Preservation of loss in linear-optical processing

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We propose a measure of quantum efficiency of a multimode state of light that quantifies the amount of optical loss this state has experienced, and prove that this efficiency cannot increase in any linear-optical processing with destructive conditional measurements. Any loss that has affected a state can neither be removed nor redistributed so as to further increase the efficiency in higher-efficiency modes at the expense of lower-efficiency modes. This result eliminates the possibility of catalytically improving photon sources.

I. INTRODUCTION

A leading approach to quantum information processing is via linear-optical quantum computing (LOQC), first proposed in 2001 by Knill, Laflamme, and Milburn [1]. Major progress has been made both on the theoretical and experimental fronts toward implementation of LOQC. Modifications have been proposed that greatly reduce the overhead costs [2], a quantum error correction protocol has been introduced [3,4], and experimental implementation of primary gates has been demonstrated [5].

In spite of this progress, practical LOQC is still out of reach. Many of the difficulties arise because the single-photon sources required for LOQC, as well as computational circuits themselves, suffer from losses. Although a certain degree of tolerance to losses does exist in some LOQC schemes [4], the efficiency of existing single-photon sources [6] as well as the quality of individual circuit elements and waveguides are far below the required minima.

Under these circumstances it appears beneficial to develop a procedure that would reverse the effect of losses, perhaps at a cost of introducing extra resources. It would be useful, for example, to employ the outputs of N imperfect single-photon sources to obtain $K < N$ single-photon sources of improved quantum efficiency. Accomplishing this task would be straightforward if nonlinear-optical interactions with single photons were readily available: for example, one could employ nondemolition photon number measurements to select only those modes that contain photons. However, achieving such interactions is extremely technically challenging [7], whereas linear-optical (LO) processing is easily achieved in the laboratory.

It is therefore important to investigate whether elimination of losses is possible under LO processing. Under this processing we understand arbitrary interferometric transformations and conditioning on results of arbitrary *destructive* measurements on some of the optical modes involved. The efforts to construct such a scheme began in 2004, mostly ending with various no-go results [8–11]. The most general result to date was obtained in Ref. [12]. In that work we quantified the *efficiency* of a quantum optical state by the amount of loss that state might have experienced. We then proved that the efficiency in any single-mode optical state obtained through

LO processing cannot exceed the quantum efficiency of the best available single-mode input [12].

However, those previous results had limited application to multimode states. First, as we show below, extending the definition of the efficiency of a quantum state to the multimode case is not straightforward, particularly when the loss has been "mixed" among the modes by interferometric transformations. Second, our earlier results do not provide any information on how the efficiencies can be distributed among the output modes, aside from the general upper bound mentioned above. For example, they leave open the possibility of a "catalytic" scheme, in which some high-efficiency single photons are used to obtain additional high-efficiency single photons.

In the present work we generalize our study of the dynamics of optical losses to the multimode case. We introduce the notion of quantum efficiency of a (possibly entangled) multimode state which quantifies the amount of loss this state may have experienced. We show that this efficiency cannot increase under LO processing. That is, any loss that has occurred at the input can neither be removed nor redistributed so as to improve the efficiency in some of the modes at the expense of lower-efficiency modes. This means that there is a majorization relation between the efficiencies at the input and the output. The LO processing can act to average the efficiencies, but not to concentrate them. This rules out, in particular, any possibility of catalytic efficiency improvement.

II. SINGLE-MODE MEASURES OF EFFICIENCY

Before describing multimode measures of efficiency, we discuss the properties and relationships for single-mode measures of efficiency that have been previously proposed. Usually efficiency is used to describe a process for producing a state. However, it is also convenient to regard efficiency as a measure on the state itself, regardless of the process used to produce it [8–12]. Specifically, Ref. [12] uses the efficiency to quantify *the maximum amount of loss an optical mode carrying the given state might have previously experienced*:

$$E(\hat{\rho}) := \inf\{p | \exists \hat{\rho}_0 \geqslant 0 : \mathcal{E}_p(\hat{\rho}_0) = \hat{\rho}\}, \qquad (1)$$

where $\mathcal{E}_p$ is a loss channel with transmissivity p. That is, one considers all hypothetical methods of producing the given state $\hat{\rho}$ via loss from some valid initial quantum state $\hat{\rho}_0$.

