introducing the displacement

$$\mathbf{x}'=\mathbf{x}-\mathbf{G}(R), \quad (62)$$

where $\mathbf{G}(R)$ is a real (coordinate) vector function of the adiabatic parameter $R=R(t)$. The adiabatically translated state is now given by

$$\phi_n(\mathbf{x})=\phi_n^{(0)}(\mathbf{x}') \quad (63)$$

with $\phi_n^{(0)}$ the eigenstate for $\mathbf{G}[R(t=0)]=0$.

As before, we see

$$\frac{\partial\phi_n(\mathbf{x})}{\partial R}\bigg/\phi_n(\mathbf{x})=-\frac{d\mathbf{G}}{dR}\cdot\frac{\nabla_{\mathbf{x}'}\phi_n^{(0)}(\mathbf{x}')}{\phi_n^{(0)}(\mathbf{x}')} \quad (64)$$

and

$$\frac{\nabla_{\mathbf{x}}\phi_n(\mathbf{x})}{\phi_n(\mathbf{x})}=\frac{\nabla_{\mathbf{x}'}\phi_n^{(0)}(\mathbf{x}')}{\phi_n^{(0)}(\mathbf{x}')}. \quad (65)$$

Using Eqs. (64) and (65) in Eq. (39), we find

$$\nabla_{\mathbf{x}}^2\theta+2\left(\nabla_{\mathbf{x}}\theta-\frac{m_0}{\hbar}\frac{d\mathbf{G}}{dR}\right)\cdot\mathrm{Re}\left[\frac{\nabla_{\mathbf{x}'}\phi_n^{(0)}(\mathbf{x}')}{\phi_n^{(0)}(\mathbf{x}')}\right]=0, \quad (66)$$

which has the solution

$$\theta=\frac{m_0}{\hbar}\frac{d\mathbf{G}}{dR}\cdot\mathbf{x}. \quad (67)$$

For this solution, the apparently divergent terms in Eq. (49) becomes

$$-\alpha\varepsilon\left(\nabla_{\mathbf{x}}\theta-\frac{m_0}{\hbar}\frac{d\mathbf{G}}{dR}\right)\cdot\mathrm{Im}\left[\frac{\nabla_{\mathbf{x}}\phi_n^{(0)}(\mathbf{x}')}{\phi_n^{(0)}(\mathbf{x}')}\right]=0. \quad (68)$$

Therefore, again, we see that both θ and V_{FF} are free from singularities. A concrete example in the case of $\mathbf{G}=(R(t),0,0)$ is given in Sec. V.

In this way, for other prototype of adiabatic dynamics such as rotations and boosts, one can show that the solutions θ and V_{FF} can be available analytically and be free from singularities.

C. The cases of general nodes

Sometimes the adiabatic states cannot be expressed in terms of the simple transformations described in Secs. IV A and IV B and both θ and V_{FF} should be solved numerically. In such cases, wave function can also have nodes. In the vicinity of each node, wave function is given by the first-order term in Taylor expansion of ϕ_n around it as

$$\Psi=\sum_{j=1}^{d}W_j(R)[x_j-X_j(R)], \quad (69)$$

where $W_j(R)\equiv\frac{\partial\phi_n^{(0)}}{\partial x_j}|_{x_j=X_j(R)}(j=1,2,\ldots,d)$ are complex coefficients, and $X_j(R)$ stand for the location of the node. The expression in Eq. (69) indicates a combination of anisotropic complex scaling and translations investigated so far.

In the vicinity of the above node, one can obtain θ and V_{FF} which are free from the singularity. Below we shall give a proof in the case

$$\frac{d\ln W_j(R)}{dR}=\text{real} \quad (j=1,2,\ldots,d) \quad (70)$$

which still keeps a complex nature of $\{W_j\}$. Equation (70) corresponds to a widely applicable case that $\arg(W_j)$ is arbitrary but remains unchanged against a small change of R.

