Remote information concentration and multipartite entanglement in multilevel systems

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Remote information concentration (RIC) in d-level systems (qudits) is studied. It is shown that the quantum information initially distributed in three spatially separated qudits can be remotely and deterministically concentrated to a single qudit via an entangled channel without performing any global operations. The entangled channel can be different types of genuine multipartite pure entangled states which are inequivalent under local operations and classical communication. The entangled channel can also be a mixed entangled state, even a bound entangled state which has a similar form to the Smolin state, but has different features from the Smolin state. A common feature of all these pure and mixed entangled states is found; i.e., they have d^2 common commuting stabilizers. The differences of qudit-RIC and qubit-RIC ($d = 2$) are also analyzed.

I. INTRODUCTION

Although an unknown quantum state cannot be perfectly copied [1,2], quantum cloning, functioning as copying approximately quantum states as well as possible, has attracted considerable attention [3] since Bužek and Hillery [4] first introduced such a concept, due to its potential applications in quantum-information science (see, e.g., [5–8]). Although the fidelities of clones relative to the original state are less than one, the quantum information of the input system is not degraded but only distributed into a larger quantum system. That is, the quantum-cloning process can be regarded as the distribution of quantum information from an initial system to final ones. Thus, quantum cloning combined with other quantum-information processing (QIP) tasks may have potential applications in multiparty quantum communication and distributed quantum computation. This leads to the advent of the concept of quantum telecloning [9–11], which is the combination of quantum cloning and quantum teleportation [12]. Telecloning functions as transmitting many copies of an unknown quantum state of the input system to many distant quantum systems, i.e., realizing one-to-many remote cloning, via previously shared multipartite entangled states. As the reverse process of telecloning, remote information concentration (RIC) was also presented by Murao and Vedral [13]. They demonstrated that the quantum information originally distributed into three spatially separated qubits from a single qubit can be remotely concentrated back to a single qubit via a four-qubit unlockable bound entangled state [14,15] without performing any global operations. Telecloning and concentrating processes could be regarded as, respectively, remote information depositing and withdrawing, or remote information encoding and decoding, which is expected to find useful applications in network-based QIP [13]. Yu *et al.* [16] showed that a four-qubit GHZ state can also be used to implement three-to-one RIC. Not long before, RIC was generalized to the $N \rightarrow 1$ case in two-level systems [17,18].

In recent years, encoding and manipulating quantum information with high-dimensional systems, or qudits, instead of two-state systems, or qubits, has attracted considerable attention. This is due to the fact that significant fundamental and practical advantages can be gained by employing high-dimensional quantum states. For instance, higher dimensional entangled states exhibit stronger violation of local realism [19] and can lower the detection efficiencies required for closing the detection loophole in Bell tests [20], higher dimensional states are more robust against isotropic noise [21], qudit-based quantum cryptographic protocols may enhance the security against eavesdropping attacks [22], qudits can simplify quantum logic [23] and have higher capacity to carry information, and so on.

In this paper, we investigate RIC for d-level ($d \geqslant 2$) quantum systems, called qudits for short (when $d = 2$, they reduce to qubits). It will be shown that the quantum information originally distributed into three spatially separated qudits from a single qudit by the telecloning procedure can be remotely concentrated back to a single qudit via a previously shared entangled channel assisted by local operations and classical communication (LOCC). The entangled channel can be mixed entangled states as well as pure ones. All these entangled states have d^2 common commuting stabilizers. We also show that there are minor constraints on the distribution of the general entangled channel, in contrast to qubit-RIC which has no constraint on the distribution of the entangled channel.

It can be seen that entanglement, a very important physical resource for QIP, plays an essential role in quantum cloning, telecloning, and RIC. Quantum cloning is in fact creating entanglement among the involved quantum systems, and the fidelities of clones are inherently linked with the entanglement among them. Both telecloning and RIC protocols need special structures of entangled states acting as the quantum channel. In a word, all the aforementioned tasks cannot be achieved without entanglement.

On the other hand, the quantum tasks mentioned above can reveal some peculiar entanglement characteristics [13,16–18,24–26], in addition to their practical applications. In this paper, we reveal other interesting phenomena that appear in the RIC. A counterintuitive phenomenon is that inequivalent

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genuine four-partite pure entangled states, i.e., they cannot be transformed into each other by LOCC, can implement deterministically a same *multiparty* QIP task, three-to-one RIC. Another phenomenon is that a single asymmetric unlockable bound entangled state can be competent for implementing RIC in multilevel systems. Such a multilevel bound entangled state has a similar form to the Smolin bound entangled state [14] (a four-qubit unlockable bound entangled state), but has some different features from the Smolin state.