We emphasize that the loss is just a way of mathematically quantifying the efficiency of the state. It is not necessary that the state were created by such a process. The efficiency is a measure on the state, and should not be regarded as an intrinsic feature of the mode.

Let us study a few examples. Ideally, single-photon sources would produce a single-photon state $|1\rangle$ on demand. In practice, such sources may with some probability fail to produce a photon, and there is no way to detect this failure without destructive measurement. Therefore the state produced by a generic single-photon source may be approximated as

$$\hat{\rho} = p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0|. \qquad (2)$$

Here the quantity p is commonly referred to as the efficiency of the single-photon source. In the context of our definition, state (2) can be obtained from the single-photon state by transmitting a (perfect) single photon through a loss channel with transmissivity p, and hence its efficiency equals p. In this way, the efficiency of state (2) according to our new definition is consistent with the traditional definition of the efficiency of a single-photon source.

Coherent states have efficiency exactly equal to zero, regardless of their amplitude. This is because coherent states remain coherent states under loss. A coherent state of amplitude α can be obtained from one of amplitude $\alpha/\sqrt{p}$ under a loss channel of transmissivity p. Although one cannot take $p = 0$ (because complete loss always results in the vacuum state), possible values of p form an open set with zero infimum.

On the other hand, any pure state *other* than a coherent state (or the vacuum state) must have efficiency 1. This is because a state under loss is a mixture of the original state, and the state with different numbers of photons lost. That is, a pure state $|\chi\rangle$ becomes a mixture of $|\chi\rangle$, $\hat{a}|\chi\rangle$, $\hat{a}^2|\chi\rangle$, and so forth. The only way in which the state after loss can remain pure is if $|\chi\rangle \propto \hat{a}|\chi\rangle$. The only states for which this is true are eigenstates of the annihilation operator, that is, coherent states.

Determining the efficiency of a known single-mode state is a straightforward computational task. The loss channel $\mathcal{E}_p$ corresponds to a linear transformation known as the generalized Bernoulli transformation. Provided the state $\hat{\rho}$ can be obtained via loss channel $\mathcal{E}_p$ from some initial operator, we can define the inverse map $\mathcal{E}_p^{-1}$, which can be calculated as in Ref. [13]. Therefore, we need to find the infimum of the values of p such that the inverse Bernoulli mapping $\mathcal{E}_p^{-1}(\hat{\rho})$ exists and yields a valid quantum state, that is, can be represented by a positive semidefinite density matrix.

A further interesting feature of a state's efficiency is that it equals zero if and only if the state is classical; that is, it can be written as a statistical mixture of coherent states, or, equivalently, its Glauber-Sudarshan P function has the properties of a probability density. As discussed above, any coherent state has efficiency zero, and hence so does any statistical mixture of coherent states. To prove the converse, let us suppose there exists a nonclassical state $\hat{\rho}$ such that $E(\hat{\rho}) = 0$. Let $\Phi_{\hat{\rho}}(\eta)$ denote the Fourier transform of this

state's P function $P(\alpha)$ over the phase space. According to Bochner's theorem [14], because $P(\alpha)$ is not a probability density, there exist two sets of n complex numbers η_k and z_k, such that

$$\sum_{i,j=1}^{n} \Phi_{\hat{\rho}}(\eta_i - \eta_j) z_i z_j^* < 0. \qquad (3)$$