We start from a general adiabatic state as before:

$$\phi=\left(\prod_{j=1}^{d}W_j\right)^{1/2}\Psi. \quad (71)$$

We see

$$\frac{d\phi}{dR}\bigg/\phi=\frac{1}{2}\sum_{j=1}^{d}\frac{dW_j}{dR}\bigg/W_j+\left\{\sum_{j=1}^{d}\frac{\partial W_j}{\partial R}[x_j-X_j(R)]-\sum_{j=1}^{d}W_j(R)\frac{dX_j}{dR}\right\}\bigg/\Psi \quad (72)$$

and

$$\frac{\partial\phi}{\partial x_j}\bigg/\phi=\frac{W_j(R)}{\Psi}. \quad (73)$$

Using Eqs. (72) and (73) in Eq. (39), we find the equation for θ, which can be solved in the case of Eq. (70) as

$$\theta = \frac{m_0}{\hbar}\left[-\sum_{j=1}^{d}\left(\frac{dW_j}{dR}\Big/W_j\right)\frac{(x_j-X_j)^2}{2}+\sum_{j=1}^{d}\frac{dX_j}{dR}x_j\right]. \tag{74}$$

This solution makes two divergent terms in Eq. (49) vanishing. While one should numerically solve θ and V_{FF} in areas other than the vicinity of nodes, the analytical expression in Eq. (74) can be used in their vicinity and is useful to see the absence of the divergence in θ and V_{FF}. In the case of a nodal set made up of curves, $x_j = X_j(\tau,R)\,(j=1,2,\ldots,d)$ give a parametric representation of the nodal curve; of surfaces, $x_j = X_j(\kappa,\tau,R)$ $(j=1,2,\ldots,d)$ give the nodal surface, etc., where τ with κ are suitable parameters characterizing the nodal set. Then one can as well apply the argument from Eq. (69) through Eq. (74). In other cases when $\arg(W_j)$ changes with R, the proof in this subsection is not justified, and the problem of removing the singularity remains unsolved.

V. SOME EXPLICIT EXAMPLES

Section IV elucidated how the space $\mathbf{x}$ and parameter R derivatives of adiabatic eigenstates are interrelated and therefore sweep away the problem of singularities. Now we shall show prototype examples of the fast forward of adiabatic dynamics in EMF. As adiabatic states we choose the excited states of the Landau states with nonzero angular momentum in the uniform magnetic field which have nodal points with the phase singularity. Then we investigate the acceleration of the adiabatic dilation (expansion or contraction) and the adiabatic transport of wave function and see that θ and V_{FF} are free from singularities.

A. Acceleration of adiabatic dilation in EMF

We consider the fast forward of adiabatic-isotropic dilation of wave function. Suppose $c\varphi_n(\mathbf{x})$ is energy eigenstate under vector potential $\mathbf{A}(\mathbf{x})$ and scalar potential $V(\mathbf{x})$. c is a normalization constant. Let $\phi_n(\mathbf{x})$ to be an energy eigenstate which represents adiabatic expansion or contraction. The normalized state is given by

$$\phi_n = C(R)\varphi_n(\sqrt{R}\mathbf{x}), \tag{75}$$

with the adiabatic parameter R and the normalization constant $C(R) = cR^{d/4}$, where d is the space dimension. Then, the rescaled wave function ϕ_n is an energy eigenstate of Hamiltonian with

$$\mathbf{A}_0(\mathbf{x}) = \sqrt{R}\mathbf{A}(\sqrt{R}\mathbf{x}), \tag{76}$$

$$V_0(\mathbf{x}) = RV(\sqrt{R}\mathbf{x}). \tag{77}$$

The corresponding energy is represented as $E_n(R) = RE_n(R=1)$, where $E_n(R=1)$ is equal to the energy of $\varphi_n(\mathbf{x})$. (Here we see no problem of anomalous mass scaling.) The additional phase θ is now obtained by putting $\lambda = \sqrt{R}$ in the formula in Eq. (57) as

$$\theta = -\frac{m_0}{4\hbar R}\mathbf{x}^2. \tag{78}$$

In a similar way the driving scalar potential is expressed as

$$\frac{V_{\mathrm{FF}}}{\hbar} = \left(\frac{d\alpha}{dt}\varepsilon\frac{m_0}{4\hbar R} - \alpha^2\varepsilon^2\frac{3m_0}{8\hbar R^2}\right)\mathbf{x}^2 + \frac{\alpha\varepsilon}{2R}\frac{\mathbf{A}_0}{\hbar}\cdot\mathbf{x} + \frac{V_0}{\hbar}. \tag{79}$$