II. PROTOCOLS FOR REMOTE INFORMATION CONCENTRATION VIA DIFFERENT TYPES OF ENTANGLEMENT

Before describing our RIC protocols, we briefly summarize the forward process, telecloning. We focus on the $1 \rightarrow 2$ universal telecloning in d-level systems and its reverse in this paper. Such a telecloning scheme [11] allows direct distribution of optimal clones from a single original qudit state

$$|\varphi\rangle_t = \sum_{j=0}^{d-1} x_j |j\rangle_t \tag{1}$$

($\sum_{j=0}^{d-1} |x_j|^2 = 1$) to two spatially separated parties (Bob and Charlie) with only LOCC. The quantum channel is a four-qudit entangled state

$$|\Phi\rangle_{t'12a} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_{t'} |\phi_j\rangle_{12a}, \tag{2}$$

where

$$|\phi_j\rangle_{12a} = Q\left[|j\rangle_1 |j\rangle_2 |j\rangle_a + \sum_{r=1}^{d-1} (p|j\rangle_1 |\overline{j+r}\rangle_2 \right.$$
$$\left. + q|\overline{j+r}\rangle_1 |j\rangle_2) |\overline{j+r}\rangle_a \right] \tag{3}$$

with

$$Q = 1/\sqrt{1 + (d-1)(p^2 + q^2)}, \quad p + q = 1,$$

and

$$\overline{j+r} = j + r$$

modulo d. Here qudit t' is an input port of the distributor, qudit a is an output port for the ancilla held by Alice, and qudits 1 and 2 are output ports for the clones held, respectively, by Bob and Charlie (throughout the paper, if necessary, the subscripts outside the kets or the operators denote the qudit index). The distributor performs a generalized (or qudit) Bell-basis [see Eq. (5)] measurement (GBM) on qudits t and t'. Depending on the distributor's measurement outcome, Alice, Bob, and Charlie perform local operations on the qudits they hold, and obtain the cloning state of $|\varphi\rangle$ represented by the three-qudit state

$$|\psi\rangle_{12a} = \sum_{j=0}^{d-1} x_j |\phi_j\rangle_{12a}. \tag{4}$$

The aforementioned generalized (d-level) Bell basis is given by

$$|B^{0,0}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle|j\rangle,$$
$$|B^{m,n}\rangle = I \otimes U^{m,n}|B^{0,0}\rangle, \tag{5}$$
$$U^{m,n} = \sum_{k=0}^{d-1} \omega^{km} |\overline{k+n}\rangle\langle k|$$

for $0 \leqslant m,n \leqslant d - 1$, where $\omega = e^{2\pi i/d}$. In the telecloning scheme above, when $p = q = 1/2$, the cloning is symmetric (two clones have the same fidelity) [27], and otherwise, it is asymmetric (two clones have different fidelities) [28].

Using the equality

$$|j\rangle|k\rangle = \frac{1}{\sqrt{d}} \sum_{r=0}^{d-1} \omega^{-jr} |B^{r,\overline{k-j}}\rangle (0 \leqslant j,k \leqslant d-1) \tag{6}$$

with $\overline{k-j} = k - j + d$ modulo d, we can rewrite the cloning state of Eq. (4) as

$$|\psi\rangle_{12a} = \alpha |B^{0,0}\rangle_{1a}|\varphi\rangle_2 + \beta \sum_{m=1}^{d-1} |B^{m,0}\rangle_{1a} U_2^{\overline{-m},0}|\varphi\rangle_2$$
$$+ \gamma \sum_{m=0,n=1}^{d-1} |B^{m,n}\rangle_{1a} U_2^{\overline{-m},n}|\varphi\rangle_2, \tag{7}$$

where

$$\alpha = \frac{Q[1+(d-1)p]}{\sqrt{d}}, \quad \beta = \frac{Q(1-p)}{\sqrt{d}}, \quad \gamma = \frac{Qq}{\sqrt{d}}. \tag{8}$$

Because of the permutability of qudits 1 and 2, the cloning state can also be expressed as

$$|\psi\rangle_{12a} = \alpha |B^{0,0}\rangle_{2a}|\varphi\rangle_1 + \beta \sum_{m=1}^{d-1} |B^{m,0}\rangle_{2a} U_1^{\overline{-m},0}|\varphi\rangle_1$$
$$+ \gamma \sum_{m=0,n=1}^{d-1} |B^{m,n}\rangle_{2a} U_1^{\overline{-m},n}|\varphi\rangle_1. \tag{9}$$

When $d = 2$, the results reduce to that for qubits. In other words, the formulas of Eqs. (7) and (9) can be directly generalized from qubits to qudits. However,

$$|\psi\rangle_{12a} \neq \sum_{m,n=0}^{d-1} C_{mn} |B^{m,n}\rangle_{12} U_a^{\overline{-m},n}|\varphi\rangle_a \tag{10}$$

for $d > 2$, which can also be verified by the equality of Eq. (6). That is, the formulation of Eq. (10) cannot be generalized from qubits to qudits. Such a minor difference will lead to the results of RIC for qudits and qubits also having differences. Particularly, there are minor constraints on the distribution of the general entangled channel for qudit-RIC, but none for qubit-RIC.