Because $E(\hat{\rho}) = 0$, for any $p > 0$ there exists state $\hat{\rho}_0$ such that $\hat{\rho}$ is obtained from $\hat{\rho}_0$ by means of attenuation by factor p. Because attenuation corresponds to "shrinkage" of the P function in the phase space [15], we have $\Phi_{\hat{\rho}_0}(\eta) = \Phi_{\hat{\rho}}(\eta/\sqrt{p})$ and hence

$$\sum_{i,j=1}^{n} \Phi_{\hat{\rho}_0}(\eta_i' - \eta_j') z_i z_j^* < 0, \qquad (4)$$

where $\eta_k' = \eta_k \sqrt{p}$. By choosing p close to zero, the set of arguments of function $\Phi_{\hat{\rho}_0}$ in the above equation can be upper bounded by an arbitrarily small value A.

Now let us recall that the Husimi Q function of any quantum state does have the properties of a probability density. This means that the Fourier transform $\Psi_{\hat{\rho}_0}(\eta)$ of the Q function of state $\hat{\rho}_0$ must obey

$$\sum_{i,j=1}^{n} \Psi_{\hat{\rho}_0}(\eta_i' - \eta_j') z_i z_j^* \geqslant 0. \qquad (5)$$

But the Q function is obtained from the P function by convolving the latter with a Gaussian, $e^{-|\alpha|^2}/\pi$ [15]. This means that the Fourier transforms of these functions are connected by multiplication,

$$\Psi_{\hat{\rho}}(\eta) = \Phi_{\hat{\rho}}(\eta) e^{-|\eta|^2}. \qquad (6)$$

By choosing p close to zero, one can make the factor $e^{-|\eta|^2}$ arbitrarily close to 1 within radius A. Accordingly, the left-hand sides of Eqs. (4) and (5) are equal in the limit $p \to 0$. We arrive at a contradiction, which means that any nonclassical state $\hat{\rho}$ must have a finite efficiency $E(\hat{\rho}) > 0$.

III. MULTIMODE MEASURES OF EFFICIENCY

Let us now generalize the notion of efficiency to an optical state carried by multiple modes. A direct generalization can be obtained by assuming that each mode has propagated through its own loss channel, and taking the sum of the transmissivities:

$$E_{\mathrm{d}}(\hat{\rho}, K) := \inf \left\{ \sum_{\ell=1}^{K} p_\ell^{\downarrow} \,\middle|\, \exists\, \hat{\rho}_0 \geqslant 0 : \mathcal{E}_{\vec{p}}(\hat{\rho}_0) = \hat{\rho} \right\}. \qquad (7)$$

The notation $p_\ell^{\downarrow}$ indicates the elements of the vector $\vec{p}$ sorted in nonincreasing order. The value of K can be less than the number of modes constituting state $\hat{\rho}$. In this way, the efficiency is defined not only for the entire state, but also for a subset of K modes with the lowest losses. This extension facilitates comparison of efficiencies of states with different number of modes.

A drawback of this definition is that it does not adequately take account of loss that has been mixed between modes. For example, consider two polarization modes carrying a single-photon qubit in the state $|\psi\rangle = |1_H\rangle|0_V\rangle$. The efficiency of the

state in the horizontally polarized mode is 1, and that in the vertically polarized mode 0, so $E_{\mathrm{d}}(|\psi\rangle\langle\psi|,2) = 1$. On the other hand, writing the same state in terms of diagonal polarization modes, we find $|\psi'\rangle = (|1_{+45^\circ}\rangle|0_{-45^\circ}\rangle + |0_{+45^\circ}\rangle|1_{-45^\circ}\rangle)/\sqrt{2}$. This state cannot be obtained by independent loss in the two modes, and would have a different efficiency, $E_{\mathrm{d}}(|\psi'\rangle\langle\psi'|,2) = 2$, even though its utility for quantum information processing is exactly the same as that of $|\psi\rangle$.

