

## Optimal superdense coding over memory channels

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We study the superdense coding capacity in the presence of quantum channels with correlated noise. We investigate both the cases of unitary and nonunitary encoding. Pauli channels for arbitrary dimensions are treated explicitly. The superdense coding capacity for some special channels and resource states is derived for unitary encoding. We also provide an example of a memory channel where nonunitary encoding leads to an improvement in the superdense coding capacity.

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### I. INTRODUCTION

Superdense coding is one of the notable areas in which quantum entanglement plays a crucial role. By this protocol, due to the nonlocal properties of quantum entanglement, it is possible to communicate two bits of classical information by sending one qubit only [1]. The first attention, after proposing the superdense coding protocol, was given to various scenarios over noiseless channels and unitary encoding [2–4]. In this case one starts from a  $d^2$ -dimensional bipartite shared state  $\rho$  between the sender Alice and the receiver Bob. Alice performs with probability  $p_i$  a local unitary operation  $W_i$  on her subsystem to encode classical information through the state  $\rho_i = (W_i \otimes \mathbb{1})\rho(W_i^\dagger \otimes \mathbb{1})$ . Subsequently, she sends her subsystem to Bob (ideally via a noiseless channel). The ensemble that Bob receives is  $\{\rho_i, p_i\}$ . The maximal amount of classical information that can reliably be transmitted in this process is known as superdense coding capacity. It has been shown that for noiseless channels and unitary encoding, the capacity is given by  $C = \log_2 d + S(\rho_b) - S(\rho)$  [2,5]. Here,  $\rho_b$  is Bob’s reduced density operator with  $\rho_b = \text{tr}_a \rho$ , and  $S(\rho) = -\text{tr}(\rho \log_2 \rho)$  is the von Neumann entropy. Without the additional resource of entangled states, a  $d$ -dimensional quantum state can be used to transmit the information  $\log_2 d$ . Hence, quantum states for which  $S(\rho_b) - S(\rho) > 0$ , i.e., those which are more mixed locally than globally, are the useful states for superdense coding.

A realistic quantum system usually suffers from unwanted interactions with the outside world. Optical fibers and an unmodulated spin chain [6] are examples of such quantum channels which are suitable for long- and short-distance quantum communication, respectively. Superdense coding in the situation when the quantum states experience noise in the transmission channels was studied in [7]. In [7] uncorrelated noise (i.e., memoryless channels) was discussed. For those cases (channels and states) where the von Neumann entropy fulfills a specific condition, the superdense coding capacity was derived. Explicitly, for the two-dimensional uncorrelated depolarizing channel, it was shown that Alice and Bob do not

win by sending classical information via a superdense coding protocol with unitary encoding if there is too much noise.

In this paper, memory effects along the transmission channel are taken into account. In this scenario the noisy channel acting on two subsystems cannot be expressed as a product of two independent channels acting on each subsystem separately. In particular, we investigate the bipartite superdense coding scenario for a *correlated* Pauli channel and unitary or nonunitary encoding. Such kinds of channels were originally analyzed from the point of view of optimization of the classical information transmission [8–10].

The paper is organized as follows. In Sec. II we review the Holevo bound as a key concept in finding the superdense coding capacity. We discuss the mathematical definition of the Holevo quantity in the presence of an arbitrary channel  $\Lambda$ . Section III is devoted to the superdense coding capacity in the presence of a correlated Pauli channel and unitary encoding. We give examples of correlated channels and initial states for which the capacity is explicitly determined. Section IV is dedicated to the correlated Pauli channel and nonunitary encoding. We compare the capacities related to both unitary and nonunitary encoding and also discuss a case where nonunitary encoding has an advantage over unitary encoding. Finally, in Sec. V, we summarize the main results.

### II. CAPACITY OF SUPERDENSE CODING

The performance of a given composite state  $\rho$  for superdense coding is usually quantified by the Holevo quantity, maximized over all possible encodings on Alice’s side. A theorem stated by Gordon [11] and Levitin [12], and proved by Holevo [13], states that the amount of accessible classical information ( $I_{\text{acc}}$ ) contained in an ensemble  $\{\rho_i, p_i\}$  is upper bounded by the so-called  $\chi$  quantity  $\chi(\{\rho_i, p_i\})$ , often also referred to as the Holevo quantity. This upper bound holds for any measurement that can be performed on the system and is given by

$$I_{\text{acc}} \leq \chi(\{\rho_i, p_i\}) \equiv S(\bar{\rho}) - \sum_i p_i S(\rho_i), \quad (1)$$

where  $\bar{\rho} = \sum_i p_i \rho_i$  is the average ensemble state and  $S(\eta) = -\text{tr}(\eta \log_2 \eta)$  is the von Neumann entropy of  $\eta$ . From the

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concavity of the von Neumann entropy  $S(\rho)$ , it follows that the Holevo quantity is non-negative. The Holevo bound (1) is achievable in the asymptotic limit [14,15].

### A. Holevo quantity in the presence of noise

A quantum channel is a communication channel which can transmit quantum information. Physically, a noisy quantum channel is a communication channel that is affected by interaction with the environment. Mathematically, a noisy quantum channel can be described as a completely positive trace preserving (CPTP) map acting on the quantum state that is transmitted. We consider  $\Lambda : \rho_i \rightarrow \Lambda(\rho_i)$  to be a CPTP map that acts on the encoded state  $\rho_i = (W_i \otimes \mathbb{1})\rho(W_i^\dagger \otimes \mathbb{1})$ . For  $\{\Lambda(p_i, \rho_i)\}$  being the ensemble that Bob receives, the Holevo quantity is given by

$$\begin{aligned} \chi\{\Lambda(\rho_i, p_i)\} &= S(\overline{\Lambda(\rho)}) - \sum_i p_i S(\Lambda(\rho_i)) \\ &= \sum_i p_i S(\Lambda(\rho_i) \| \overline{\Lambda(\rho)}), \end{aligned} \quad (2)$$

where  $\overline{\Lambda(\rho)} = \sum_i p_i \Lambda(\rho_i)$  is the average state after transmission through the noisy channel and  $S(\rho \| \sigma) = \text{tr} \rho (\log_2 \rho - \log_2 \sigma)$  is the relative entropy. The superdense coding capacity  $C$  for a given resource state  $\rho$  and the noisy channel  $\Lambda$  is defined to be the maximum of the Holevo quantity  $\chi\{\Lambda(p_i, \rho_i)\}$  with respect to the unitary operators  $W_i$ , chosen with the probabilities  $p_i$ , namely,

$$\begin{aligned} C &= \max_{\{W_i, p_i\}} [\chi\{\Lambda(p_i, \rho_i)\}] \\ &\equiv \max_{\{W_i, p_i\}} \left[ S(\overline{\Lambda(\rho)}) - \sum_i p_i S(\Lambda(\rho_i)) \right]. \end{aligned} \quad (3)$$

In the following, we will concentrate on the optimization of the Holevo quantity in order to find the superdense coding capacity.