Now we present our RIC schemes, the reverse of the aforementioned universal $1 \rightarrow 2$ telecloning in d-level systems,

that is, concentrating the information initially distributed in three spatially separated qudits a, 1, and 2 (held by Alice, Bob, and Charlie, respectively) to a single remote qudit 6 (held by Diana) with only LOCC: $|\psi\rangle_{12a} \to |\varphi\rangle_6$. We first consider employing the following four-qudit pure entangled state as the quantum channel:

$$|\Psi^g\rangle_{3456} = \sum_{m',n'=0}^{d-1} C_{m'n'} |B^{m',n'}\rangle_{34} |B^{\overline{u-m'},\overline{v-n'}}\rangle_{56},$$
(11)

where u and v are two arbitrarily given nonnegative integers that are less than d, and $C_{m'n'}$ are normalization coefficients satisfying $\sum_{m',n'=0}^{d-1} |C_{m'n'}|^2 = 1$. We first assume that qudits 3, 4, and 5 belong to Alice, Bob, and Charlie, respectively. According to Eqs. (7) and (11), the state of the whole system $|\Omega\rangle_{12a3456} = |\psi\rangle_{12a} |\Psi^g\rangle_{3456}$ is given by

$$|\Omega\rangle_{12a3456} = \alpha \sum_{m',n'=0}^{d-1} C_{m'n'} |B^{0,0}\rangle_{1a} |B^{m',n'}\rangle_{34} |\varphi\rangle_2 |B^{\overline{u-m'},\overline{v-n'}}\rangle_{56} + \beta \sum_{\substack{m',n'=0 \\ m=1}}^{d-1} C_{m'n'} |B^{m,0}\rangle_{1a} |B^{m',n'}\rangle_{34} U_2^{\overline{-m},0} |\varphi\rangle_2 |B^{\overline{u-m'},\overline{v-n'}}\rangle_{56}$$

$$+ \gamma \sum_{\substack{m',n'=0 \\ m=0,n=1}}^{d-1} C_{m'n'} |B^{m,n}\rangle_{1a} |B^{m',n'}\rangle_{34} U_2^{\overline{-m},n} |\varphi\rangle_2 |B^{\overline{u-m'},\overline{v-n'}}\rangle_{56}.$$
(12)

With the equality of Eq. (6), we can obtain an equality on entanglement swapping

$$|B^{m,n}\rangle_{XY} |B^{m',n'}\rangle_{X'Y'} = \frac{1}{d} \sum_{m'',n''=0}^{d-1} \omega^{m''n''} |B^{\overline{m+m''},\overline{n'+n''}}\rangle_{XY'} |B^{\overline{m'-m''},\overline{n-n''}}\rangle_{X'Y}.$$
(13)

Using Eq. (13), the global state $|\Omega\rangle_{12a3456}$ can be rewritten as

$$|\Omega\rangle_{12a3456} = \frac{\alpha}{d} \sum_{\substack{m',n'=0 \\ m'',n''=0}}^{d-1} \omega^{m''n''} C_{m'n'} |B^{m'',\overline{n'+n''}}\rangle_{14} |B^{\overline{m'-m''},\overline{-n''}}\rangle_{3a} |\varphi\rangle_2 |B^{\overline{u-m'},\overline{v-n'}}\rangle_{56}$$

$$+ \frac{\beta}{d} \sum_{\substack{m',n'=0 \\ m'',n''=0 \\ m=1}}^{d-1} \omega^{m''n''} C_{m'n'} |B^{\overline{m+m''},\overline{n'+n''}}\rangle_{14} |B^{\overline{m'-m''},\overline{-n''}}\rangle_{3a} U_2^{\overline{-m},0} |\varphi\rangle_2 |B^{\overline{u-m'},\overline{v-n'}}\rangle_{56}$$

$$+ \frac{\gamma}{d} \sum_{\substack{m',n'=0 \\ m'',n''=0 \\ m=0,n=1}}^{d-1} \omega^{m''n''} C_{m'n'} |B^{\overline{m+m''},\overline{n'+n''}}\rangle_{14} |B^{\overline{m'-m''},\overline{n-n''}}\rangle_{3a} U_2^{\overline{-m},n} |\varphi\rangle_2 |B^{\overline{u-m'},\overline{v-n'}}\rangle_{56}.$$
(14)