An alternative approach to quantifying the efficiency is to treat each mode separately, and calculate the sum of single-mode efficiencies for K highest-efficiency modes:

$$E_{\mathrm{s}}(\hat{\rho},K) := \sum_{\ell=1}^{K} E(\mathrm{Tr}_{\forall k \neq \ell} \hat{\rho})^{\downarrow}. \tag{8}$$

This definition is also problematic. First, similarly to the d-efficiency [16], it depends on the choice of the mode basis. For the example above, $E_{\mathrm{s}}(|\psi\rangle\langle\psi|,1) = 1$, but $E_{\mathrm{s}}(|\psi'\rangle\langle\psi'|,1) = 1/2$. Second, it may underestimate the efficiency in many cases. For example, the s-efficiency of the state $|\phi\rangle = \sqrt{1-p}|00\rangle + \sqrt{p}|11\rangle$ equals $E_{\mathrm{s}}(|\phi\rangle\langle\phi|,1) = p$, and can be very small. On the other hand, conditioning on detection of a photon in one of the modes of $|\phi\rangle$ results in a perfect single photon in the other mode, as is the case with producing heralded single photons via parametric down-conversion. State $|\phi\rangle$ is thus much more useful than, for example, single-mode state $\hat{\sigma} = (1-p)|0\rangle\langle0| + p|1\rangle\langle1|$, which has the same s-efficiency but cannot be processed to produce a high-quality single photon.

We aim to provide a definition of efficiency that would be invariant with respect to transformation of modes and adequately reflect the state's value for quantum information purposes. To this end we modify the definition E_{d} by including an optimization over interferometers. That is, we consider simultaneous loss channels on each of the modes $\mathcal{E}_{\vec{p}}$, followed by an arbitrary interferometer W, as shown in Fig. 1(a). The efficiency is then the sum of the K largest values of p_ℓ:

$$E_{\mathrm{u}}(\hat{\rho},K) := \inf\left\{ \sum_{\ell=1}^{K} p_\ell^{\downarrow} \,\middle|\, \exists\, \hat{\rho}_0 \geqslant 0, W : W\mathcal{E}_{\vec{p}}(\hat{\rho}_0) = \hat{\rho} \right\}. \tag{9}$$

An important property of the u-efficiency (9) is its invariance with respect to interferometric transformation of modes. Indeed, if state $\hat{\rho}'$ can be obtained from state $\hat{\rho}$ by applying interferometric transformation U, so that $\hat{\rho}' = U\hat{\rho}$, and we have $W\mathcal{E}_{\vec{p}}(\hat{\rho}_0) = \hat{\rho}$ in the context of Eq. (9), we also have $UW\mathcal{E}_{\vec{p}}(\hat{\rho}_0) = \hat{\rho}'$. But transformation UW can be treated as a single interferometer, which means that $E_{\mathrm{u}}(\hat{\rho}',K) \leqslant E_{\mathrm{u}}(\hat{\rho},K)$. But because interferometric transformations are reversible, we also have $E_{\mathrm{u}}(\hat{\rho},K) \leqslant E_{\mathrm{u}}(\hat{\rho}',K)$ and hence $E_{\mathrm{u}}(\hat{\rho}',K) = E_{\mathrm{u}}(\hat{\rho},K)$.

Similar to the case for the efficiency E, the u-efficiency can be calculated via inverting the channel. In finite dimension, the channel given by the loss followed by the unitary operation W may be represented by a matrix, which may be inverted to find $\hat{\rho}_0$. The efficiency can then be found by a minimization over $\vec{p}$ and W such that $\hat{\rho}_0$ is a valid quantum state. For the other two efficiencies, the calculation is simpler. For the d-efficiency, one
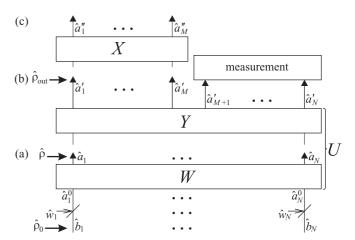

