Arbitrarily complete Bell-state measurement using only linear optical elements

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(Received 24 May 2011; published 19 October 2011)

A complete Bell-state measurement is not possible using only linear-optic elements, and most schemes achieve a success rate of no more than 50%, distinguishing, for example, two of the four Bell states but returning degenerate results for the other two. It is shown here that the introduction of a pair of ancillary entangled photons improves the success rate to 75%. More generally, the addition of $2^N - 2$ ancillary photons yields a linear-optic Bell-state measurement with a success rate of $1 - 1/2^N$.

DOI: 10.1103/PhysRevA.84.042331 PACS number(s): 03.67.Mn, 03.65.Ud

I. INTRODUCTION

The Bell-state measurement (BSM), defined as the projection of two qubits onto maximally entangled Bell states, is an essential feature of a number of quantum communication protocols, including quantum teleportation [1] and entanglement swapping [2]. The simplest photonic polarization BSMs, some of which have been demonstrated experimentally, involve single-photon detectors along with ordinary linear-optic elements such as waveplates and beam splitters. Unfortunately, none of these has risen to the level of a complete BSM in the sense that none has provided unambiguous discrimination of all four Bell states. One commonly used scheme, for example, is able to discriminate two of the four Bell states but returns a degenerate result for the other two—a success rate of only 50% [3]. In principle, a complete BSM can be achieved through the use of nonlinear optics [4] or entanglement in auxiliary degrees of freedom [5], and the techniques of linear-optic quantum computing allow for a complete BSM, but those schemes require feed-forward techniques [6]. However, within the constraints of linear-optic elements without hyperentanglement or feed-forward techniques, a complete BSM has never been demonstrated and, in fact, it has been shown that a complete BSM is not even possible [7,8].

It is shown here that the linear-optic BSM success rate can be improved with the addition of ancillary entangled photons and that, with enough additional photons, a BSM can be realized that is arbitrarily close to "complete." The technique is introduced with the review of a typical scheme that yields a 50% success rate. It is then shown that the addition of a single pair of entangled photons cuts the degeneracy rate in half, thus improving the success rate to 75%. The paper concludes with a general proof showing that the addition of $2^N - 2$ ancillary photons yields a linear-optic BSM with a success rate of $1 - 1/2^N$.

II. SIMPLE BELL-STATE MEASUREMENT

The technique described here is an extension of the BSM proposed by Braunstein and Mann [3] and demonstrated in several experiments. As shown in Fig. 1, two photons are incident on a 50:50 beam splitter and the outputs are directed to a pair of single-photon detectors (not shown). For the purposes of this discussion, the detectors are assumed to have unit efficiency and to be capable of resolving photon number and polarization (the latter can be realized with a polarization beam splitter and additional detectors). The input and output operators are related by

$$\begin{pmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix}_{\text{in}} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix}_{\text{out}}, \tag{1}$$

where the subscripts "in" and "out" refer to all operators inside the respective matrices. Where it is appropriate in subsequent expressions, the $\hat{a}^\dagger$ will be replaced by either $\hat{h}^\dagger$ or $\hat{v}^\dagger$ for horizontal or vertical polarization, respectively.

Under this transformation, the four Bell states evolve from input to output as follows:

$$|\psi^{(+)}\rangle = \frac{1}{\sqrt{2}}[\hat{h}_1^\dagger \hat{v}_2^\dagger + \hat{v}_1^\dagger \hat{h}_2^\dagger]_{\text{in}}|0\rangle \rightarrow \frac{i}{\sqrt{2}}[\hat{h}_1^\dagger \hat{v}_1^\dagger + \hat{h}_2^\dagger \hat{v}_2^\dagger]_{\text{out}}|0\rangle,$$
$$|\psi^{(-)}\rangle = \frac{1}{\sqrt{2}}[\hat{h}_1^\dagger \hat{v}_2^\dagger - \hat{v}_1^\dagger \hat{h}_2^\dagger]_{\text{in}}|0\rangle \rightarrow \frac{1}{\sqrt{2}}[\hat{h}_1^\dagger \hat{v}_2^\dagger - \hat{v}_1^\dagger \hat{h}_2^\dagger]_{\text{out}}|0\rangle,$$
$$\begin{aligned} |\phi^{(\pm)}\rangle &= \frac{1}{\sqrt{2}}[\hat{h}_1^\dagger \hat{h}_2^\dagger \pm \hat{v}_1^\dagger \hat{v}_2^\dagger]_{\text{in}}|0\rangle \\ &\rightarrow \frac{1}{\sqrt{2}}[\hat{h}_1^{\dagger 2} + \hat{h}_2^{\dagger 2} \pm \hat{v}_1^{\dagger 2} \pm \hat{v}_2^{\dagger 2}]_{\text{out}}|0\rangle. \end{aligned} \tag{2}$$

The states $|\psi^{(+)}\rangle$ and $|\psi^{(-)}\rangle$ result in two different outcomes: for $|\psi^{(+)}\rangle$, both photons end up at the same detector but with different polarizations; and, for $|\psi^{(-)}\rangle$, one photon reaches each detector. For $|\phi^{(+)}\rangle$ and $|\phi^{(-)}\rangle$, both photons end up at the same detector and with identical polarizations, an outcome that is degenerate but distinct from the $|\psi^{(+)}\rangle$ and $|\psi^{(-)}\rangle$ results. Thus, this BSM scheme can discriminate between $|\psi^{(+)}\rangle$ and $|\psi^{(-)}\rangle$ but yields degenerate results for $|\phi^{(+)}\rangle$ and $|\phi^{(-)}\rangle$. It is said that the success rate is only 50%, in the sense that an unambiguous result is obtained only half the time for an input state in which all four Bell states are equally likely.