The procedure of the RIC is as follows. (S1) Alice, Bob, and Charlie perform GBMs on the qudit pairs $(3,a)$, $(1,4)$, and $(2,5)$, respectively. (S2) Each party tells Diana the measurement outcome by sending $2 \log d$ bits of classical information. (S3) Diana performs the conditional local operation on qudit 6. A schematic picture of this protocol is shown in Fig. 1.

In (S1), the GBMs of Alice, Bob, and Charlie are independent, and thus the sequence can be arbitrary. For clarity, we here assume that Alice and Bob perform the GBMs before Charlie. For the outcomes $(\overline{m'-m''},\overline{n-n''})$ and $(\overline{m+m''},\overline{n'+n''})$, we obtain the digital values $u' = \overline{m+m'}$ and $v' = \overline{n+n'}$. Then qudits 2, 5, and 6 are projected in the state $U_2^{\overline{-m},n} |\varphi\rangle_2 |B^{\overline{u-m'},\overline{v-n'}}\rangle_{56}$, which can be rewritten as

$$U_2^{\overline{-m},n} |\varphi\rangle_2 |B^{\overline{u-m'},\overline{v-n'}}\rangle_{56} = \frac{1}{d} U_2^{\overline{-m},n} \sum_{m''',n'''=0}^{d-1} U_5^{m''',n'''} |B^{\overline{u-m'},\overline{v-n'}}\rangle_{25} U_6^{\overline{-m'''},n'''} |\varphi\rangle_6$$

$$= \frac{1}{d} \sum_{m''',n'''=0}^{d-1} \omega^{n(u'-u)+(v-v')m'''} |B^{\overline{m'''+u-u'},\overline{n'''+v-v'}}\rangle_{25} U_6^{\overline{-m'''},n'''} |\varphi\rangle_6.$$
(15)

Next Charlie performs a GBM on qudits 2 and 5, which can be regarded as being equivalent to Charlie and Diana together performing the teleportation protocol with a local error-correction operation on qudit 6. Assume that the measurement outcome is $(u'' = \overline{m'''+u-u'}, v'' = \overline{n'''+v-v'})$, and qudit 6 is projected in the state $U_6^{\overline{-m'''},n'''} |\varphi\rangle_6$. After receiving all the measurement outcomes sending from the other three parties, Diana can deduce the digital values $m''' = \overline{u''+u'-u}$ and $n''' = \overline{v''+v'-v}$. Then, Diana performs the local operation $(U_6^{\overline{-m'''},n'''})^+ = \omega^{-m'''n'''} U_6^{m''',\overline{-n'''}}$ and obtains the state $|\varphi\rangle_6$. As a consequence, the information initially distributed in three spatially separated qudits is now remotely concentrated in a single qudit.
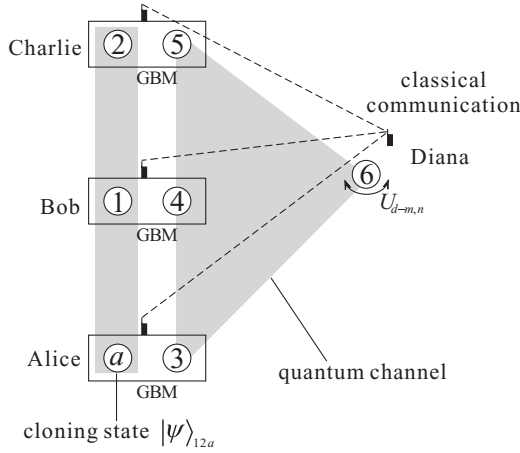
FIG. 1. Schematic picture showing the successful concentration of information from Alice, Bob, and Charlie at the remote receiver, Diana, in the case in which qudits 3, 4, and 5 of the four-qudit entangled state acting as the quantum channel are distributed to Alice, Bob, and Charlie, respectively.