## III. SUPERDENSE CODING VIA CORRELATED PAULI CHANNELS

We will now consider quantum channels with memory, where noise in consecutive uses of the channel is correlated. We specifically consider correlated Pauli channels [8–10], modeled as follows. Consider first a single Pauli channel, whose action on a  $d$ -dimensional density operator  $\xi$  is given by

$$\Lambda^P(\xi) = \sum_{m,n=0}^{d-1} q_{mn} V_{mn} \xi V_{mn}^\dagger, \quad (4)$$

where  $q_{mn}$  are probabilities (i.e.,  $q_{mn} \geq 0$  and  $\sum_{mn} q_{mn} = 1$ ). The unitary *displacement* operators  $V_{mn}$  are defined as

$$V_{mn} = \sum_{k=0}^{d-1} \exp\left(\frac{2i\pi kn}{d}\right) |k\rangle \langle k + m(\text{mod } d)|. \quad (5)$$

The above operators satisfy  $\text{tr} V_{mn} = d\delta_{m0}\delta_{n0}$  and  $V_{mn} V_{m'n'}^\dagger = \mathbb{1}$ , and commute up to a phase,

$$V_{mn} V_{\tilde{m}\tilde{n}} = \exp\left(\frac{2i\pi(\tilde{m}m - n\tilde{n})}{d}\right) V_{\tilde{m}\tilde{n}} V_{mn}. \quad (6)$$

As the operators  $V_{mn}$  in Eq. (4) are unitary, the Pauli channel is unital, i.e., it preserves the identity. Now, let  $\Lambda_a^P$  and  $\Lambda_b^P$  be two  $d$ -dimensional Pauli channels (4) with the elements  $\{q_{mn}, V_{mn}\}$  and  $\{q_{\tilde{m}\tilde{n}}, V_{\tilde{m}\tilde{n}}\}$ , respectively. Based on the elements of these two channels, a model of a correlated Pauli channel is defined as

$$\Lambda_{ab}^P(\xi) = \sum_{m,n,\tilde{m},\tilde{n}=0}^{d-1} q_{mn\tilde{m}\tilde{n}} (V_{mn} \otimes V_{\tilde{m}\tilde{n}}) \xi (V_{mn}^\dagger \otimes V_{\tilde{m}\tilde{n}}^\dagger), \quad (7)$$

where the probability  $q_{mn\tilde{m}\tilde{n}}$  is given by  $q_{mn\tilde{m}\tilde{n}} = (1 - \mu)q_{mn}q_{\tilde{m}\tilde{n}} + \mu q_{mn}\delta_{m,\tilde{m}}\delta_{n,\tilde{n}}$ , and the parameter  $\mu$  ( $0 \leq \mu \leq 1$ ) quantifies the correlation degree. For  $\mu = 0$  the two channels  $\Lambda_a^P$  and  $\Lambda_b^P$  are uncorrelated and act independently on Alice's and Bob's subsystems, respectively. For  $\mu = 1$ , the global channel (7) is called *fully correlated* and for other values of  $\mu$ , different from zero and one, the global channel is partially correlated.

For a single sender, a single receiver, and a correlated Pauli channel as well as unitary and nonunitary encoding, we derive two explicit expressions for the superdense coding capacity. We show that both unitary and nonunitary encoding problems reduce to the problem of finding a single CPTP map (in the case of unitary encoding this is a specific unitary transformation) that minimizes the output von Neumann entropy after its application and the action of the channel on the input state  $\rho$ . For the case of unitary encoding we find examples for the optimal unitary operator.

### A. Unitary encoding

This section treats the optimization of the Holevo quantity for a correlated Pauli channel and unitary encoding. We first introduce an upper bound on the Holevo quantity and we then show that this upper bound is reachable and thus coincides with the superdense coding capacity. This procedure is phrased in the following lemma.

*Lemma 1.* Let

$$\chi = S(\overline{\Lambda_{ab}^P(\rho)}) - \sum_i p_i S(\Lambda_{ab}^P(\rho_i)) \quad (8)$$

be the Holevo quantity with  $\rho_i = (W_i \otimes \mathbb{1})\rho(W_i^\dagger \otimes \mathbb{1})$ , the average state  $\overline{\Lambda_{ab}^P(\rho)} = \sum_i p_i \Lambda_{ab}^P(\rho_i)$  and  $\Lambda_{ab}^P$  the correlated Pauli channel defined via Eq. (7). Let  $U_{\min}$  be the unitary operator that minimizes the von Neumann entropy after application of this unitary operator and the channel  $\Lambda_{ab}^P$  to the initial state  $\rho$ , i.e.,  $U_{\min}$  minimizes the expression  $S(\Lambda_{ab}^P((U_{\min} \otimes \mathbb{1})\rho(U_{\min}^\dagger \otimes \mathbb{1})))$ . Then the superdense coding capacity  $C_{\text{un}}^P$  is given by

$$\begin{aligned} C_{\text{un}}^P &= \log_2 d + S(\Lambda_b^P(\rho_b)) \\ &\quad - S(\Lambda_{ab}^P((U_{\min} \otimes \mathbb{1})\rho(U_{\min}^\dagger \otimes \mathbb{1}))), \end{aligned} \quad (9)$$

where  $\rho_b = \text{tr}_a \rho$  and  $\Lambda_b^P$  is the  $d$ -dimensional Pauli channel (4) on Bob's subsystem. The subscript "un" refers to unitary encoding.