For reasons that will become clear shortly, it is noted that these outcomes can be classified in terms of n_H and n_V, the number of horizontally and vertically polarized photons at the outputs, respectively, and by $n_{[1]}$, which is the total number of photons at detector 1. That is, n_H and n_V are odd for $|\psi^{(\pm)}\rangle$ and even for $|\phi^{(\pm)}\rangle$, while the quantity $n_{[1]}$ is even for $|\psi^{(+)}\rangle$ (0 or 2) and odd for $|\psi^{(-)}\rangle$.

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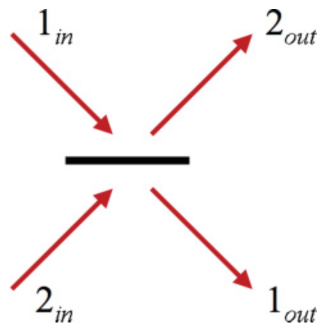

FIG. 1. (Color online) Simple linear-optic method for measuring Bell states. Polarization and photon-number resolving detectors are not shown.

III. BELL-STATE MEASUREMENT WITH TWO ANCILLARY PHOTONS

Consider now the arrangement shown in Fig. 2. The input state in paths 1 and 2 is mixed at a pair of 50:50 beam splitters with ancillary photons in paths 3 and 4, and the four beam-splitter outputs are subsequently subjected to a pair of simple BSMs, as described above. The input and output operators in this arrangement are related by

$$\begin{pmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \\ \hat{a}_3^\dagger \\ \hat{a}_4^\dagger \end{pmatrix}_{\text{in}} \rightarrow \frac{1}{2}\begin{pmatrix} 1 & i & i & -1 \\ i & 1 & -1 & i \\ i & -1 & 1 & i \\ -1 & i & i & 1 \end{pmatrix}\begin{pmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \\ \hat{a}_3^\dagger \\ \hat{a}_4^\dagger \end{pmatrix}_{\text{out}}. \quad (3)$$

The ancillary photons are in the state $|\Upsilon_1\rangle = \frac{1}{\sqrt{2}}[\hat{h}_3^\dagger \hat{h}_4^\dagger + \hat{v}_3^\dagger \hat{v}_4^\dagger]_{\text{in}}|0\rangle$ (although $\frac{1}{\sqrt{2}}[\hat{h}_3^\dagger \hat{h}_4^\dagger - \hat{v}_3^\dagger \hat{v}_4^\dagger]_{\text{in}}|0\rangle$ would also work). The rationale for this choice is that the degeneracy in the simple BSM can be broken by interfering $|\phi^{(\pm)}\rangle$ with something having a very similar form (i.e., with $|\Upsilon_1\rangle$).

With four outputs, each of which could receive between zero and four photons of either polarization, the number of detection outcomes is quite large. Nevertheless, the calculation is straightforward and the results can be organized in the following way. Just as for the simple BSM, n_H and n_V are odd for $|\psi^{(\pm)}\rangle|\Upsilon_1\rangle$ and even for $|\phi^{(\pm)}\rangle|\Upsilon_1\rangle$. The states $|\psi^{(+)}\rangle|\Upsilon_1\rangle$ and $|\psi^{(-)}\rangle|\Upsilon_1\rangle$ differ in the total number of photons in outputs 1 and 3: the quantity $n_{[1,3]} \equiv n_{[1]} + n_{[3]}$ is even for $|\psi^{(+)}\rangle|\Upsilon_1\rangle$ and odd for $|\psi^{(-)}\rangle|\Upsilon_1\rangle$. The measurement outcomes for $|\phi^{(+)}\rangle|\Upsilon_1\rangle$ and $|\phi^{(-)}\rangle|\Upsilon_1\rangle$ are of two types: either all photons have the same polarization or half are horizontally polarized and half are vertically polarized. When all the polarizations are identical, which occurs in 50% of the cases, there is no information to distinguish the two input states. In the latter case (when two horizontally and two vertically polarized photons are detected), $|\phi^{(+)}\rangle|\Upsilon_1\rangle$ and $|\phi^{(-)}\rangle|\Upsilon_1\rangle$ can be distinguished by the quantity $n_{[1,2]}$, which is even for $|\phi^{(+)}\rangle|\Upsilon_1\rangle$ and odd for $|\phi^{(-)}\rangle|\Upsilon_1\rangle$.

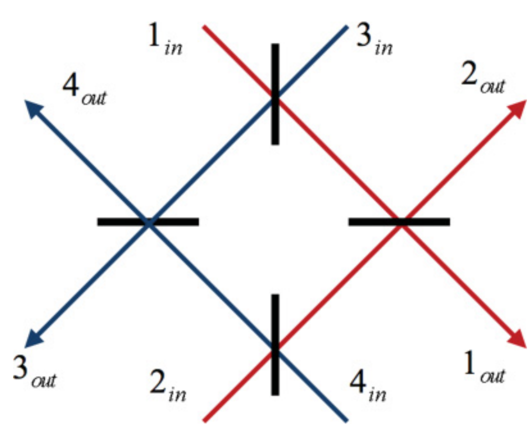

FIG. 2. (Color online) Bell-state measurement with two ancillary photons. The unknown state is in paths 1_{in} and 2_{in}, while the ancillary photons enter via paths 3_{in} and 4_{in}.