FIG. 1. A general setup for LO processing. (a) To determine the efficiency of the input state $\hat{\rho}$ we find an initial state $\hat{\rho}_0$ such that $\hat{\rho}$ may be obtained by attenuation and interferometer W according to Eq. (9). (b) LO processing of the input state. The modes pass through a general interferometer, and all but M of the output modes are detected via a measurement. The state $\hat{\rho}_{\mathrm{out}}$ of the remaining M modes can be conditioned on a particular measurement result. (c) The upper limit on the efficiency of the output state is established by choosing an interferometer X through which this state can be transmitted such that the resulting state, carried by modes $\hat{a}''_m$, can be obtained by multimode attenuation of another state.

only needs to minimize over $\vec{p}$, and for the s-efficiency one can just determine the single-mode efficiencies for the reduced density matrices in the individual modes.

Let us evaluate the multimode efficiency of the example states studied above. State $|\psi\rangle$ is a tensor product and has $E_{\mathrm{s}}(|\psi\rangle\langle\psi|,2) = E_{\mathrm{s}}(|\psi\rangle\langle\psi|,1) = 1$. As we show below, the d-, s-, and u-efficiencies coincide for tensor product states, so we also have $E_{\mathrm{u}}(|\psi\rangle\langle\psi|,2) = E_{\mathrm{u}}(|\psi\rangle\langle\psi|,1) = 1$. Since the u-efficiency is invariant under interferometric transformations, state $|\psi'\rangle$ has the same u-efficiency. Analyzing each of the modes of state $|\psi'\rangle$ on its own, we find them to carry the state $(|1\rangle\langle1| + |0\rangle\langle0|)/2$, so $E_{\mathrm{s}}(|\psi'\rangle\langle\psi'|,1) = 1/2$ and $E_{\mathrm{s}}(|\psi'\rangle\langle\psi'|,2) = 1$. For state $|\phi\rangle$, both the u- and d-efficiencies equal 2. This is because, even if subjected to an interferometric transformation, it is a pure state that is not coherent, and hence cannot be obtained by attenuating another state.

IV. PROOF THAT THE u-EFFICIENCY CANNOT INCREASE UNDER LO PROCESSING

In this section we show that it is impossible to increase the u-efficiency using LO processing. A general LO scheme is shown in Fig. 1(b). The input state $\hat{\rho}$, carried by N optical modes with annihilation operators $\hat{a}_1, \ldots, \hat{a}_N$, is passed through a general interferometer which performs a unitary operation Y on these mode operators. We retain M of the output modes $\hat{a}'_i$, and the remaining $N - M$ modes are subjected to a generalized destructive quantum measurement. We consider postselection on a particular result of this measurement, and determine the u-efficiency of the state $\hat{\rho}_{\mathrm{out}}$ carried by the remaining output modes. Our goal is to prove that

$$E_{\mathrm{u}}(\hat{\rho}_{\mathrm{out}},K) \leqslant E_{\mathrm{u}}(\hat{\rho},K) \tag{10}$$

for any $K \leqslant M$.

In accordance with definition (9), we model the state $\hat{\rho}$ as being obtained from some initial state $\hat{\rho}_0$ by combining each of its modes $\hat{b}_j$ with vacuum $\hat{w}_j$ on a beam splitter with transmissivity p_j [Fig. 1(a)]:

$$\hat{a}_j^0 = \sqrt{p_j}\hat{b}_j + \sqrt{1 - p_j}\hat{w}_j, \qquad (11)$$

followed by interferometer W. We assume that the settings are chosen such that, for some $\epsilon > 0$,

$$\sum_{\ell=1}^{K} p_\ell^\downarrow \leqslant E_{\mathrm{u}}(\hat{\rho}, K) + \epsilon. \qquad (12)$$

The introduction of ϵ takes account of the possibility that there does not exist a setting which achieves the infimum.