*Proof:* We start by introducing an upper bound on the Holevo quantity (8). Since  $U_{\min}$  is a unitary operator that leads to the minimum of the output von Neumann entropy, for  $\chi$  we have

$$\chi \leq S(\overline{\Lambda_{ab}^P(\rho)}) - S(\Lambda_{ab}^P((U_{\min} \otimes \mathbb{1})\rho(U_{\min}^\dagger \otimes \mathbb{1}))).$$

The von Neumann entropy is subadditive and the maximum entropy of a  $d$ -dimensional system is  $\log_2 d$ . Therefore we can write

$$\begin{aligned} \chi &\leq \log_2 d + S(\text{tr}_a \overline{\Lambda_{ab}^P(\rho)}) \\ &\quad - S(\Lambda_{ab}^P((U_{\min} \otimes \mathbb{1})\rho(U_{\min}^\dagger \otimes \mathbb{1}))) \\ &= \log_2 d + S(\Lambda_b^P(\rho_b)) \\ &\quad - S(\Lambda_{ab}^P((U_{\min} \otimes \mathbb{1})\rho(U_{\min}^\dagger \otimes \mathbb{1}))), \end{aligned} \quad (10)$$

where we have used that  $\text{tr}_a \overline{\Lambda_{ab}^P(\rho)} = \Lambda_b^P(\rho_b)$ . This statement can be proved as follows. By using the definition of a correlated Pauli channel (7) and of the average state, and noting that  $W_i$  acts on Alice's side, we have

$$\begin{aligned} \text{tr}_a \overline{\Lambda_{ab}^P(\rho)} &= \sum_i p_i \sum_{m,n,\bar{m},\bar{n}=0}^{d-1} q_{mn\bar{m}\bar{n}} \text{tr}_a [(V_{mn} \otimes V_{\bar{m}\bar{n}}) \\ &\quad \times (W_i \otimes \mathbb{1})\rho(W_i^\dagger \otimes \mathbb{1})(V_{mn}^\dagger \otimes V_{\bar{m}\bar{n}}^\dagger)] \\ &= \sum_i p_i \sum_{m,n,\bar{m},\bar{n}=0}^{d-1} q_{mn\bar{m}\bar{n}} V_{\bar{m}\bar{n}} \rho_b V_{\bar{m}\bar{n}}^\dagger \\ &= \sum_{\bar{m}\bar{n}} q_{\bar{m}\bar{n}} V_{\bar{m}\bar{n}} \rho_b V_{\bar{m}\bar{n}}^\dagger = \Lambda_b^P(\rho_b), \end{aligned} \quad (11)$$

which completes this part of the proof. To show that the upper bound (10) is achievable, we consider the ensemble  $\{\tilde{p}_i = \frac{1}{d^2}, \tilde{U}_i = V_i U_{\min}\}$  with  $V_{i(=mn)}$  being the displacement operators of Eq. (5). The Holevo quantity for this ensemble is denoted by  $\tilde{\chi}$  and is given by

$$\begin{aligned} \tilde{\chi} &= S\left(\sum_i \frac{1}{d^2} \Lambda_{ab}^P((\tilde{U}_i \otimes \mathbb{1})\rho(\tilde{U}_i^\dagger \otimes \mathbb{1}))\right) \\ &\quad - \sum_i \frac{1}{d^2} S(\Lambda_{ab}^P((\tilde{U}_i \otimes \mathbb{1})\rho(\tilde{U}_i^\dagger \otimes \mathbb{1}))). \end{aligned} \quad (12)$$

In [2], for an arbitrary bipartite state  $\tau$ , it was shown that  $\frac{1}{d^2} \sum_i (V_i \otimes \mathbb{1})\tau(V_i^\dagger \otimes \mathbb{1}) = \frac{1}{d} \text{tr}_a \tau$ . By using this property, and noting that  $U_{\min}$  acts only on Alice's side, we find that the argument in the first term on the right-hand side of Eq. (12) is given by

$$\begin{aligned} &\sum_i \frac{1}{d^2} \Lambda_{ab}^P((\tilde{U}_i \otimes \mathbb{1})\rho(\tilde{U}_i^\dagger \otimes \mathbb{1})) \\ &= \Lambda_{ab}^P \sum_i \frac{1}{d^2} [(V_i U_{\min} \otimes \mathbb{1})\rho(U_{\min}^\dagger V_i^\dagger \otimes \mathbb{1})] \\ &= \Lambda_{ab}^P \left( \frac{1}{d} \otimes \rho_b \right) = \frac{1}{d} \otimes \Lambda_b^P(\rho_b). \end{aligned} \quad (13)$$

Furthermore, the second term on the right-hand side of Eq. (12) can be expressed in terms of the unitary operator  $U_{\min}$  and the channel. By inserting the action of the correlated Pauli channel, using Eq. (6), from which follows that  $V_{i(=jk)}$  and  $V_{mn}$  commute up to a phase, and since the von Neumann entropy is invariant under unitary transformation, we can write

$$\begin{aligned} &\sum_i \frac{1}{d^2} S(\Lambda_{ab}^P(\tilde{U}_i \otimes \mathbb{1})\rho(\tilde{U}_i^\dagger \otimes \mathbb{1})) \\ &= \frac{1}{d^2} \sum_{kj} S\left((V_{jk} \otimes \mathbb{1}) \left[ \sum_{m,n,\bar{m},\bar{n}} q_{mn\bar{m}\bar{n}} (V_{mn} \otimes V_{\bar{m}\bar{n}}) \right. \right. \\ &\quad \left. \left. \times (U_{\min} \otimes \mathbb{1})\rho(U_{\min}^\dagger \otimes \mathbb{1})(V_{mn}^\dagger \otimes V_{\bar{m}\bar{n}}^\dagger) \right] (V_{jk}^\dagger \otimes \mathbb{1})\right) \\ &= S(\Lambda_{ab}^P((U_{\min} \otimes \mathbb{1})\rho(U_{\min}^\dagger \otimes \mathbb{1}))). \end{aligned} \quad (14)$$

Inserting Eqs. (13) and (14) into Eq. (12), one finds that the Holevo quantity  $\tilde{\chi}$  is equal to the upper bound given in Eq. (10) and consequently, this is the superdense coding capacity. ■

By Lemma 1, we proved that in order to determine the superdense coding capacity, it is enough to find an optimal  $U_{\min}$  that minimizes the channel output von Neumann entropy  $S(\Lambda_{ab}^P((U_{\min} \otimes \mathbb{1})\rho(U_{\min}^\dagger \otimes \mathbb{1})))$ . In the next two sections we give examples of channels and initial states for which  $U_{\min}$  can explicitly be determined.