Hence, the introduction of an ancillary pair of photons in the state $|\Upsilon_1\rangle = \frac{1}{\sqrt{2}}[\hat{h}_3^\dagger \hat{h}_4^\dagger + \hat{v}_3^\dagger \hat{v}_4^\dagger]_{\text{in}}|0\rangle$ has made it possible to distinguish some of the $|\phi^{(\pm)}\rangle$ states. Whereas the two were completely indistinguishable with the simple BSM, the ancillary entangled photon pair yields unambiguous results for half of the $|\phi^{(+)}\rangle$ outcomes and half of the $|\phi^{(-)}\rangle$ outcomes. Moreover, the states $|\psi^{(+)}\rangle$ and $|\psi^{(-)}\rangle$, which could be distinguished by the simple BSM, remain distinguishable in the presence of the ancillary photons. Thus, the success rate improves from 50% to 75% with the introduction of a single pair of ancillary photons.

IV. BELL-STATE MEASUREMENT WITH ADDITIONAL ANCILLARY PHOTONS

Given that the BSM success rate is improved with the addition of one pair of ancillary photons, it is not surprising that more ancillaries will bring even more improvement. A scheme is presented in this section for the introduction of additional entangled states with the result of an improved success rate with each set of ancillaries. The additional entangled photons are introduced in the same way as in the preceding section. That is, each of the inputs is mixed at a 50:50 beam splitter with one photon of an entangled state and the beam-splitter outputs are sent to identical arrangements of beam splitters. The total number of photons doubles with each stage, as does the total number of outputs.

The arrangement of beam splitters is defined by the recursive relation

$$\begin{pmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \\ \vdots \\ \hat{a}_{2^N}^\dagger \end{pmatrix}_{\text{in}} \rightarrow \mathbf{S}^{(N)} \begin{pmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \\ \vdots \\ \hat{a}_{2^N}^\dagger \end{pmatrix}_{\text{out}}, \quad (4)$$

where $\mathbf{S}^{(N)}$ is a $2^N \times 2^N$ matrix given by

$$\mathbf{S}^{(N)} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{S}^{(N-1)} & i\mathbf{S}^{(N-1)} \\ i\mathbf{S}^{(N-1)} & \mathbf{S}^{(N-1)} \end{pmatrix}, \quad (5)$$

with $\mathbf{S}^{(0)} = 1$. As before, the $\hat{a}^\dagger$ will be replaced with $\hat{h}^\dagger$ or $\hat{v}^\dagger$, as appropriate. The input state is taken to be $|\zeta\rangle|\Upsilon_1\rangle \cdots |\Upsilon_{N-1}\rangle$, the product of the unknown state $|\zeta\rangle$ and $N-1$ ancillary entangled states given by

$$|\Upsilon_j\rangle \equiv \frac{1}{\sqrt{2}}\left[\hat{h}_{2^j+1}^\dagger \cdots \hat{h}_{2^{j+1}}^\dagger + \hat{v}_{2^j+1}^\dagger \cdots \hat{v}_{2^{j+1}}^\dagger\right]_{\text{in}}|0\rangle. \quad (6)$$

It is noted that the expressions in Eqs. (1) and (3) are consistent with the notational scheme of Eqs. (4)–(6).

Just as with $|\Upsilon_1\rangle$, it is still possible to distinguish $|\psi^{(\pm)}\rangle$ from $|\phi^{(\pm)}\rangle$ with the introduction of additional ancilla. Recall that, in the two schemes discussed in the preceding sections, n_H and n_V are odd for $|\psi^{(\pm)}\rangle$ and for $|\psi^{(\pm)}\rangle|\Upsilon_1\rangle$, but even for $|\phi^{(\pm)}\rangle$ and for $|\phi^{(\pm)}\rangle|\Upsilon_1\rangle$. Since each $|\Upsilon_j\rangle$ contributes an even number of photons, all with the same polarization, this parity remains unchanged with additional ancillary states. Hence, n_H and n_V are odd for $|\psi^{(\pm)}\rangle|\Upsilon_1\rangle\cdots|\Upsilon_N\rangle$ and even for $|\phi^{(\pm)}\rangle|\Upsilon_1\rangle\cdots|\Upsilon_N\rangle$.

The distinguishability of $|\psi^{(+)}\rangle$ and $|\psi^{(-)}\rangle$ is also preserved with the introduction of the ancillary states. It is shown in the Appendix that, if a 2^N-photon state $|\Theta_N\rangle$ leads to an even (odd) number of photons in the set $A^{(0,N)} = \{1,3,\ldots,2^N-1\}$ of output ports, then the state $|\Theta_N\rangle|\Upsilon_N\rangle$ leads to an even (odd) number of photons in the set $A^{(0,N+1)} = \{1,3,\ldots,2^{N+1}-1\}$ of output ports. As an example, recall that $n_{[1]}$ is even for $|\psi^{(+)}\rangle$ (0 or 2) and odd for $|\psi^{(-)}\rangle$. With the introduction of the first pair of ancillary photons, it was found that $n_{[1,3]}$ is always even for $|\psi^{(+)}\rangle|\Upsilon_1\rangle$ and odd for $|\psi^{(-)}\rangle|\Upsilon_1\rangle$. Making use of the result in the Appendix, it follows that $n_{[1,3,5,7]}$ is always even for $|\psi^{(+)}\rangle|\Upsilon_1\rangle|\Upsilon_2\rangle$ and odd for $|\psi^{(-)}\rangle|\Upsilon_1\rangle|\Upsilon_2\rangle$ and, more generally, that n_{odd} is always even for $|\psi^{(+)}\rangle|\Upsilon_1\rangle\cdots|\Upsilon_N\rangle$ and odd for $|\psi^{(-)}\rangle|\Upsilon_1\rangle\cdots|\Upsilon_N\rangle$, where n_{odd} is the total number of photons in the odd-numbered outputs. Hence, the states $|\psi^{(+)}\rangle$ and $|\psi^{(-)}\rangle$ remain distinguishable upon the incorporation of additional ancillary states.