Because interferometers W and Y are adjacent to each other, we can without loss of generality treat them as a single interferometer, corresponding to unitary transformation $U = YW$. The action of this interferometer can be written as

$$\hat{a}_i' = \sum_{j=1}^{N} U_{ij}\hat{a}_j^0 = \sum_{j=1}^{N} U_{ij}\sqrt{p_j}\hat{b}_j + \sum_{j=1}^{N} U_{ij}\sqrt{1 - p_j}\hat{w}_j. \qquad (13)$$

We see that each vacuum mode contributes to each of the output modes, including those that are subjected to conditional measurements. These measurements may "compromise" the vacuum contributions to the output state [17], so the output efficiency cannot be calculated directly from the matrix elements U_{ij}. We address this issue by performing an RQ decomposition on the matrix $U_{ij}\sqrt{1 - p_j}$ such that

$$U_{ij}\sqrt{1 - p_j} = \sum_{\ell=1}^{N} R_{i\ell} Q_{\ell j}, \qquad (14)$$

where Q is unitary and R is an upper triangular matrix, so $R_{i\ell} = 0$ for $\ell < i$. Then we get

$$\hat{a}_i' = \sum_{\ell=1}^{N} U_{i\ell}\sqrt{p_\ell}\hat{b}_\ell + \sum_{\ell=1}^{N} R_{i\ell}\hat{v}_\ell, \qquad (15)$$

where

$$\hat{v}_\ell := \sum_{j=1}^{N} Q_{\ell j}\hat{w}_j \qquad (16)$$

are obtained by transforming modes $\hat{w}_j$ in a fictitious interferometer Q. Because all the $\hat{w}_j$ correspond to vacuum states, so do the $\hat{v}_\ell$. The subset $\{\hat{v}_1, \ldots, \hat{v}_M\}$ of these modes does not contribute to the set of output modes $\{\hat{a}_{M+1}', \ldots, \hat{a}_N'\}$ that is subjected to measurement, and thus leads directly to the loss of efficiency in the output state.

Without loss of generality, we append another interferometer X acting on the M output modes. Because the u-efficiency is independent of linear interferometers, this interferometer does not affect the u-efficiency at the output. To determine the interferometer to use, we perform a singular value decomposition on the upper left $M \times M$ block of R such that

$$R = X^\dagger R' Q', \qquad (17)$$

where the upper left $M \times M$ block of R' is diagonal, and unitaries X and Q' are equal to the identity outside the upper left $M \times M$ block. We choose the unitary matrix X for the final interferometer to be that given by this decomposition.

Denoting the annihilation operators for the modes after the interferometer X by $\hat{a}_k''$, we have, for $k \leqslant M$,

$$\begin{aligned} \hat{a}_k'' &= \sum_{i=1}^{M} X_{ki}\hat{a}_i' \\ &= \sum_{i=1}^{M} X_{ki}\left(\sum_{\ell=1}^{N} U_{i\ell}\sqrt{p_\ell}\hat{b}_\ell + \sum_{\ell=1}^{N} R_{i\ell}\hat{v}_\ell\right) \\ &= \sum_{i=1}^{M}\sum_{\ell=1}^{N} X_{ki}U_{i\ell}\sqrt{p_\ell}\hat{b}_\ell + \sum_{i=1}^{M}\sum_{\ell=M+1}^{N} X_{ki}R_{i\ell}\hat{v}_\ell \\ &\quad + \sum_{i=1}^{M}\sum_{\ell=1}^{M} X_{ki}\sum_{k',n=1}^{M} [X^\dagger]_{ik'}R'_{k'n}Q'_{n\ell}\hat{v}_\ell \\ &= \sum_{i=1}^{K}\sum_{\ell=1}^{N} X_{ki}U_{i\ell}\sqrt{p_\ell}\hat{b}_\ell \\ &\quad + \sum_{i=1}^{M}\sum_{\ell=M+1}^{N} X_{ki}R_{il}\hat{v}_\ell + R'_{kk}\hat{v}_k'', \end{aligned} \qquad (18)$$