## B. Correlated quasiclassical channel

A  $d$ -dimensional quasiclassical depolarizing channel (or simply quasiclassical channel) is a particular form of a  $d$ -dimensional Pauli channel [9,10], as given in Eq. (4). For this channel, the probabilities of the displacement operators  $V_{mn}$  are equal for  $m = 0$  and any phase shift labeled by  $n$ , and they differ from the rest of the probabilities which are also equal, i.e.,

$$q_{mn} = \begin{cases} \frac{1-p}{d}, & m = 0 \\ \frac{p}{d(d-1)}, & \text{otherwise.} \end{cases} \quad (15)$$

The quasiclassical channel is characterized by a single probability parameter  $0 \leq p \leq 1$ . With probability  $p$ , a displacement occurs and with probability  $1 - p$ , no displacement occurs to the quantum signal. As in the classical case,  $p$  can also be seen as the amount of noise in a channel.

In the following we will consider as a resource state a Werner state  $\rho_w = \eta|\Phi^+\rangle\langle\Phi^+| + \frac{1-\eta}{4}\mathbb{1}$  with  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ . In the presence of a correlated quasiclassical channel, as defined via Eqs. (7) and (15), we find  $U_{\min}$ . Thus the dimension is  $d = 2$ . For two-dimensional systems the displacement operators, defined in Eq. (5), are either the identity or the Pauli operators, i.e.,

$$\begin{aligned} \sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (16)$$

The correlated quasiclassical channel is in this case

$$\Lambda_{\text{ab}}^Q(\xi) = \sum_{m,n} q_{mn} \sigma_m \otimes \sigma_n(\xi) \sigma_m \otimes \sigma_n, \quad (17)$$

where  $q_{mn} = (1 - \mu)q_m q_n + \mu q_n \delta_{mn}$  with  $q_0 = q_3 = \frac{1-p}{2}$  and  $q_1 = q_2 = \frac{p}{2}$ . To find  $U_{\text{min}}$ , we start with the most general  $2 \times 2$  unitary operator  $U$

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad (18)$$

where  $a$  and  $b$  are complex variables which satisfy  $|a|^2 + |b|^2 = 1$ . The output of a correlated quasiclassical channel for an arbitrary input state  $\rho$  is invariant under the unitary transformation  $\sigma_3 \otimes \sigma_3$  [9], i.e.,

$$\Lambda_{\text{ab}}^Q(\rho) = \Lambda_{\text{ab}}^Q((\sigma_3 \otimes \sigma_3)\rho(\sigma_3 \otimes \sigma_3)). \quad (19)$$

By the above property (19), the channel outputs for the input states  $(U \otimes \mathbb{1})\rho^W(U^\dagger \otimes \mathbb{1})$  and  $(\sigma_3 U \otimes \sigma_3)\rho^W(U^\dagger \sigma_3 \otimes \sigma_3)$  are equal and therefore we can conveniently use the average of both states instead of only  $(U \otimes \mathbb{1})\rho^W(U^\dagger \otimes \mathbb{1})$ . This average is given by

$$\begin{aligned} & \frac{1}{2}(U \otimes \mathbb{1})\rho^W(U^\dagger \otimes \mathbb{1}) + \frac{1}{2}(\sigma_3 U \otimes \sigma_3)\rho^W(U^\dagger \sigma_3 \otimes \sigma_3) \\ &= \begin{pmatrix} \eta \frac{aa^*}{2} + \frac{1-\eta}{4} & 0 & 0 & \eta \frac{a^2}{2} \\ 0 & \eta \frac{bb^*}{2} + \frac{1-\eta}{4} & \eta \frac{-b^2}{2} & 0 \\ 0 & \eta \frac{-(b^*)^2}{2} & \eta \frac{bb^*}{2} + \frac{1-\eta}{4} & 0 \\ \eta \frac{(a^*)^2}{2} & 0 & 0 & \eta \frac{aa^*}{2} + \frac{1-\eta}{4} \end{pmatrix} \\ &= |a|^2 \left( \eta |\Phi_1\rangle\langle\Phi_1| + \frac{1-\eta}{4} \mathbb{1} \right) \\ &+ |b|^2 \left( \eta |\Phi_2\rangle\langle\Phi_2| + \frac{1-\eta}{4} \mathbb{1} \right), \end{aligned} \quad (20)$$

with  $|\Phi_1\rangle$  and  $|\Phi_2\rangle$  being

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}} \left( \frac{a}{|a|} |00\rangle + \frac{a^*}{|a|} |11\rangle \right), \quad (21a)$$

$$|\Phi_2\rangle = \frac{1}{\sqrt{2}} \left( \frac{b}{|b|} |01\rangle - \frac{b^*}{|b|} |10\rangle \right). \quad (21b)$$

After applying the quasiclassical channel, using Eqs. (19) and (20), and the concavity of the von Neumann entropy, we find the following lower bound:

$$\begin{aligned} & S(\Lambda_{\text{ab}}^Q((U \otimes \mathbb{1})\rho^W(U^\dagger \otimes \mathbb{1}))) \\ & \geq |a|^2 S \left( \Lambda_{\text{ab}}^Q \left( \eta |\Phi_1\rangle\langle\Phi_1| + \frac{1-\eta}{4} \mathbb{1} \right) \right) \\ & + |b|^2 S \left( \Lambda_{\text{ab}}^Q \left( \eta |\Phi_2\rangle\langle\Phi_2| + \frac{1-\eta}{4} \mathbb{1} \right) \right) \\ & \geq S \left( \Lambda_{\text{ab}}^Q \left( \eta |\Phi^+\rangle\langle\Phi^+| + \frac{1-\eta}{4} \mathbb{1} \right) \right). \end{aligned} \quad (22)$$

In the last line we have used that both  $S(\Lambda_{\text{ab}}^Q(\eta |\Phi_{1,2}\rangle\langle\Phi_{1,2}| + \frac{1-\eta}{4} \mathbb{1}))$  are lower bounded by  $S(\Lambda_{\text{ab}}^Q(\eta |\Phi^+\rangle\langle\Phi^+| + \frac{1-\eta}{4} \mathbb{1}))$ . The proof for this statement is as follows. We can rewrite

$|\Phi_1\rangle$  (a similar argument holds for  $|\Phi_2\rangle$ ) up to a global phase as

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}} [|00\rangle + \exp(i\phi) |11\rangle]. \quad (23)$$