If qudit 4 is distributed to Charlie but not Bob, and qudit 5 to Bob but not Charlie, the information initially distributed in qudits 1, 2, and a can also be concentrated to qudit 6 via the entangled channel of Eq. (11). In this case, the procedure of RIC is as follows. (S1) Alice, Charlie, and Bob perform GBMs on the qudit pairs $(3,a)$, $(2,4)$, and $(1,5)$, respectively. (S2) Each party tells Diana the measurement outcome by sending $2 \log d$ bits of classical information. (S3) Diana performs the conditional local operation on qudit 6. This can be easily verified by Eqs. (9), (11), and (13). A schematic picture for this case is shown in Fig. 2. There are also other cases of distribution of the entangled channel with which the RIC can be achieved. However, if qudits 3 and 4 are simultaneously distributed to Bob and Charlie (see, e.g., Fig. 3), RIC cannot be achieved generally for $d > 2$ by the same entangled channel of Eq. (11) without special superposition coefficients as shown later, which can be understood from Eq. (10). Note that there is no such constraint for qubit-RIC, because the inequality of Eq. (10) does not hold for $d = 2$. Thus this is a minor difference between qudit-RIC and qubit-RIC.

Equation (11) contains a broad family of pure entangled states. We now consider some special cases. Assuming $u = v = 0$, $n' = c$ (an arbitrary nonnegative integer that is less than d), and $C_{m'c} = 1/\sqrt{d}$ for all m', Eq. (11) reduces to

$$|\Psi^{s_1}\rangle_{3456} = \frac{1}{\sqrt{d}} \sum_{m'=0}^{d-1} |B^{m',c}\rangle_{34} |B^{\overline{-m'},\overline{-c}}\rangle_{56}$$
$$= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_3 |\overline{j+c}\rangle_4 |j\rangle_5 |\overline{j-c}\rangle_6, \quad (16)$$

i.e., a multilevel (or generalized) Greenberger-Horne-Zeilinger (GHZ) state [29,30]. In this case, there is no constraint on the channel distribution; i.e., qudits 3, 4, and 5 can be arbitrarily distributed to Alice, Bob, and Charlie, each party one qudit. Assuming $u = v = 0$, $C_{00} = \alpha$, $C_{m'0} = \beta$ $(m' = 1, 2, \dots,$
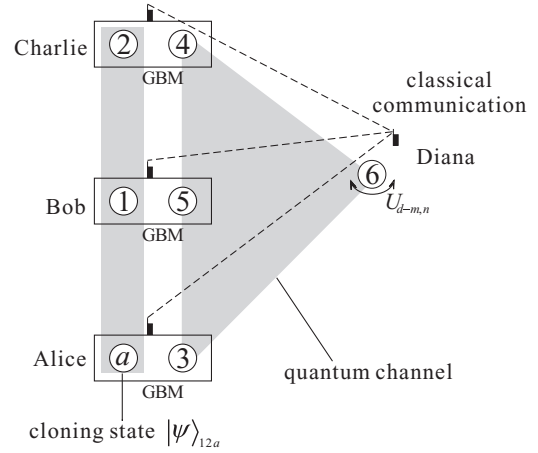

FIG. 2. Schematic picture showing the successful concentration of information from Alice, Bob, and Charlie at the remote receiver, Diana, in the case in which qudits 3, 4, and 5 of the four-qudit entangled state acting as the quantum channel are distributed to Alice, Charlie, and Bob, respectively.

$d - 1)$, and $C_{m'n'} = \gamma$ $(m' = 0, 1, \dots, d - 1; n' = 1, 2, \dots, d - 1)$, the entangled channel of Eq. (11) reduces to

$$|\Psi^{s_2}\rangle_{3456} = \alpha |B^{0,0}\rangle_{34} |B^{0,0}\rangle_{56} + \beta \sum_{m'=1}^{d-1} |B^{m',0}\rangle_{34} |B^{\overline{-m'},0}\rangle_{56}$$
$$+ \gamma \sum_{m'=0, n'=1}^{d-1} |B^{m',n'}\rangle_{34} |B^{\overline{-m'},\overline{-n'}}\rangle_{56}. \quad (17)$$

For the case $d = 2$, it can be proved that the state of Eq. (17) is the same as that of Eq. (2). This indicates that the four-qubit entangled state of Eq. (2) can be competent for implementing both telecloning and RIC, two inverse processes. In other words, the aforementioned telecloning and RIC for $d = 2$ (qubit) can be achieved by using the same entangled channel. However, such a result is not applicable to $d > 2$
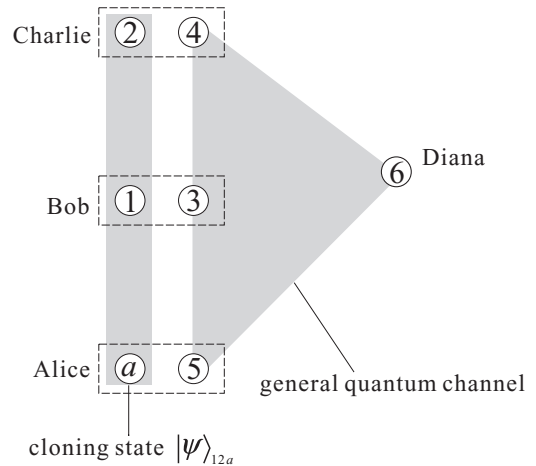