Thus far, it has been shown that the ancillary photons do no harm, in the sense that none of the functionality of the simple BSM is lost. However, the real advantage of the ancillaries is that the degeneracy rate for the states $|\phi^{(+)}\rangle$ and $|\phi^{(-)}\rangle$ is reduced by half for each additional ancillary state. To show how this happens, it is instructive to express the $|\phi^{(\pm)}\rangle$ input state as follows:

$$
\begin{aligned}
&|\phi^{\pm}\rangle|\Upsilon_1\rangle\cdots|\Upsilon_{N-1}\rangle \\
&\quad = |\Xi_N^{(\pm)}\rangle + |\Xi_{N-1}^{(\pm)}\rangle|\Upsilon_{N-1}\rangle + |\Xi_{N-2}^{(\pm)}\rangle|\Upsilon_{N-2}\rangle|\Upsilon_{N-1}\rangle + \cdots \\
&\qquad + |\Xi_2^{(\pm)}\rangle|\Upsilon_2\rangle|\Upsilon_3\rangle\cdots|\Upsilon_{N-1}\rangle + |\Gamma_N^{(\pm)}\rangle,
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
|\Xi_j^{(\pm)}\rangle = \left(\frac{1}{\sqrt{2}}\right)^j &\left[\left(\hat{h}_1^\dagger\cdots\hat{h}_{2^{j-1}}^\dagger\right)\left(\hat{v}_{2^{j-1}+1}^\dagger\cdots\hat{v}_{2^j}^\dagger\right)\right. \\
&\left.\pm\left(\hat{v}_1^\dagger\cdots\hat{v}_{2^{j-1}}^\dagger\right)\left(\hat{h}_{2^{j-1}+1}^\dagger\cdots\hat{h}_{2^j}^\dagger\right)\right]_{\text{in}}|0\rangle
\end{aligned} \tag{8}
$$

and

$$
|\Gamma_N^{(\pm)}\rangle = \left(\frac{1}{\sqrt{2}}\right)^N \left[\left(\hat{h}_1^\dagger\cdots\hat{h}_{2^N}^\dagger\right)\pm\left(\hat{v}_1^\dagger\cdots\hat{v}_{2^N}^\dagger\right)\right]_{\text{in}}|0\rangle. \tag{9}
$$

With the input state expressed as in Eq. (7), the terms are organized by $n_H - n_V$. Each of the states $|\Xi_j^{(\pm)}\rangle$ is made up of two terms, each leading to equal numbers of horizontally and vertically polarized photons. Each of the states $|\Upsilon_j\rangle$ leads to either 2^j horizontally polarized photons or 2^j vertically polarized photons. By expanding the products in Eq. (7), it is easy to see that the state $|\Xi_N^{(\pm)}\rangle$ leads to $n_H - n_V = 0$, the state $|\Xi_{N-1}^{(\pm)}\rangle|\Upsilon_{N-1}\rangle$ leads to $n_H - n_V = \pm 2^{N-1}$, the state $|\Xi_{N-1}^{(\pm)}\rangle|\Upsilon_{N-1}\rangle|\Upsilon_{N-2}\rangle$ leads to $n_H - n_V = \pm 2^{N-1} \pm 2^{N-2}$, and so forth, up to the state $|\Xi_2^{(\pm)}\rangle|\Upsilon_2\rangle|\Upsilon_3\rangle\cdots|\Upsilon_{N-1}\rangle$, which yields $n_H - n_V = \pm 2^{N-1} \pm 2^{N-2} \pm\cdots\pm 2^3 \pm 2^2$. For the final term, $|\Gamma_N^{(\pm)}\rangle$, the photons are either all horizontally polarized or all vertically polarized, yielding $n_H - n_V = \pm 2^N$.

Hence, each term in the expansion in Eq. (7) can be distinguished from all others by the quantity $n_H - n_V$. But in order to distinguish $|\phi^{(+)}\rangle$ from $|\phi^{(-)}\rangle$, it must be possible to distinguish each of the (+) terms from each of the (−) terms. To see how this is done consider the state $|\Xi_N^{(\pm)}\rangle$. Each of the input raising operators can be expressed as a linear combination of the 2^N output operators according to Eq. (4). With the exception of very small N, this very quickly leads to an unwieldy number of terms. Note, however, that each term in the expansion of $(\hat{h}_1^\dagger\cdots\hat{h}_{2^{j-1}}^\dagger)_{\text{in}}(\hat{v}_{2^{j-1}+1}^\dagger\cdots\hat{v}_{2^j}^\dagger)_{\text{in}}$ has the form