As an example we consider an adiabatically squeezed wave function in two dimensions (xy plane) under the MF in z direction adiabatically increasing as $B_z = R(t)$, where $R(t) = R_0 + \varepsilon t$ as given in Eq. (24). The vector potential corresponding to the MF can be taken as

$$\mathbf{A}_0 = B\left[-\frac{R(t)y}{2},\frac{R(t)x}{2}\right]. \tag{80}$$

In this example the scalar potential V_0 is put to zero. Energy eigenstates of the instantaneous Hamiltonian, which satisfies $\Psi \to 0$ at $|\mathbf{x}| \to \infty$, are represented in polar coordinates as

$$\Psi_{N,M}(r,\theta) = \frac{1}{\sqrt{2\pi}}\exp(iM\theta)\phi_{N,M}(r), \tag{81}$$

with

$$\phi_{N,M}(r) = \sqrt{\frac{N!}{(N+|M|)!}}\frac{1}{l}\left(\frac{r}{\sqrt{2}l}\right)^{|M|} \times \exp\left(-\frac{r^2}{4l^2}\right)L_N^{(|M|)}\left(\frac{r^2}{2l^2}\right) \tag{82}$$

with integers N and M. N is equal or greater than 0, while M can be negative. $L_N^{(|M|)}(r)$ is the generalized Laguerre polynomials. l is defined by $l \equiv \sqrt{\hbar/BR}$. The eigenenergy of the eigenstate in Eq. (81) is given by $E_{N,M} = (N + \frac{|M|-M}{2} + \frac{1}{2})\hbar\omega_c$ with $\omega_c = BR/m$. We choose the eigenstate in Eq. (81) as a squeezed state, which is squeezed when MF is increased. We accelerate the manipulation which controls the width of wave function. The driving scalar potential is obtained from Eq. (79) as

$$\frac{V_{\mathrm{FF}}}{\hbar} = \left(\frac{d\alpha}{dt}\varepsilon\frac{m_0}{4\hbar R} - \alpha^2\varepsilon^2\frac{3m_0}{8\hbar R^2}\right)r^2, \tag{83}$$

where we omitted a spatially uniform term concerned with the spatially uniform phase. The driving EMF can be obtained from Eqs. (7a) and (7b) with the use of Eqs. (46) and (83). The driving potential in (83) does not show divergence while there is a singularity in the phase of wave function in (81) at $r = 0$.

For numerical calculation we choose a first excited state

$$\Psi_{1,1}(r,\theta) = \frac{1}{2\sqrt{2\pi}}\frac{r}{l^2}\exp(i\theta)\exp\left(-\frac{r^2}{4l^2}\right)L_1^{(1)}\left(\frac{r^2}{2l^2}\right) \tag{84}$$

with $N = M = 1$. The magnification factor is chosen (for $0 \leqslant t \leqslant T_F$) in the form

$$\alpha(t)\varepsilon = \bar{v}\left[1-\cos\left(\frac{2\pi}{T_F}t\right)\right], \tag{85}$$

where $\bar{v}$ is time average of $\alpha(t)\varepsilon$ during the fast forwarding, and the final time of the fast forward T_F is related to the standard final time T with $\bar{v}$ as $T_F = \varepsilon T/\bar{v}$ [see Eq. (9)]. εT and T_F are taken as any finite value, although ε is infinitesimal and T is infinitely large. Namely we aim to