where

$$\hat{v}_k'' := \sum_{\ell=1}^{N} Q'_{k\ell}\hat{v}_\ell. \qquad (19)$$

As the set $\{\hat{v}_k''\}$ may be regarded as being obtained from initial vacuum modes $\{\hat{w}_k\}$ via a unitary transformation, they represent an orthonormal set of bosonic modes in the vacuum state. Furthermore, those $\hat{v}_k''$ that contribute to $\hat{a}_k''$ do not contain any contribution from the compromised vacuum modes. Indeed, they only contain contributions from $\hat{v}_\ell$ for $\ell \leqslant M$, whereas the operators for the measured modes only contain contributions from $\hat{v}_\ell$ for $\ell > M$. As a result, these vacuum contributions are equivalent to loss.

To make this result explicit, we write the annihilation operator in the form $\hat{a}_k'' = \hat{B}_k'' + \hat{V}_k''$, where

$$\hat{B}_k'' = \sum_{i=1}^{M}\sum_{\ell=1}^{N} X_{ki}U_{i\ell}\sqrt{p_\ell}\hat{b}_\ell + \sum_{i=1}^{M}\sum_{\ell=M+1}^{N} X_{ki}R_{il}\hat{v}_\ell \qquad (20)$$

and

$$\hat{V}_k'' = R'_{kk}\hat{v}_k''. \qquad (21)$$

We then find that

$$[\hat{V}_k'', (\hat{V}_{k'}'')^\dagger] = \delta_{kk'}|R'_{kk}|^2, \qquad (22)$$

$$[\hat{B}_k'', (\hat{B}_{k'}'')^\dagger] = \delta_{kk'}(1 - |R'_{kk}|^2). \qquad (23)$$

The first line follows immediately from Eq. (21). The second line is obtained because $\hat{B}_k'' = \hat{a}_k'' - \hat{V}_k''$ and $[\hat{a}_k'', (\hat{a}_{k'}'')^\dagger] = \delta_{kk'}$. Defining

$$p_k'' := 1 - |R'_{kk}|^2, \quad \hat{b}_k'' := \hat{B}_k''/\sqrt{p_k''}, \qquad (24)$$
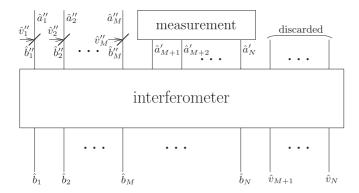
FIG. 2. A rearrangement of the interferometer. The vacuum modes $\{\hat{v}''_1, \ldots, \hat{v}''_M, \hat{v}_{M+1} \ldots \hat{v}_N\}$ can be obtained via an interferometer (not shown) from the original vacuum modes $\hat{w}_\ell$. The modes $\hat{b}_\ell$ and $\{\hat{v}_{M+1}, \ldots, \hat{v}_N\}$ are combined in the interferometer to produce $\{\hat{b}''_1, \ldots, \hat{b}''_M\}$, as well as $\{\hat{a}'_{M+1}, \ldots, \hat{a}'_N\}$, which are measured, and some modes which are discarded. The modes $\{\hat{b}''_1, \ldots, \hat{b}''_M\}$ are then combined with vacuum modes $\{\hat{v}''_1, \ldots, \hat{v}''_M\}$ to generate the output state.

we have [18]

$$\hat{a}''_k = \sqrt{p''_k}\,\hat{b}''_k + \sqrt{1 - p''_k}\,\hat{v}''_k. \tag{25}$$

Therefore, the output state may be obtained by an interferometer that produces the modes with annihilation operators $\hat{b}''_k$, then combining with vacua on beam splitters with transmissivities p''_k, as shown in Fig. 2.