FIG. 3. Schematic picture showing the failure of concentrating information from Alice, Bob, and Charlie to the remote receiver, Diana, using the general entangled channel $|\Psi^g\rangle_{3456}$ [see Eq. (11)] or ρ_{3456} [see Eq. (19)], in the case in which qudits 3, 4, and 5 are distributed to Bob, Charlie, and Alice, respectively.

(qudit). This is another difference between qudit-RIC and qubit-RIC. According to Ref. [9], the states of Eqs. (16) and (17) with $d = 2$ are not equivalent to each other, i.e., cannot be transformed into each other by LOCC. It can be verified that the states of Eqs. (16) and (17) with $d > 2$ are also LOCC inequivalent. This implies that Eq. (11) contains at least two inequivalent types of genuine four-partite pure entangled states. In other words, different types of genuine four-partite pure entangled states can implement a same multiparty QIP task, three-to-one RIC. Such a phenomenon is counterintuitive, since a given QIP task can be achieved by only typical structure of entangled states and different types of entangled states are usually competent for implementing different QIP tasks. It has been shown [31,32] that quantum teleportation can be deterministically implemented by using both multiqubit W and GHZ states, two inequivalent genuine multiqubit entangled states [33]. However, teleportation is a two-party communication, and the W and GHZ states in fact play the same role as the bipartite entangled state; i.e., only the bipartite entanglement of them is exploited. In contrast, RIC is a multiparty communication (each party holds one particle of the entangled channel), and the states of Eqs. (16) and (17) play a role of multipartite entanglement.

We now show that the quantum channel of our RIC can also be a broad family of mixed entangled states. Let $C_{m'n'} = \delta_{m',M}\delta_{n',N}$, where M and N are two arbitrarily chosen nonnegative integers that are less than d. Then the quantum channel of Eq. (11) reduces to a product state of two generalized Bell states,

$$|\Psi^{s_3}\rangle_{3456} = |B^{M,N}\rangle_{34}|B^{\overline{u-M},\overline{v-N}}\rangle_{56}. \tag{18}$$

Because the two constants M and N are arbitrary, we can deduce that the quantum channel of our RIC can also be the following form of mixed entangled states:

$$\rho_{3456} = \sum_{m',n'=0}^{d-1} |C_{m'n'}|^2 |B^{m',n'}\rangle_{34}\langle B^{m',n'}|$$
$$\otimes |B^{\overline{u-m'},\overline{v-n'}}\rangle_{56}\langle B^{\overline{u-m'},\overline{v-n'}}|. \tag{19}$$

This can be easily proved by resorting to a purified state of ρ_{3456},

$$|\Psi^\rho\rangle_{3456XY} = \sum_{m',n'=0}^{d-1} C_{m'n'}|B^{m',n'}\rangle_{34}|B^{\overline{u-m'},\overline{v-n'}}\rangle_{56}|B^{m',n'}\rangle_{XY}. \tag{20}$$

Particularly, by carrying out the same procedure as before [see Eqs. (12)–(15)], the information of $|\psi\rangle_{12a}$ can also be concentrated in qudit 6 via the entangled channel $|\Psi^\rho\rangle_{3456XY}$. In the whole process, qudits X and Y are not touched, and thus can be traced out at any time. This finishes the proof that the mixed state ρ_{3456} can be competent for our RIC. For the case $d > 2$, and using the entangled channel ρ_{3456} with $|C_{m'n'}| \neq 1/d$, qudits 3 and 4 can also not be simultaneously distributed to Bob and Charlie; otherwise, the information of $|\psi\rangle_{12a}$ cannot be successfully concentrated to qudit 6. This can be understood from Eq. (10) and that ρ_{3456} with $|C_{m'n'}| \neq 1/d$ cannot be expanded as the same form as Eq. (19) with respect

to the 2 : 2 partition $\{\{3,5\},\{4,6\}\}$ or $\{\{3,6\},\{4,5\}\}$. However, there is no such a constraint for qubit-RIC [13,17].