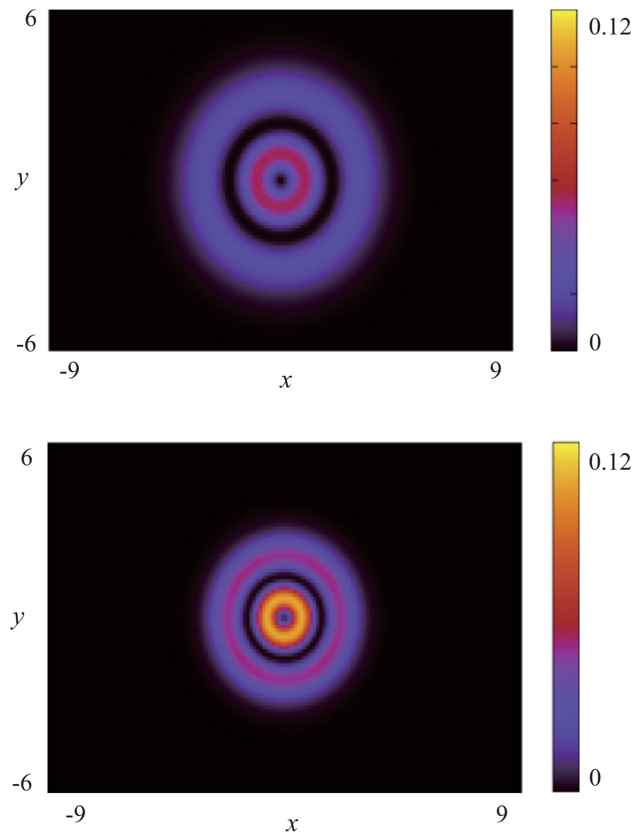

FIG. 1. (Color online) Wave function profile $|\Psi_{\rm FF}|$ before (upper figure) and after (lower figure) the squeezing. Space coordinates and time are scaled by $L = 10^{-2}\times$ *the linear dimension of a device* and $\tau = 10^{-2}\times$ *the phase coherent time*, respectively. The parameters are taken as $\frac{m_0}{\hbar} = 1.0(\times \tau L^{-2})$, $T_F = 1.0(\times\tau)$, $\bar{v} = 1.0(\times\tau^{-1})$, $B = 1.0(\times m_e^{1/2}\tau^{-1}L^{-1/2})$ and $R_0 = 1.0$ (dimensionless).

generate the target state in finite time, while the state is supposed to be obtained after infinitely long time T in the original adiabatic dynamics. $\alpha\varepsilon$ starts from zero, takes time-dependent value of $O(1)$, and comes back to $zero$ at the end of the fast forward. With the use of typical space and time scales like $L = 10^{-2}\times$ *the linear dimension of a device* and $\tau = 10^{-2}\times$ *the phase coherent time*, the parameters are chosen as $\frac{m_0}{\hbar} = 1.0(\times\tau L^{-2})$, $T_F = 1.0(\times\tau)$, $\bar{v} = 1.0(\times\tau^{-1})$, $B = 1.0(\times m_e^{1/2}\tau^{-1}L^{-1/2})$, and $R_0 = 1.0$(dimensionless), where m_e is the mass of an electron. The wave function profile is shown in Fig. 1 at the initial (upper figure) and final (lower figure) time of the fast forward. It can be seen the wave function is squeezed successfully. To check the accuracy of the acceleration we evaluated the fidelity which is defined by

$$F = |\langle\Psi_{FF}(t)|\Psi_0(\Lambda(t))\rangle|, \tag{86}$$

that is, the overlap between the fast-forwarded state $\Psi_{\rm FF}(t)$ and the corresponding standard one $\Psi_0(\Lambda(t))$. It is unity when $\Psi_{\rm FF}(t) = \Psi_0(\Lambda(t))$. We confirmed that the fidelity first decreases from unity due to the additional phase f of the fast-forwarded state, but at the final time it becomes unity again (see Fig. 2), which means the exact fast forward of the adiabatic state aside from the spatially uniform phase factor.