After applying the correlated quasiclassical channel  $\Lambda_{\text{ab}}^Q$  to the state  $\eta |\Phi_1\rangle\langle\Phi_1| + \frac{1-\eta}{4} \mathbb{1}$ , we arrive at

$$\begin{aligned} & \Lambda_{\text{ab}}^Q \left( \eta |\Phi_1\rangle\langle\Phi_1| + \frac{1-\eta}{4} \mathbb{1} \right) = \frac{1-\eta}{4} \mathbb{1} \otimes \mathbb{1} \\ & + \frac{\eta}{4} \{ [\mu + (1-\mu)(1-2p)^2] \sigma_3 \otimes \sigma_3 \\ & + \mu(1-2p) \sin \phi (\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1) \\ & + \mu \cos \phi (\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2) + \mathbb{1} \otimes \mathbb{1} \}. \end{aligned} \quad (24)$$

The von Neumann entropy of a quantum state is defined via its eigenvalues. The eigenvalues of Eq. (24) are

$$\begin{aligned} v_{1,2} &= \eta(1-\mu)p(1-p) + \frac{1-\eta}{4}, \\ v_{3,4} &= \frac{\eta}{2} \{ 1 - 2(1-\mu)p(1-p) \\ & \pm \sqrt{\mu^2 [1 - 4p(1-p) \sin^2 \phi]} \} + \frac{1-\eta}{4}. \end{aligned} \quad (25)$$

To minimize the von Neumann entropy  $S(\Lambda_{\text{ab}}^Q(\eta |\Phi_1\rangle\langle\Phi_1| + \frac{1-\eta}{4} \mathbb{1})) = -\sum_i v_i \log_2 v_i$ , the eigenvalues should diverge as much as possible with respect to the parameter  $\phi$ . The eigenvalues  $v_{1,2}$  are independent of  $\phi$ . Thus, the von Neumann entropy is minimized when we maximize  $v_3$  while we minimize  $v_4$ . This is the case for  $\phi = 0$  and it leads  $|\Phi_{1,2}\rangle$  to be the Bell state  $|\Phi^+\rangle$  which proves the above statement, namely,

$$\begin{aligned} & S \left( \Lambda_{\text{ab}}^Q \left( \eta |\Phi_{1,2}\rangle\langle\Phi_{1,2}| + \frac{1-\eta}{4} \mathbb{1} \right) \right) \\ & \geq S \left( \Lambda_{\text{ab}}^Q \left( \eta |\Phi^+\rangle\langle\Phi^+| + \frac{1-\eta}{4} \mathbb{1} \right) \right). \end{aligned} \quad (26)$$

■

The lower bound on the von Neumann entropy (22) is reachable. It is not difficult to see that the variables  $a = 1$  and  $b = 0$ , which correspond to  $U$  being the identity operator.

We have thus found a unitary operator that minimizes the output entropy. Therefore, the superdense coding capacity for a Werner state in a correlated quasiclassical channel, according to Eq. (9), is given by

$$C_{\text{un}}^{Q,W} = 2 - S(\Lambda_{\text{ab}}^Q(\rho^W)). \quad (27)$$

For  $\eta = 1$ , the Werner state  $\rho^W$  reduces to a Bell state  $|\Phi^+\rangle$ . Therefore, the superdense coding capacity, according to Eq. (27), for a Bell state and in the presence of a correlated quasiclassical channel is given by

$$C_{\text{un}}^{Q,B} = 2 - S(\Lambda_{\text{ab}}^Q(|\Phi^+\rangle\langle\Phi^+|)). \quad (28)$$

In Figs. 1 and 2, we visualize the superdense coding capacity for the correlated quasiclassical channel as a function of the parameters  $\mu$ ,  $\eta$ , and  $p$  [Eq. (27)]. In Fig. 1, we consider a Bell state, i.e.,  $\eta = 1$ , as a function of the noise parameter  $p$  and the correlation degree  $\mu$ . In Fig. 2, the noise parameter is fixed to  $p = 0.05$  and we vary  $\mu$  and the parameter  $\eta$  characterizing the Werner state.

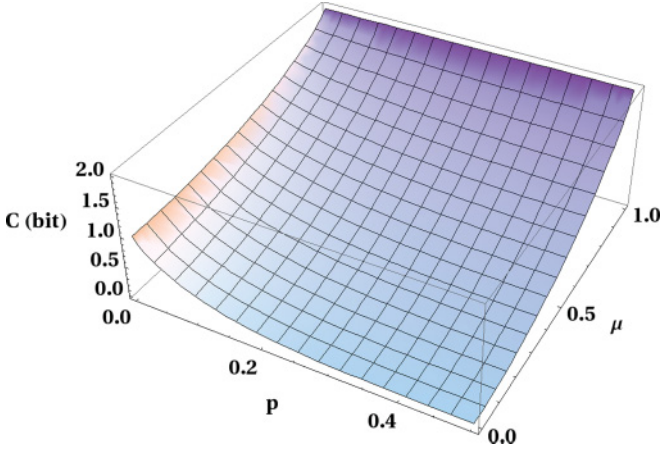


FIG. 1. (Color online) Superdense coding capacity for a correlated quasiclassical channel and a Bell state ( $\eta = 1$ ), as a function of the noise parameter  $p$  and the correlation degree  $\mu$ .

### C. Fully correlated Pauli channel

In this section we give another example for which we determined  $U_{\min}$ . That is the case of a fully correlated Pauli channel and a Werner state. As mentioned above, a fully correlated Pauli channel is a special form of a correlated Pauli channel (7) when  $\mu = 1$ . For  $d = 2$ , it is given by

$$\Lambda_{ab}^f(\xi) = \sum_m q_m(\sigma_m \otimes \sigma_m)(\xi)(\sigma_m \otimes \sigma_m), \quad (29)$$

where  $\sum_m q_m = 1$  and  $\sigma_m$  are either the identity or the Pauli operators.