If we set $u = v = 0$ and $|C_{m'n'}| = 1/d$, Eq. (19) reduces to

$$\rho'_{3456} = \frac{1}{d^2} \sum_{m',n'=0}^{d-1} |B^{m',n'}\rangle_{34}\langle B^{m',n'}| \otimes |B^{\overline{-m'},\overline{-n'}}\rangle_{56}\langle B^{\overline{-m'},\overline{-n'}}|. \tag{21}$$

By Eq. (13), we can rewrite ρ'_{3456} as

$$\rho'_{3456} = \frac{1}{d^2} \sum_{m',n'=0}^{d-1} |B^{m',n'}\rangle_{36}\langle B^{m',n'}| \otimes |B^{\overline{-m'},\overline{-n'}}\rangle_{54}\langle B^{\overline{-m'},\overline{-n'}}|. \tag{22}$$

For $d = 2$, ρ'_{3456} is exactly the Smolin state [14], a four-qubit unlockable bound entangled state. The Smolin state is fully symmetric; i.e., it is unchanged under permutation of any two qubits. This leads to the Smolin state being separable with respect to any 2 : 2 partition of $\{3,4,5,6\}$. For $d > 2$, ρ'_{3456} also describes an unlockable bound entangled state. It can be seen from Eqs. (21) and (22) that for any two qudits $x \neq y \in \{3,4,5,6\}$, there exists at least one partition $\{G_1,G_2\}$ ($G_1 \cap G_2 = \emptyset$ and $G_1 \cup G_2 = \{3,4,5,6\}$) with $x \in G_1$ and $y \in G_2$ such that ρ'_{3456} is separable with respect to this partition, which implies that it is impossible to distill out pure entanglement between x and y, even between G_1 and G_2, by LOCC, as long as G_1 and G_2 remain spatially separated. Thus ρ'_{3456} is undistillable when the four particles are spatially separated. The unlockability or activability of ρ'_{3456} is obvious. Particularly, it can be unlocked as follows. Let qudits 3 and 4 (3 and 6) join together and perform a GBM on them. Then depending on the measurement outcome qudits 5 and 6 (4 and 5) is projected in a generalized Bell state. That is, pure entanglement is distilled out between qudits 5 and 6 (4 and 5). However, ρ'_{3456} with $d > 2$ is an asymmetric but not symmetric unlockable bound entangled state, because it is not separable with respect to the 2 : 2 partition $\{\{3,5\},\{4,6\}\}$. In addition, it can be verified that ρ'_{3456} cannot be superactivated for $d > 2$, which presents a striking contrast to the Smolin bound entangled state being superactivable [34,35]. These results indicate that there exists an analog to the Smolin state in multilevel systems; however, it has some different features. Note that the asymmetric four-qudit unlockable bound entangled state ρ'_{3456} is not contained in Ref. [36]. Therefore, it is a "new" asymmetric unlockable bound entangled state.

As shown above, many different types of entangled states, including mixed entangled states as well as pure ones, can be exploited as the quantum channel of three-to-one RIC. The pure states can be double-Bell states and LOCC inequivalent genuine four-partite entangled states. The mixed states can even be bound entangled states. However, it can be verified that all these states have a common feature that they have d^2 common commuting stabilizers $\{S^{jk} = U_3^{\overline{-j},k} \otimes U_4^{j,k} \otimes U_5^{\overline{-j},k} \otimes U_6^{j,k} : j,k = 0,1,\ldots,d-1\}$. That is, for any j and k, $\mathrm{tr}(S^{jk}|\Psi^g\rangle_{3456}\langle \Psi^g|) = \mathrm{tr}(S^{jk}\rho_{3456}) = 1$.