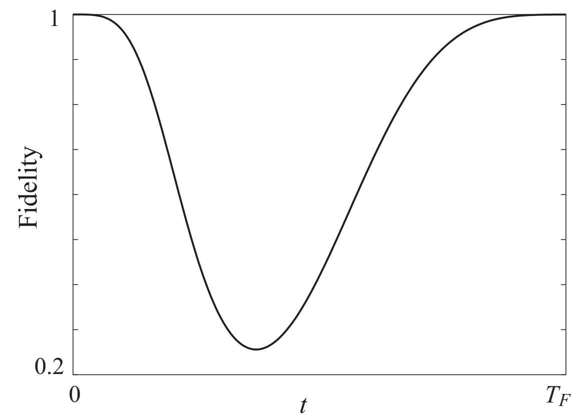

FIG. 2. Time dependence of fidelity F defined by Eq. (86). The fidelity is calculated during the acceleration of squeezing of wave function of which initial and final wave function distributions are shown in Fig. 1. The time is scaled by $\tau = 10^{-2}\times$ *the phase coherent time*. The same prescription is also employed in Fig. 4.

The result given here may also be obtained by applying the method of transitionless quantum driving (TLQD) [21]. The details are given in the Appendix. As already emphasized at the end of Sec. III, however, the method of TLQD is limited to a direct acceleration of the adiabatic states. And sometimes the formal result [see Eq. (A2)] leads to unrealistic Hamiltonians with time-dependent mass, which cannot be accepted experimentally. By contrast, the present theory combines opposite ideas of the infinitely fast forward and infinitesimally slow adiabatic dynamics, and against any time dependence of the magnification factor $\alpha(t)\epsilon$, gives the realistic driving potential by having recourse to a space-dependent additional phase. Therefore the present scheme is much more flexible than the method of TLQD.

B. Acceleration of adiabatic transport in EMF

Then we shall investigate another prototype, that is, the fast forward of adiabatic transport of wave function under EMF, without leaving any disturbance on the wave function at the end of the transport.

The wave function takes a form $\psi(\mathbf{x})e^{-\frac{i}{\hbar}E_n t}$, which is stationary except for the spatially uniform phase, in the presence of vector potential $\mathbf{A}(x,y,z)$ and scalar potential $U(x,y,z)$ at the initial time. The EMF is adiabatically shifted with infinitesimal velocity ε in x direction. The shifted vector potential $\mathbf{A}_0(\mathbf{x},t)$ and scalar potential $V_0(\mathbf{x},t)$ are represented with use of $\mathbf{A}$ and U as

$$\mathbf{A}_0 = \mathbf{A}(x-\varepsilon t,y,z), \tag{87a}$$
$$V_0 = U(x-\varepsilon t,y,z), \tag{87b}$$

respectively. The nth energy eigenstate of the instantaneous Hamiltonian is written as

$$\phi_n = \psi(x-\varepsilon t,y,z) = \psi[x-R(t),y,z]. \tag{88}$$

$R(t)$ which characterizes the position of wave function in x direction is adiabatically changed as $R(t) = R_0 + \varepsilon t$ with

$\varepsilon \ll 1$ and $R_0 = 0$. Using $\mathbf{G} = (R,0,0)$ in the formula in Eq. (67), we obtain the additional phase as

$$\theta = \frac{m_0}{\hbar} x. \tag{89}$$

Similarly the driving scalar potential is given by

$$V_{\rm FF}(x,y,z,t) = -m_0 \frac{d\alpha}{dt} \varepsilon x - \alpha\varepsilon A_x + U[x - R(\Lambda(t)),y,z]. \tag{90}$$

The driving MF is obtained from the vector potential determined from Eq. (46) as

$$\mathbf{A}_{\rm FF}(x,y,z,t) = \mathbf{A}[x - R(\Lambda(t)),y,z]. \tag{91}$$

The driving electric field is given by

$$\mathbf{E}_{\rm FF}(x,y,z,t) = \varepsilon\alpha \left(\frac{\partial \mathbf{A}}{\partial R}\right) - \nabla V_{\rm FF}. \tag{92}$$

It should be emphasized that since we have derived θ without giving any specific profile on ϕ_n, the formulas for the driving scalar potential in Eq. (90) and the driving vector potential in Eq. (91) are independent of the profile of wave function and are free from the problem of divergence. The resultant electric field due to the second term in Eq. (90) and the time derivative of $\mathbf{A}_{\rm FF}$ can be interpreted as balancing with Lorenz force perpendicular to the transport.