We again consider the Werner states  $\rho^W = \eta|\Phi^+\rangle\langle\Phi^+| + \frac{1-\eta}{4}\mathbb{1}$  as resource states. For a fully correlated Pauli channel (29) we determine the operation  $U_{\min}$ . To do so, we derive a lower bound on  $S(\Lambda_{ab}^f((U \otimes \mathbb{1})\rho^W(U^\dagger \otimes \mathbb{1})))$ , where  $U$  is an arbitrary unitary operator. By using the concavity of the von Neumann entropy and also by using the invariance of the von

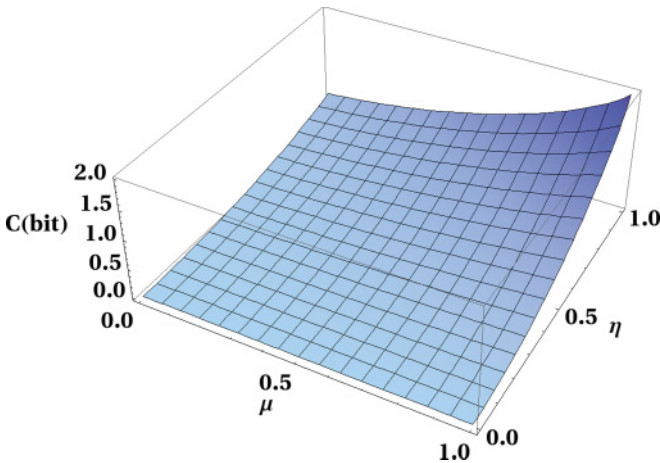


FIG. 2. (Color online) Superdense coding capacity for a correlated quasiclassical channel and a Werner state, as a function of the correlation degree  $\mu$  and the parameter  $\eta$ . The noise parameter is  $p = 0.05$ .

Neumann entropy under unitary transformations, the lower bound on  $S(\Lambda_{ab}^f((U \otimes \mathbb{1})\rho^W(U^\dagger \otimes \mathbb{1})))$  takes the form

$$\begin{aligned} & S(\Lambda_{ab}^f((U \otimes \mathbb{1})\rho^W(U^\dagger \otimes \mathbb{1}))) \\ &= S\left(\sum_m q_m(\sigma_m \otimes \sigma_m)(U \otimes \mathbb{1})\right. \\ &\quad \times \left.\left(\eta\rho^+ + \frac{1-\eta}{4}\mathbb{1}\right)(U^\dagger \otimes \mathbb{1})(\sigma_m \otimes \sigma_m)\right) \\ &\geq S\left(\eta\rho^+ + \frac{1-\eta}{4}\mathbb{1}\right), \end{aligned} \quad (30)$$

where we use the notation  $\rho^+ = |\Phi^+\rangle\langle\Phi^+|$ .

By using the invariance of a Bell state under the action of a fully correlated Pauli channel, i.e.,  $\Lambda_{ab}^f(\rho^+) = \rho^+$ , it follows that the lower bound (30) is reachable by the identity operator. Then  $U_{\min} = \mathbb{1}$  and the superdense coding capacity, according to Eq. (9), is given by

$$C_{\text{un}}^{f,W} = 2 - S(\Lambda_{ab}^f(\rho^W)). \quad (31)$$

The Werner state  $\rho^W$  reduces to a Bell state  $\rho^+$  for  $\eta = 1$ . Since the Bell state is invariant under the action of a fully correlated Pauli channel, its von Neumann entropy  $S(\Lambda_{ab}^f(\rho^+))$  is zero. Therefore, using Eq. (31), the superdense coding capacity for a shared Bell state and a fully correlated Pauli channel (29) is two bits. It is the maximum information transfer for  $d = 2$ . This shows that no information at all is lost to the environment and this class of channels behaves like a noiseless one. This behavior corresponds to the results of [8,9].

### IV. NON-UNITARY ENCODING

So far, we have assumed that the encoding in the superdense coding protocol is unitary. The superdense coding protocol with nonunitary encoding for noiseless channels has been discussed by Horodecki and Piani [16], Horodecki *et al.* [17], and Winter [18]. In this section we consider the possibility of performing nonunitary encoding in the presence of a correlated Pauli channel. Let us consider  $\Gamma_i$  to be a CPTP map. Alice applies the map  $\Gamma_i$  on her side of the shared state  $\rho$ , thereby encoding  $\rho$  as  $\rho_i = [\Gamma_i \otimes \mathbb{1}](\rho) := \Gamma_i(\rho)$ . The rest of the scheme is similar to the case of unitary encoding. Alice sends the encoded state  $\rho_i = \Gamma_i(\rho)$  with the probability  $p_i$  to Bob through the correlated Pauli channel  $\Lambda_{ab}^P$ . Now, the question is, which ensemble of CPTP maps achieves the superdense coding capacity? In other words, what is the optimum Holevo quantity with respect to the encoding  $\Gamma_i$  and  $p_i$ ? To answer this question, first we give the definition for the superdense coding capacity with a correlated Pauli channel and nonunitary encoding:

$$\begin{aligned} C &= \max_{\{\Gamma_i, p_i\}} \left[ S\left(\sum_i p_i \Lambda_{ab}^P(\Gamma_i(\rho))\right) \right. \\ &\quad \left. - \sum_i p_i S(\Lambda_{ab}^P(\Gamma_i(\rho))) \right], \end{aligned} \quad (32)$$

where  $\Lambda_{ab}^P(\rho)$  is defined via Eq. (7). Similar to the unitary encoding case in Sec. III A, we find an upper bound on the

Holevo quantity (32) and then we show that this upper bound is reachable by a preprocessing before unitary encoding. The above statement will be expressed in the following lemma.

*Lemma 2.* Let  $\chi$  be the Holevo quantity (32), and let  $\Gamma_{\min}(\rho) := [\Gamma_{\min} \otimes \mathbb{1}](\rho)$  be the map that minimizes the von Neumann entropy after application of this map and the correlated Pauli channel  $\Lambda_{ab}^P$  to the initial state  $\rho$ , i.e.,  $\Gamma_{\min}$  minimizes the expression  $S(\Lambda_{ab}^P(\Gamma_{\min}(\rho)))$ . Then the superdense coding capacity is given by

$$C_{\text{nonun}}^P = \log_2 d + S(\Lambda_b^P(\rho_b)) - S(\Lambda_{ab}^P(\Gamma_{\min}(\rho))), \quad (33)$$

where  $\rho_b = \text{tr}_a \rho$  and  $\Lambda_b^P$  is the  $d$ -dimensional Pauli channel Eq. (4) on Bob's subsystem.