$$
\begin{aligned}
{}_{HV}\hat{\Xi}_{\{k1,\ldots,k2^N\}} &= \left({}_{[1]}S_{[k1]}\hat{h}_{k1}^\dagger\cdots{}_{[2^{N-1}]}S_{[k2^{N-1}]}\hat{h}_{k2^{N-1}}^\dagger\right)_{\text{out}} \\
&\quad\times\left({}_{[2^{N-1}+1]}S_{[k(2^{N-1}+1)]}\hat{v}_{k(2^{N-1}+1)}^\dagger\cdots{}_{[2^N]}S_{[k2^N]}\hat{v}_{k2^N}^\dagger\right)_{\text{out}},
\end{aligned} \tag{10}
$$

where ${}_{[p]}S_{[q]}$ is the matrix element in the pth row and qth column of $\mathbf{S}^{(N)}$ and where each kj is some integer in the set $\{1,2,\ldots,2^N\}$. Note that each row of $\mathbf{S}^{(N)}$ shows up in this term, since there is one photon in each input. In general, however, not all columns are represented, owing to the fact that most terms result in one or more outputs with no photons. The specific set $\{k1,\ldots,k2^N\}$ for each term determines the number of photons in each output.

For each term of the form in Eq. (10), there is a corresponding term in the expansion of $(\hat{v}_1^\dagger\cdots\hat{v}_{2^{j-1}}^\dagger)_{\text{in}}(\hat{h}_{2^{j-1}+1}^\dagger\cdots\hat{h}_{2^j}^\dagger)_{\text{in}}$ that has the same set of raising-operator labels $\{k1,\ldots,k2^N\}$:

$$
\begin{aligned}
{}_{VH}\hat{\Xi}_{\{k1,\ldots,k2^N\}} &= \left({}_{[1]}S_{[k(2^{N-1}+1)]}\hat{v}_{k(2^{N-1}+1)}^\dagger\cdots{}_{[2^{N-1}]}S_{[k2^N]}\hat{v}_{k2^N}^\dagger\right)_{\text{out}} \\
&\quad\times\left({}_{[2^{N-1}+1]}S_{[k1]}\hat{h}_{k1}^\dagger\cdots{}_{[2^N]}S_{[k2^{N-1}]}\hat{h}_{k2^{N-1}}^\dagger\right)_{\text{out}}.
\end{aligned} \tag{11}
$$

Putting the two together yields

$$
\begin{aligned}
{}_{HV}\hat{\Xi}_{\{k1,\ldots,k2^N\}} \pm {}_{VH}\hat{\Xi}_{\{k1,\ldots,k2^N\}} &= \begin{bmatrix} \left({}_{[1]}S_{[k1]}\cdots{}_{[2^{N-1}]}S_{[k2^{N-1}]}\right)\left({}_{[2^{N-1}+1]}S_{[k(2^{N-1}+1)]}\cdots{}_{[2^N]}S_{[k2^N]}\right) \\ \pm\left({}_{[1]}S_{[k(2^{N-1}+1)]}\cdots{}_{[2^{N-1}]}S_{[k2^N]}\right)\left({}_{[2^{N-1}+1]}S_{[k1]}\cdots{}_{[2^N]}S_{[k2^{N-1}]}\right) \end{bmatrix} \\
&\quad\times\left(\hat{h}_{k1}^\dagger\cdots\hat{h}_{k2^{N-1}}^\dagger\right)_{\text{out}}\left(\hat{v}_{k(2^{N-1}+1)}^\dagger\cdots\hat{v}_{k2^N}^\dagger\right)_{\text{out}} \\
&= i^{2^N}\left({}_{[1]}S_{[k1]}\cdots{}_{[2^{N-1}]}S_{[k2^{N-1}]}\right)\left({}_{[1]}S_{[k(2^{N-1}+1)]}\cdots{}_{[2^{N-1}]}S_{[k2^N]}\right) \\
&\quad\times[(-1)^q \pm (-1)^p]\left(\hat{h}_{k1}^\dagger\cdots\hat{h}_{k2^{N-1}}^\dagger\right)_{\text{out}}\left(\hat{v}_{k(2^{N-1}+1)}^\dagger\cdots\hat{v}_{k2^N}^\dagger\right)_{\text{out}},
\end{aligned} \tag{12}
$$

where p and q are the number of horizontally and vertically polarized photons, respectively, in the outputs $\{2^{N-1}+1,\ldots,2^N\}$. This result follows from the specific structure of $\mathbf{S}^{(N)}$; namely, that

$$_{[j+2^{N-1}]}S_{[kx]}=\begin{cases} i_{[j]}S_{[kx]} & \text{if} \quad kx\in\{1,\ldots,2^{N-1}\} \\ -i_{[j]}S_{[kx]} & \text{if} \quad kx\in\{2^{N-1}+1,\ldots,2^N\}. \end{cases} \tag{13}$$

It is clear from Eq. (12) that different values of p and q lead to different results. Specifically, $(-1)^p$ and $(-1)^q$ have the same sign when p and q are both even or both odd, and they have opposite signs when only one is odd. This result holds for all sets $\{k1,\ldots,k2^N\}$ and so it follows that $n_{[1,\ldots,2^{N-1}]}$ is always even for $|\Xi_N^{(+)}\rangle$ and always odd for $|\Xi_N^{(-)}\rangle$. Hence, $|\Xi_N^{(+)}\rangle$ can be distinguished from $|\Xi_N^{(-)}\rangle$ on the 2^N-port system by the quantity $n_{[1,\ldots,2^{N-1}]}$.