As a concrete example of the fast forward of adiabatic transport, we consider a wave function in two dimensions (xy plane) under uniform MF $B_z = B$ without the original scalar potential ($V_0 = U = 0$). We choose the vector potential as

$$\mathbf{A}_0 = \left[-\frac{By}{2}, \frac{B(x - \varepsilon t)}{2}\right], \tag{93}$$

which leads to B_z and electric field of $O(\varepsilon)$ in y direction:

$$\mathbf{E}_0 = -\frac{d\mathbf{A}_0}{dt} = \left(0, \frac{\varepsilon B}{2}\right). \tag{94}$$

From Eqs. (91) and (93) it is obvious that we do not have to change MF for the fast forward. A first excited state of the instantaneous Hamiltonian with energy $E_0 = \frac{3\hbar B}{2m_0}$ is given as

$$\Psi_{1,1}(x,y) = \frac{1}{2\hbar}\sqrt{\frac{1}{2\pi}} B[(x - R) + iy] \times \exp\left\{-\frac{B}{4\hbar}[(x - R)^2 + y^2]\right\} \times L_1^{(1)}\left(\frac{B[(x - R)^2 + y^2]}{2\hbar}\right). \tag{95}$$

Note that ϕ_n in Eq. (95) is a stationary state with an instantaneous value of R. We transport this state by the driving field. In this case, the driving scalar potential in Eq. (90) is represented as

$$V_{\rm FF}(x,y,z,t) = -m_0 \frac{d\alpha}{dt} \varepsilon x + \frac{\alpha\varepsilon B y}{2}. \tag{96}$$

In the numerical calculation $\alpha(t)\varepsilon$ is chosen as in Eq. (85) and the parameters are chosen as $\frac{m_0}{\hbar} = 1.0(\times \tau L^{-2})$, $T_F = 1.0(\times\tau)$, $\bar{v} = 8.0(\times L\tau^{-1})$, $B = 2.0(\times m_e^{1/2}\tau^{-1}L^{-1/2})$, and $R_0 = -4(\times L)$. By applying the driving potential in Eq. (96), we accelerate wave function. In Fig. 3 the wave function profile $|\Psi_{\rm FF}|$ is shown at the initial and final time of the fast forward. The wave function is transported by distance $8.0(\times L)$ in time $1.0(\times\tau)$ and becomes stationary at the end. We confirmed that wave function is moved without changing its amplitude profile during the acceleration. We evaluated the fidelity defined by Eq. (86) and confirmed that it becomes back to unity at the end of the fast forward (see Fig. 4). Thus we have obtained the adiabatically accessible target state in a finite time $T_F = 1.0(\times\tau)$.