III. DISCUSSION AND CONCLUSION

We now give a brief discussion on the physical or experimental realization of the RIC presented in Sec. II. Light quantum states can be utilized for implementing qudits by exploiting various degrees of freedom of photons, such as polarization [37–39], orbital angular momentum (OAM) [40,41], path mode [42–44], time bin [45], or a combination of different degrees of freedom (see, e.g., [46,47]), and so on. Indeed, many optical realizations, manipulations, and applications of qudits and entangled qudits with the aforementioned degrees of freedom have been experimentally demonstrated [40,43,45,47–52]. As to the experimental implementation of RIC for qudits, one mainly needs to consider three points as follows: (i) preparation of the entangled channel, i.e., preparing d-level Bell states (bipartite maximally entangled states) or GHZ states, or the unlockable bound entangled states of Eq. (21); (ii) realization of $1 \rightarrow 2$ optimal tele-cloning (or cloning) of a d-level arbitrary quantum state; (iii) implementation of GBM in d-level systems. All these building blocks are achievable in quantum optics as illustrated below. Many schemes for generating high-dimensional entangled states of photonic qudits have been proposed and demonstrated. Experimental realization of two-qutrit ($d = 3$) maximally entangled states (generalized Bell-basis states, or can be transformed into any one of d^2 Bell-basis states by local operations, *to be uniformly referred to as generalized or qudit Bell states*) with each qutrit encoded by three polarization states of two frequency-degenerate photons in the same spatiotemporal mode (biphoton) has already been reported [53,54]. A flexible scheme for generating various entangled states (including generalized Bell states) of two ququarts ($d = 4$) using polarization degrees of freedom of the frequency-nondegenerate biphoton was put forward [55], which is scalable to generating various multiququart entangled states. Simple schemes for creating h-color entangled states (including generalized Bell states or GHZ states) of N qudits ($1 \leqslant h \leqslant N$) with multiphoton polarization were also proposed [39], in which N and the dimension d can be arbitrarily large with sacrifice of success probability in principle. By using OAM of photons, the Zeilinger research group and co-workers realized qutrit Bell states of two photons with different methods [49,56], and also showed that two-qudit photonic entanglement up to $d = 21$ are experimentally realizable via a spatial light modulator [57]; Torres *et al.* presented another method to generate two-photon high-dimensional maximally entangled states and demonstrated the preparation of nine Bell-basis states of two qutrits, which is based on the use of a coherent and engineerable superposition of modes as a pump signal [58]; these methods together with OAM beam splitter [59] make it possible to create multi-qudit entangled states, e.g., multilevel GHZ states. Four- and eight-level Bell states of two photons with path mode have recently been reported [43,60]; we conjecture that these techniques together with $2d$-port beam splitter [42] could be used to create d-level GHZ states, as a natural extension of 2×2-port beam splitter synthesizing qubit GHZ states from qubit Bell states. Energy-time or time-bin generalized Bell states of two

photonic qutrits have also been experimentally realized [61]. The d-level unlockable bound entangled state of Eq. (21) can be created from two identical d-level Bell-basis states by randomly (with equal probability) and <u>simultaneously</u> performing the pairwise operations $\{U^{m,n}, U^{-m,-n}\}$ on two qudits belonging to, respectively, different Bell pairs [62]. Recently, a flexible scheme for $1 \rightarrow 2$ optimal universal cloning of a photonic ququart has been proposed and experimentally demonstrated by Nagali *et al.* [63], which is generally applicable to quantum states of arbitrarily high dimension and is scalable to an arbitrary number of copies [63,64]. As to the optical implementation of GBM, two schemes have also been put forward. Halevy *et al.* proposed and experimentally demonstrated a realization of three-level GBM, with each qutrit being represented by the polarization of biphoton [65]. Dušek presented a method to implement GBM of path-mode-encoded qudits [66]. The aforementioned schemes of cloning and GBM could also be generalized or applied to other optical systems mentioned above because of the permission of mapping or converting between different degrees of freedom [67,68]. The illustrations and analysis given above appear possible for experimental implementation of RIC in multilevel systems.

In conclusion, we have studied the RIC in multilevel systems, and shown that the information of the three-qudit universal cloning state can be remotely and deterministically concentrated to a single qudit via an entangled channel with LOCC. Minor differences of qudit-RIC with qubit-RIC have also been analyzed. It has been shown that there are minor constraints on the distribution of the general entangled channel for qudit-RIC, but none for qubit-RIC. Moreover, telecloning and RIC for qubits can be achieved by using the same entangled channel, but there is no such a feature for qudits.

We investigated many types of entangled states as the quantum channel, including mixed entangled states as well as pure ones, and found some interesting phenomena. Similar to qubit-RIC, qudit-RIC can also be implemented by an unlockable bound entangled state. Though such a multilevel bound entangled state has a similar form to the Smolin bound entangled state, it has some different features. As a matter of fact, they belong to different types of unlockable bound entangled states: The former one is asymmetric and the latter one is symmetric. It has been shown that the quantum channel of RIC can be different types of genuine four-partite pure entangled states which are LOCC inequivalent. Moreover, we found that all these states, which can act as the quantum channel of RIC, have d^2 common commuting stabilizers. This implies that the states which have common stabilizers could be competent for implementing (deterministically) some same QIP tasks. We hope these phenomena will stimulate more research into the topic of dividing or classifying entangled states by the usefulness for typical QIP tasks. Maybe this needs resorting to the stabilizers. Then the *genuine* multipartite pure entangled states which can be competent for implementing (deterministically) one or more same *multiparty* tasks may be LOCC inequivalent. In view of the fact that entanglement is a very important physical *resource* for QIP, this topic should be meaningful and important.

ACKNOWLEDGMENTS

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