This result, along with the result from the Appendix, can be used to show that every $(+)$ state in Eq. (7) can be distinguished from every $(-)$ state (with the exception of $|\Gamma_N^{(\pm)}\rangle$). Note first that each state has the form $|\Xi_j^{(\pm)}\rangle|\Upsilon_j\rangle\cdots|\Upsilon_{N-1}\rangle$. Given the result of Eq. (12) and the subsequent discussion, it must be the case that $|\Xi_j^{(+)}\rangle$ can be distinguished from $|\Xi_j^{(-)}\rangle$ on the 2^j-port system by the quantity $n_{[1,\ldots,2^{j-1}]}$. Using the result from the Appendix, it must also be the case that $|\Xi_j^{(+)}\rangle|\Upsilon_j\rangle$ can be distinguished from $|\Xi_j^{(-)}\rangle|\Upsilon_j\rangle$ on the 2^{j+1}-port system by the quantity $n_{[1,\ldots,2^{j-1}]}+n_{[2\times2^{j-1}+1,\ldots,3\times2^{j-1}]}$ and, likewise, that $|\Xi_j^{(+)}\rangle|\Upsilon_j\rangle|\Upsilon_{j+1}\rangle$ can be distinguished from $|\Xi_j^{(-)}\rangle|\Upsilon_j\rangle|\Upsilon_j\rangle$ on the 2^{j+2}-port system by the quantity $n_{[1,\ldots,2^{j-1}]}+n_{[2\times2^{j-1}+1,\ldots,3\times2^{j-1}]}+n_{[4\times2^{j-1}+1,\ldots,5\times2^{j-1}]}+n_{[6\times2^{j-1}+1,\ldots,7\times2^{j-1}]}$. Continuing in this manner, it follows that, for every j, $|\Xi_j^{(+)}\rangle|\Upsilon_j\rangle\cdots|\Upsilon_{N-1}\rangle$ leads to a different outcome than $|\Xi_j^{(-)}\rangle|\Upsilon_j\rangle\cdots|\Upsilon_{N-1}\rangle$. And since each j leads to a different set of values for n_H-n_V, the only term in Eq. (7) that leads to degenerate outcomes is $|\Gamma_N^{(\pm)}\rangle$.

The general results for $|\zeta\rangle|\Upsilon_1\rangle\cdots|\Upsilon_{N-1}\rangle$ are summarized in Table I. If n_H and n_V are odd, then $|\zeta\rangle$ must be either $|\psi^{(+)}\rangle$ or $|\psi^{(-)}\rangle$. These two possibilities are distinguished by the quantity n_{odd}, which is even for $|\psi^{(+)}\rangle$ and odd for $|\psi^{(-)}\rangle$. If n_H and n_V are even, then $|\zeta\rangle$ must be either $|\phi^{(+)}\rangle$ or $|\phi^{(-)}\rangle$. Each value of n_H-n_V corresponds to exactly one $(\pm)$ term in the expansion in Eq. (7). For each of these terms, there is a set of detectors to distinguish $(+)$ from $(-)$. The total number of photons at these detectors is even for $|\phi^{(+)}\rangle$ and odd for $|\phi^{(-)}\rangle$.

The lone term in Eq. (7) for which this scheme does not provide a definitive result is the final term, $|\Gamma_N^{(\pm)}\rangle$. As noted previously, $n_H-n_V=\pm2^N$ for this term, thus making it distinct from all others. But there is no measure that will distinguish the $(+)$ term from the $(-)$ term. The probability of obtaining this result (all photons horizontally polarized or all photons vertically polarized) is $1/2^{N-1}$ when the input is either $|\phi^{(+)}\rangle$ or $|\phi^{(-)}\rangle$. Thus, the overall probability of obtaining an inconclusive result when the input is an equal mixture of all four Bell states is $1/2^N$, which means that the BSM success rate approaches unity as $2^N\to\infty$. Although this method makes an arbitrarily complete BSM possible, the number of entangled photons required makes it impractical to reach even 95% (30 entangled photons would yield 96.875%). However, moving from 50% to 75% requires only a single

TABLE I. General results for $|\zeta\rangle|\Upsilon_1\rangle\cdots|\Upsilon_{N-1}\rangle$.

Input State		n_H,n_V	n_H-n_V	$+\vert-$ Discriminator
$\vert\psi^{(+)}\rangle\vert\Upsilon_1\rangle\cdots\vert\Upsilon_{N-1}\rangle$		odd		n_{odd} is even
$\vert\psi^{(-)}\rangle\vert\Upsilon_1\rangle\cdots\vert\Upsilon_{N-1}\rangle$		odd		n_{odd} is odd
$\vert\phi^{(+)}\rangle\vert\Upsilon_1\rangle\cdots\vert\Upsilon_{N-1}\rangle=$				
	$\vert\Xi_N^{(+)}\rangle$	even	0	$n_{[1,\ldots,2^{N-1}]}$ is even
	$+\vert\Xi_{N-1}^{(+)}\rangle\vert\Upsilon_{N-1}\rangle$	$\vdots$	$\pm2^{N-1}$	$n_{[1,\ldots,2^{N-2}]}+n_{[2\times2^{N-2}+1,\ldots,3\times2^{N-2}]}$ is even
	$+\cdots$	$\vdots$	$\vdots$	$\vdots$
	$+\vert\Xi_2^{(+)}\rangle\vert\Upsilon_2\rangle\vert\Upsilon_3\rangle\cdots\vert\Upsilon_{N-1}\rangle$	$\vdots$	$\pm2^{N-1}\pm2^{N-2}\pm\cdots\pm2^3\pm2^2$	$n_{[1,2]}+n_{[5,6]}+n_{[9,10]}+\cdots$ is even
	$+\vert\Gamma_N^{(+)}\rangle$	$\vdots$	$\pm2^N$	$(+)$ and $(-)$ are degenerate
$\vert\phi^{(-)}\rangle\vert\Upsilon_1\rangle\cdots\vert\Upsilon_{N-1}\rangle=$				
	$\vert\Xi_N^{(+)}\rangle$	even	0	$n_{[1,\ldots,2^{N-1}]}$ is odd
	$+\vert\Xi_{N-1}^{(-)}\rangle\vert\Upsilon_{N-1}\rangle$	$\vdots$	$\pm2^{N-1}$	$n_{[1,\ldots,2^{N-2}]}+n_{[2\times2^{N-2}+1,\ldots,3\times2^{N-2}]}$ is odd
	$+\cdots$	$\vdots$	$\vdots$	$\vdots$
	$+\vert\Xi_2^{(-)}\rangle\vert\Upsilon_2\rangle\vert\Upsilon_3\rangle\cdots\vert\Upsilon_{N-1}\rangle$	$\vdots$	$\pm2^{N-1}\pm2^{N-2}\pm\cdots\pm2^3\pm2^2$	$n_{[1,2]}+n_{[5,6]}+n_{[9,10]}+\cdots$ is odd
	$+\vert\Gamma_N^{(-)}\rangle$	$\vdots$	$\pm2^N$	$(+)$ and $(-)$ are degenerate