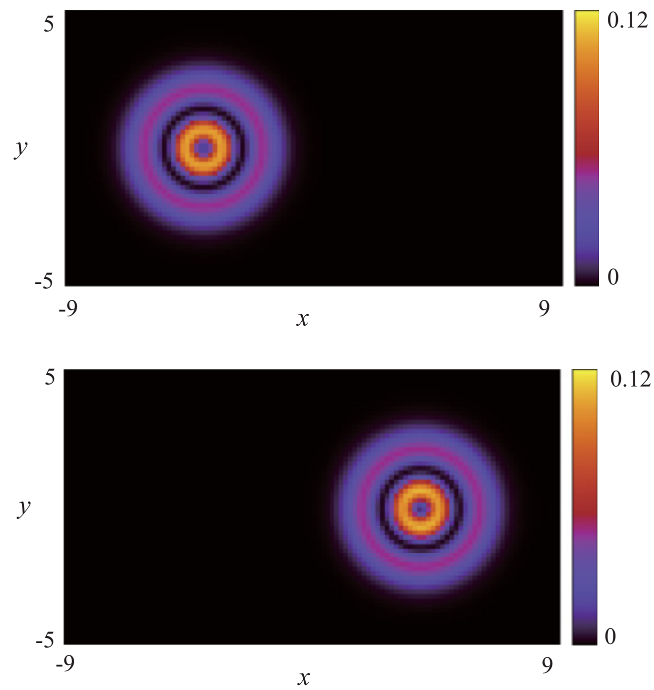

FIG. 3. (Color online) Wave function profile $|\Psi_{\rm FF}|$ at initial (upper figure) and final (lower figure) time of the fast forward. The parameters are taken as $\frac{m_0}{\hbar} = 1.0(\times\tau L^{-2})$, $T_F = 1.0(\times\tau)$, $\bar{v} = 8.0(\times L\tau^{-1})$, $B = 2.0(\times m_e^{1/2}\tau^{-1}L^{-1/2})$, and $R_0 = -4(\times L)$.

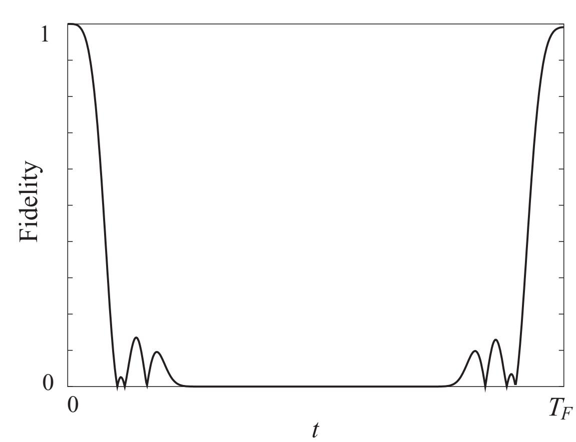

FIG. 4. Time dependence of fidelity F defined by Eq. (86). The fidelity is calculated during the acceleration of transport of wave function of which initial and final wave function distributions are shown in Fig. 3.

VI. CONCLUSION

We have presented the theory of the fast forward of quantum adiabatic dynamics in electromagnetic field (EMF) by developing our preceding work. We combined opposite ideas of the infinitely fast forward and infinitesimally slow adiabatic dynamics. We derived the driving force which accelerates the adiabatic dynamics and enables us to obtain the final adiabatic states aside from the spatially uniform phase in any desired short time, while the final state is accessible after infinite time in the original adiabatic dynamics. For the fast forward of adiabatic dynamics in EMF, we must control the driving field, but there is no need to magnify the magnetic field, while in the standard fast forward, the magnification of the magnetic field is inevitable. The scheme is consolidated by elucidating a way to overcome possible singularities in both the additional phase and driving potential due to nodes of wave functions proper to systems under EMF. As typical examples we showed fast forward of adiabatic squeezing and transport of the excited Landau states which have phase singularities around the nodes. The issue of the present work is consistent with the scheme of transitionless quantum driving applied to the orbital dynamics of a charged particle in EMF.

ACKNOWLEDGMENTS

S.M. thanks global COE program “Weaving Science Web beyond Particle-Matter Hierarchy” for its financial support. We thank B. Abdullaev, B. Baizakov, M. Berry, D. Matraslov, J. Muga, S. Sawada, and S. Tanimura for useful discussions and comments.

APPENDIX: TRANSITIONLESS QUANTUM DRIVING APPLIED TO THE ORBITAL DYNAMICS IN ELECTROMAGNETIC FIELD

The driving field in Eq. (79) may also be obtained by applying the scheme of transitionless quantum driving (TLQD) proposed by Berry [21]. In this Appendix we shall closely follow the scheme of TLQD, and construct the driving Hamiltonian $\tilde{H}_{\mathrm{FF}}$ so that the fast-forwarded adiabatic state should satisfy the time-dependent Schrödinger equation. Let $|\phi\rangle \equiv e^{-\frac{i}{\hbar}E_n t}|\phi_n\rangle$ denote a stationary state with E_n and $|\phi_n\rangle$ the eigenvalue and eigenstate of the time-independent electronic Hamiltonian $H = (\mathbf{p} + \frac{e}{c}\mathbf{A}_0)^2/2m_0 + V_0$, respectively. With prescription $e = c = 1$, we then consider