*Proof:* The von Neumann entropy is subadditive and the maximum entropy of a  $d$ -dimensional system is  $\log_2 d$ , and since  $\Gamma_{\min}$  is a map that leads to the minimum of the entropy after applying it and the channel to the initial state  $\rho$ , we have the upper bound

$$\begin{aligned} \chi &\leq S\left(\sum_i p_i \Lambda_{ab}^P(\Gamma_i(\rho))\right) - S(\Lambda_{ab}^P(\Gamma_{\min}(\rho))), \\ &\leq \log_2 d + S\left(\text{tr}_a \left(\sum_i p_i \Lambda_{ab}^P(\Gamma_i(\rho))\right)\right) \\ &\quad - S(\Lambda_{ab}^P(\Gamma_{\min}(\rho))). \end{aligned}$$

By using  $\text{tr}_a (\sum_i p_i \Lambda_{ab}^P(\Gamma_i(\rho))) = \Lambda_b^P(\rho_b)$ , we find the upper bound

$$\chi \leq \log_2 d + S(\Lambda_b^P(\rho_b)) - S(\Lambda_{ab}^P(\Gamma_{\min}(\rho))). \quad (34)$$

We now show that the ensemble  $\{\tilde{p}_i, \tilde{\Gamma}_i(\rho)\}$  with  $\tilde{p}_i = \frac{1}{d^2}$  and  $\tilde{\Gamma}_i(\rho) = (V_i \otimes \mathbb{1})\Gamma_{\min}(\rho)(V_i^\dagger \otimes \mathbb{1})$  where  $V_{i(=mn)}$  is defined in Eq. (5), reaches the upper bound (34). In other words, the optimal encoding consists of a fixed preprocessing with  $\Gamma_{\min}$  and a subsequent unitary encoding. This is analogous to the case of noiseless channels and uncorrelated Pauli channels [7,16]. Below we prove the above claim.

The Holevo quantity of the ensemble  $\{\tilde{p}_i, \tilde{\Gamma}_i(\rho)\}$  is

$$\tilde{\chi} = S\left(\sum_i \frac{1}{d^2} \Lambda_{ab}^P(\tilde{\Gamma}_i(\rho))\right) - \sum_i \frac{1}{d^2} S(\Lambda_{ab}^P(\tilde{\Gamma}_i(\rho))). \quad (35)$$

With an argument similar to the case of unitary encoding, the first term on the right-hand side of Eq. (35) is given by

$$\sum_i \frac{1}{d^2} \Lambda_{ab}^P(\tilde{\Gamma}_i(\rho)) = \frac{\mathbb{1}}{d} \otimes \Lambda_b^P(\rho_b). \quad (36)$$

Furthermore, for the second term on the right-hand side of Eq. (35) we have

$$\sum_i \frac{1}{d^2} S(\Lambda_{ab}^P(\tilde{\Gamma}_i(\rho))) = S(\Lambda_{ab}^P(\Gamma_{\min}(\rho))). \quad (37)$$

Inserting Eqs. (36) and (37) into Eq. (35), one finds that the Holevo quantity  $\tilde{\chi}$  is equal to the upper bound given in Eq. (34). Consequently, the superdense coding capacity with nonunitary encoding is determined by Eq. (33). ■

A comparison of Eqs. (33) and (9) shows that applying an appropriate preprocessing  $\Gamma_{\min}$  on the initial state  $\rho$  before the unitary encoding  $\{V_i\}$  may increase the superdense coding capacity, with respect to only using unitary encoding for the case of a correlated Pauli channel. However, for some examples, no better encoding than unitary encoding is possible. For instance, since two bits is the highest superdense coding capacity for  $d = 2$ , our results derived in Sec. III C for fully correlated Pauli channel and the Bell state provide an example where no preprocessing can improve the capacity. However, examples exist for which nonunitary preprocessing is useful to increase the superdense coding capacity. In the next section we provide an explicit example.

### A. Preprocessing can improve capacity

Here, we show that for a two-dimensional Bell state in the presence of a correlated quasiclassical channel, a nonunitary preprocessing  $\Gamma$ , which is not necessarily  $\Gamma_{\min}$ , can improve the superdense coding capacity. To show this claim, consider the completely positive trace preserving preprocessing  $\Gamma$ , with the Kraus operators  $E_1 = |0\rangle\langle 1|$  and  $E_2 = |0\rangle\langle 0|$ . Alice applies  $\Gamma$  on her side of the Bell state  $\rho^+ = |\Phi^+\rangle\langle\Phi^+|$  and transforms the Bell state to  $\Gamma(\rho^+) = |0\rangle\langle 0| \otimes \frac{\mathbb{1}}{2}$ . Therefore, according to Eq. (33), for a correlated quasiclassical channel, a Bell state, and a preprocessing  $\Gamma$ , the amount of information that is transmitted by this process is given by

$$C_{\Gamma}^{Q,B} = 1 + p \log_2 p + (1-p) \log_2(1-p), \quad (38)$$

where  $p$  is the noise parameter for a quasiclassical channel (15). Since  $\Gamma$  is not necessarily the optimal preprocessing,  $C_{\Gamma}^{Q,B}$  is not also necessarily the capacity. We name Eq. (38) the *transferred information*. We now compare the transferred information (38) with the capacity (28) which is achieved by applying only unitary encoding. In the range of  $0.3 \leq p \leq 1$  we find that the capacity  $C_{\text{un}}^{Q,B}$  is always higher than the transferred information  $C_{\Gamma}^{Q,B}$ , i.e.,  $C_{\Gamma}^{Q,B} < C_{\text{un}}^{Q,B}$ . Therefore,

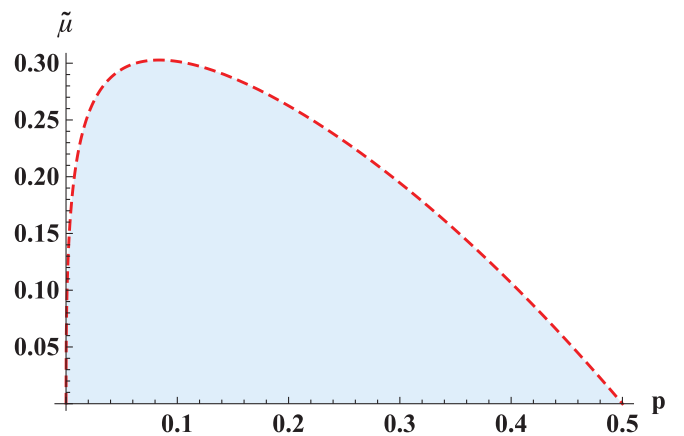


FIG. 3. (Color online) The dashed red curve is the correlation degree  $\tilde{\mu}(p)$  as a function of the noise parameter  $p$ . The superdense coding capacity  $C_{\text{un}}^{Q,B}$  and the transferred information  $C_{\Gamma}^{Q,B}$  coincide for  $\mu < \tilde{\mu}(p)$  (see main text). For  $\mu < \tilde{\mu}(p)$ , the blue (light gray) area, the nonunitary preprocessing  $\Gamma$  increases the superdense coding capacity of a quasiclassical channel and a Bell state, in comparison to just unitary encoding.