pair of ancillary photons—certainly possible with today's technology. Moreover, the scheme presented here constitutes only one approach, and it is reasonable to expect that more efficient schemes may be discovered.

Note added in proof. Recently, similar results appeared in Ref. [9]. As in the present work, it is shown there that the Bell-state measurement can be improved with additional linear resources. Here, entangled photons are added, and in [9], it is additional interferometers. Besides the difference in resources, there is a subtle difference in the way that the measurement results can be classified. For the method described in the present work, the additional resources (entangled photons) yield five distinct types of measurement results, four of which unambiguously identify the four Bell states. In contrast, the additional resources (interferometers) described in [9] yield four distinct types of results, only two of which unambiguously identify Bell states. The other two correspond to $|\phi^{(+)}\rangle$ and $|\phi^{(-)}\rangle$, respectively, but both are degenerate with $|\psi^{(+)}\rangle$. While the degree of uncertainty is reduced with additional interferometers, unambiguous identification of $|\phi^{(+)}\rangle$ and $|\phi^{(-)}\rangle$ is possible only in the limit of infinite resources. Nevertheless, the two approaches share a number of important similarities, and it would be interesting to consider some type of hybrid approach to further reduce the resource requirements.

ACKNOWLEDGMENTS

This work was sponsored by the Laboratory Directed Research and Development Program of Oak Ridge National Laboratory, managed by UT-Battelle, LLC, for the US Department of Energy. W.G. also acknowledges Government's support in the publication of this paper under AFRL Contract No. F4HBKC1013G001.

APPENDIX

At several points in the preceding text, reference is made to a theorem which asserts that a state's parity on a set of output ports is preserved with the addition of ancillary states as long as the set is enlarged to include a subset of the new output ports. A formal statement of that theorem, along with its proof, is given here. Consider a BSM system comprising 2^M input ports and 2^M output ports, as described in Sec. IV. It will be helpful in the following to group the outputs in blocks of size 2^p, where $0 \leqslant p \leqslant M-1$. Specifically, the outputs can be grouped in blocks described by the sets $A_m^{(p,M)} = \{m2^p + 1, \ldots, (m+1)2^p\}$, where $0 \leqslant m \leqslant 2^{M-p}-1$. The parity of a state is defined to be even (odd) on the set A of outputs if it has the property that it always results in an even (odd) number of photons in the outputs A.

Theorem. If the 2^M-photon input state $|\Theta_M\rangle$ has even (odd) parity on the set of output ports $A^{(p,M)} = A_0^{(p,M)} \cup A_2^{(p,M)} \cup \cdots \cup A_{2^{M-p}}^{(p,M)}$ in the 2^M-photon BSM apparatus, then the input state $|\Theta_M\rangle|\Upsilon_M\rangle$ will have even (odd) parity on the set of output ports $A^{(p,M+1)} = A_0^{(p,M+1)} \cup A_2^{(p,M+1)} \cup \cdots \cup A_{2^{M+1-p}}^{(p,M+1)}$ in the 2^{M+1}-photon BSM apparatus. That is, the parity is preserved as long as the output-port set is extended to include the additional ports using the same pattern (every other block) as the original set.

Proof. For the 2^M-photon BSM apparatus, each input raising operator is related to the output operators according to Eq. (4), (i.e., $[\hat{a}_j^\dagger]_{\rm in} \to \sum_{k=1}^{2^M} {}_{[j]}S_{[k]}[\hat{a}_k^\dagger]_{\rm out}$). In the scheme outlined in Sec. IV, the 2^M-photon state becomes the input for the first 2^M ports of the 2^{M+1}-photon BSM apparatus. The relationship between the first 2^M input operators and the output operators is unchanged in the larger system, except there are now twice as many outputs:

$$[\hat{a}_j^\dagger]_{\rm in} \to \sum_{k=1}^{2^{M+1}} {}_{[j]}S_{[k]}[\hat{a}_k^\dagger]_{\rm out} = \sum_{k=1}^{2^M} {}_{[j]}S_{[k]}[\hat{a}_k^\dagger + i\hat{a}_{k+2^M}^\dagger]_{\rm out}. \tag{A1}$$