$$|\Psi(t)\rangle = U(t)|\phi\rangle, \tag{A1}$$

where $U(t)$ is a unitary operator generating a fast-forwarded state. $|\Psi(t)\rangle$ satisfies the time-dependent Schrödinger equation with Hamiltonian

$$\tilde{H}_{\mathrm{FF}}(t) = i\hbar\dot{U}(t)U^\dagger(t) + U(t)HU^\dagger(t). \tag{A2}$$

For simplicity, we tune the origin of energy and assume E_n to be zero, that is,

$$H|\phi\rangle = 0. \tag{A3}$$

Then the second term on the right-hand side of Eq. (A2) becomes vanishing, and $\tilde{H}_{\mathrm{FF}}(t)$ can be rewritten in a more general form as

$$\tilde{H}_{\mathrm{FF}}(t) = i\hbar\dot{U}(t)U^\dagger(t) + \beta(t)U(t)HU^\dagger(t), \tag{A4}$$

where $\beta(t)$ is an arbitrary dimensionless scalar function of time. Now we shall obtain the driving potential for the fast forward of adiabatic dilation. This can be done by having recourse to the unitary operator $U(t)$ generating dilations that were originally cultivated in the context of the expanding or contracting cavity (see, e.g., [28]):

$$U(t) = \exp\left[\frac{i}{4\hbar}\ln R(t)(\mathbf{x}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{x})\right], \tag{A5}$$

with

$$\beta(t) = R(t) = R_0 + \varepsilon\int_0^t \alpha(t')dt'. \tag{A6}$$

Straightforward calculation of Eq. (A4) gives

$$\tilde{H}_{\mathrm{FF}}(t) = \frac{1}{2m_0}[\mathbf{p} + \tilde{\mathbf{A}}_{\mathrm{FF}}(\mathbf{x},t)]^2 + \tilde{V}_{\mathrm{FF}}(\mathbf{x},t), \tag{A7}$$

where

$$\tilde{\mathbf{A}}_{\mathrm{FF}}(\mathbf{x},t) = \sqrt{R}\mathbf{A}_0(\sqrt{R}\mathbf{x}) - \frac{m_0\dot{R}}{2R}\mathbf{x}, \tag{A8}$$

$$\tilde{V}_{\mathrm{FF}}(\mathbf{x},t) = RV_0(\sqrt{R}\mathbf{x}) + \frac{\dot{R}}{2R}\mathbf{x}\cdot\mathbf{A}_0(\sqrt{R}\mathbf{x}) - \frac{m_0\dot{R}^2}{8R^2}|\mathbf{x}|^2. \tag{A9}$$

Finally we make a time-dependent gauge transformation

$$\Psi(\mathbf{x},t) \to \Psi_{\mathrm{FF}} = \exp\left(-i\frac{m_0}{\hbar}\frac{\dot{R}}{4R}|\mathbf{x}|^2\right)\Psi(\mathbf{x},t). \tag{A10}$$

This gauge transformation removes the second term from Eq. (A8) and adds to Eq. (A9) an extra potential $m_0[(\ddot{R}R - \dot{R}^2)/(4R^2)]\mathbf{x}^2$. Then noting that $\dot{R} = \alpha(t)\varepsilon$, we recover the fast-forward potential in Eq. (79). The phase in Eq. (A10) coincides with the additional phase of the fast-forwarded state Ψ_{FF} in Eq. (50) with θ given in Eq. (78). Thus the argument here justifies our predictions in the case of the fast forward of adiabatic dilation of wave function in EMF. In the case of more general fast forwarding, however, the invention of the unitary operator $U(t)$ is far from being easy and one must have recourse to our theoretical scheme in the main text.

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