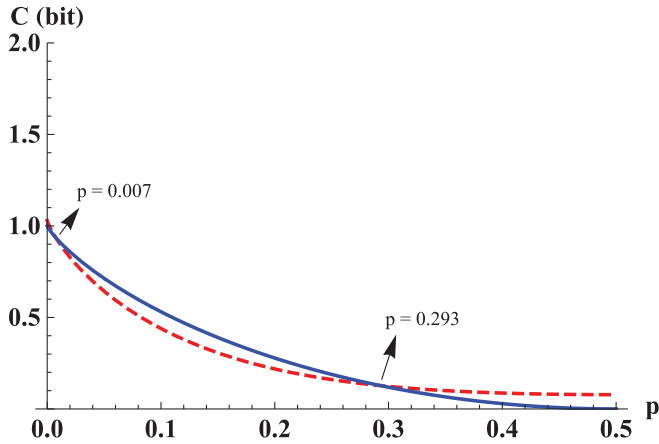


FIG. 4. (Color online) Comparison between the superdense coding capacity (28) and the transferred information (38) as functions of the noise parameter  $p$ , with the correlation degree  $\mu = 0.2$ . The dashed red curve corresponds to the capacity  $C_{\text{un}}^{Q,B}$  given by Eq. (28), while the solid blue curve represents the transferred information  $C_{\Gamma}^{Q,B}$  given by Eq. (38). As we can see, for  $\mu = 0.2$ , in the range of the noise parameter  $0.007 < p < 0.293$ , we reach a higher capacity by applying the nonunitary preprocessing  $\Gamma$ , the solid blue curve.

in this range, the chosen preprocessing  $\Gamma$  does not improve the capacity. In the range of  $0 \leq \mu < 0.3$ , the capacity with unitary encoding (28) and the transferred information with the preprocessing  $\Gamma$  (38) coincide for  $\mu = \tilde{\mu}(p)$ , shown as the dashed red curve in Fig. 3. Note that  $\tilde{\mu}(p)$  corresponds to the Root $[C_{\text{un}}^{Q,B} - C_{\Gamma}^{Q,B}]$ . The function  $\tilde{\mu}(p)$  is invariant under the simultaneous exchange  $p \leftrightarrow 1 - p$  since both functions  $C_{\text{un}}^{Q,B}$  and  $C_{\Gamma}^{Q,B}$  are symmetric under the exchange  $p \leftrightarrow 1 - p$ . Our results show that for  $\mu < \tilde{\mu}(p)$ , the blue (light gray) area in Fig. 3, the transferred information (38) leads to a higher value, in comparison to the capacity given by Eq. (28), i.e.,  $C_{\Gamma}^{Q,B} > C_{\text{un}}^{Q,B}$ . In Figs. 4 and 5, we visualize the superdense coding capacity corresponding to unitary encoding and the transferred information corresponding to the preprocessing  $\Gamma$ , Eqs. (28) and (38). In Fig. 4, the correlation degree is  $\mu = 0.2$ , while we vary the noise parameter  $p$ . In Fig. 5, the noise parameter is  $p = 0.05$  and  $\mu$  is varied.

## V. CONCLUSIONS

In summary, we discussed the superdense coding protocol in the presence of a correlated Pauli channel, considering both unitary and nonunitary encoding. Regarding unitary encoding,

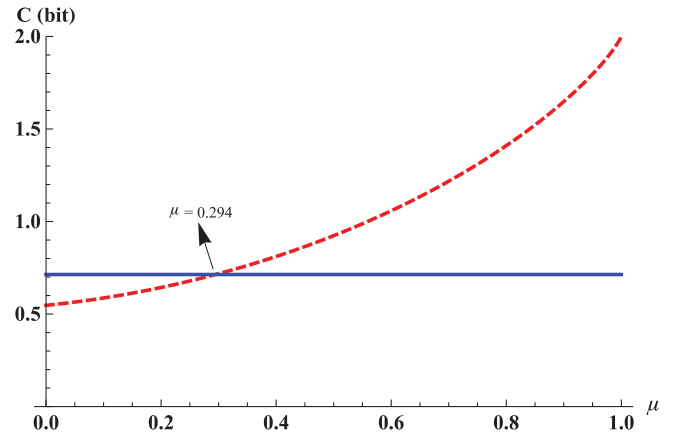


FIG. 5. (Color online) Another comparison between the superdense coding capacity (28) and the transferred information (38) as functions of the correlation degree  $\mu$ , with the noise parameter  $p = 0.05$ . The dashed red curve corresponds to the capacity  $C_{\text{un}}^{Q,B}$  given by Eq. (28). The solid blue line represents the transferred information  $C_{\Gamma}^{Q,B}$  given by Eq. (38). As we can see, for  $p = 0.05$  and  $\mu < 0.294$ , the nonunitary preprocessing  $\Gamma$  is useful to enhancing the capacity, compared to only unitary encoding.

it was shown that the problem of finding the superdense coding capacity reduces to the easier problem of finding a unitary operator which is applied to the initial state such that it minimizes the von Neumann entropy after the action of the channel. It was proven that for the two-dimensional quasiclassical channel and two-dimensional fully correlated Pauli channel with Bell states and Werner states as resources the unitary operator which minimizes the von Neumann entropy is the identity. For those examples, the superdense coding capacities were analytically derived. We also showed that when considering nonunitary encoding, the optimal strategy is to apply a preprocessing before unitary encoding. If the map that minimizes the von Neumann entropy is known, we found an expression for the superdense coding capacity. We also found an explicit example of nonunitary preprocessing  $\Gamma$  which can improve the superdense coding capacity, in comparison to only unitary encoding, for a range of the correlation degree of the channel. We also provided an example for which no better encoding than unitary encoding is possible. Therefore, based on the current results, the usefulness of nonunitary preprocessing in superdense coding depends on the quantum channel (e.g., its type, noise parameter, and correlation degree) and the resource state.

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