This last result, which arises from the fact that ${}_{[j]}S_{[k+2^M]} = i\,{}_{[j]}S_{[k]}$ for $j \leqslant 2^M$, shows that each of these input operators leads to the same type of output in the 2^{M+1}-photon system as in the 2^M-photon system, except that $[\hat{a}_k^\dagger]_{\rm out} \to [\hat{a}_k^\dagger + i\hat{a}_{k+2^M}^\dagger]_{\rm out}$. Note that the expression in Eq. (A1) is consistent with the mapping $A^{(p,M)} \to A^{(p,M+1)}$, and so the parity is preserved for $|\Theta_M\rangle$—. That is, if $|\Theta_M\rangle$ results in an even (odd) number of photons in $A^{(p,M)}$ in the BSM system with 2^M inputs and outputs, then $|\Theta_M\rangle$— results in an even (odd) number of photons in $A^{(p,M+1)}$ in the BSM system with 2^{M+1} inputs and outputs. All that remains to be shown is that this parity remains unchanged with the addition of the ancillary state $|\Upsilon_M\rangle$.

The state $|\Upsilon_M\rangle$ is a superposition of two terms, each of which is a product of 2^M raising operators acting on the vacuum, with one operator for each of the inputs in the set $\{2^M+1, \ldots, 2^{M+1}\}$. Each of these input operators is related to the output operators according to Eq. (4); that is,

$$[\hat{a}_j^\dagger]_{\rm in} \to \sum_{k=1}^{2^{M+1}} {}_{[j]}S_{[k]}[\hat{a}_k^\dagger]_{\rm out} = \sum_{k\in A^{(p,M+1)}}^{2^{M+1}} {}_{[j]}S_{[k]}[\hat{a}_k^\dagger]_{\rm out} + \sum_{k\notin A^{(p,M+1)}}^{2^{M+1}} {}_{[j]}S_{[k]}[\hat{a}_k^\dagger]_{\rm out}. \tag{A2}$$

In this expression, the terms have simply been grouped according to whether or not the photon ends up in an output in the set $A^{(p,M+1)}$. The product of two adjacent input operators is

$$\begin{aligned}
&[\hat{a}_j^\dagger \hat{a}_{j+1}^\dagger]_{\rm in} \\
&\to \left[\sum_{k\in A^{(p,M+1)}}^{2^{M+1}} {}_{[j]}S_{[k]}\hat{a}_k^\dagger + \sum_{k\notin A^{(p,M+1)}}^{2^{M+1}} {}_{[j]}S_{[k]}\hat{a}_k^\dagger\right] \\
&\quad\times \left[\sum_{k'\in A^{(p,M+1)}}^{2^{M+1}} {}_{[j+1]}S_{[k']}\hat{a}_{k'}^\dagger + \sum_{k'\notin A^{(p,M+1)}}^{2^{M+1}} {}_{[j+1]}S_{[k']}\hat{a}_{k'}^\dagger\right]_{\rm out} \\
&= i\left[\sum_{k\in A^{(p,M+1)}}^{2^{M+1}} {}_{[j]}S_{[k]}\hat{a}_k^\dagger + \sum_{k\notin A^{(p,M+1)}}^{2^{M+1}} {}_{[j]}S_{[k]}\hat{a}_k^\dagger\right] \\
&\quad\times \left[\sum_{k'\in A^{(p,M+1)}}^{2^{M+1}} {}_{[j]}S_{[k']}\hat{a}_{k'}^\dagger - \sum_{k'\notin A^{(p,M+1)}}^{2^{M+1}} {}_{[j]}S_{[k']}\hat{a}_{k'}^\dagger\right]_{\rm out},
\end{aligned} \tag{A3}$$

where, without loss of generality, j has been taken to be odd. The index shift for the matrix elements in the second equality arises from the structure of **S** given in Eq. (5); namely, that

$$ {}_{[j+1]}S_{[k]} = \begin{cases} i\,{}_{[j]}S_{[k]} & \text{if} \quad k \in A^{(p,M+1)} \\ -i\,{}_{[j]}S_{[k]} & \text{if} \quad k \notin A^{(p,M+1)}. \end{cases} \tag{A4} $$

Noting that this expression has the form $i(x+y)(x-y) = ix^2 - iy^2$, Eq. (A3) can be rewritten as

$$ [\hat{a}_j^\dagger \hat{a}_{j+1}^\dagger]_{\text{in}} \rightarrow i \left[\sum_{k \in A^{(p,M+1)}}^{2^{M+1}} {}_{[j]}S_{[k]} \hat{a}_k^\dagger \right]^2 - i \left[\sum_{k \notin A^{(p,M+1)}}^{2^{M+1}} {}_{[j]}S_{[k]} \hat{a}_k^\dagger \right]^2 . \tag{A5} $$

When written in this form, it is clear that each pair of adjacent inputs leads to two possibilities: both photons end up in $A^{(p,M+1)}$ or neither photon ends up in $A^{(p,M+1)}$. That is, each pair of $|\Upsilon_M\rangle$ input operators contributes only an even number of photons to the outputs in $A^{(p,M+1)}$. It follows that the state $|\Upsilon_M\rangle$ leads exclusively to an even number of photons in the outputs in $A^{(p,M+1)}$.

The following is a summary of these results: (i) The parity for $|\Theta_M\rangle$ is the same in the 2^{M+1}-port BSM system as it is in the 2^M-port BSM system; (ii) The ancillary state $|\Upsilon_M\rangle$ does nothing to change that parity. ■

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