Without loss of generality we can assume X and Q' to have been chosen such that the numbers p''_k are in nonincreasing order. The u-efficiency at the output is therefore upper bounded by

$$E_u(\hat{\rho}_{\text{out}}, K) \leqslant \sum_{i=1}^{K} p''_k. \tag{26}$$

To determine the sum (26), we can define the unitaries

$$U' := XU, \quad Q'' := Q'Q. \tag{27}$$

It follows from Eq. (17) that $R' = XR(Q')^\dagger$. Therefore, according to Eq. (14),

$$R'_{k\ell} = \sum_{m=1}^{N} U'_{km}\sqrt{1 - p_m}(Q''_{\ell m})^*. \tag{28}$$

Then we obtain

$$\sum_{k=1}^{K} p''_k \leqslant K - \sum_{k=1}^{K}\sum_{\ell=1}^{K}|R'_{k\ell}|^2$$

$$= K - \sum_{k=1}^{N}\sum_{\ell=1}^{K}\sum_{m,j=1}^{N} U'_{km}\sqrt{1 - p_m}(Q''_{\ell m})^*$$

$$\times (U'_{kj})^*\sqrt{1 - p_j}\,Q''_{\ell j}$$

$$= K - \sum_{\ell=1}^{K}\sum_{m,j=1}^{N} \delta_{mj}\sqrt{1 - p_m}(Q''_{\ell m})^*\sqrt{1 - p_j}\,Q''_{\ell j}$$

$$= K - \sum_{\ell=1}^{K}\sum_{j=1}^{N}(1 - p_j)|Q''_{\ell j}|^2$$

$$= \sum_{\ell=1}^{K}\sum_{j=1}^{N} p_j|Q''_{\ell j}|^2 \leqslant \sum_{\ell=1}^{K} p_\ell^\downarrow. \tag{29}$$

The last inequality in Eq. (29) is obtained because Q''_{ij} is unitary, so $|Q''_{ij}|^2$ is a doubly stochastic matrix, and thus vector p_ℓ majorizes vector [19]

$$q_\ell := \sum_{j=1}^{N} p_j|Q''_{ij}|^2. \tag{30}$$

Now, according to Eqs. (12), (26), and (29), and because we can choose ϵ to be arbitrarily close to zero, we obtain

$$E_u(\hat{\rho}_{\text{out}}, K) \leqslant E_u(\hat{\rho}, K). \tag{31}$$

This is the main result of this work: the universal measure of quantum efficiency of a multimode state, the u-efficiency, cannot increase under LO processing.

V. COMPARISON OF EFFICIENCY MEASURES

We now use the above result to prove some additional properties of the different measures of multimode efficiency defined in Sec. III. First, we show that these efficiencies are related according to

$$E_s(\hat{\rho}, K) \leqslant E_u(\hat{\rho}, K) \leqslant E_d(\hat{\rho}, K). \tag{32}$$

To examine the s-efficiency we can again assume that state $\hat{\rho}$ is obtained via a set of beam splitters with transmissivities p_j and an interferometer W as in Fig. 1(a), such that the sum of the K largest values of p_j is no more than $E_u(\hat{\rho}, K) + \epsilon$. Then the operators for the state $\hat{\rho}$ are given by

$$\hat{a}_j = \hat{B}_j + \hat{V}_j, \tag{33}$$

with

$$\hat{B}_j := \sum_{\ell=1}^{N} W_{j\ell}\sqrt{p_\ell}\,\hat{b}_\ell, \quad \hat{V}_j := \sum_{\ell=1}^{N} W_{j\ell}\sqrt{1 - p_\ell}\,\hat{w}_\ell. \tag{34}$$

corresponding to operators carrying signal and vacuum fields, respectively.

To determine the s-efficiency we determine the efficiency for each mode individually. When determining the efficiency for mode $\hat{a}_j$, we can regard modes $\hat{a}_k$ for $k \neq j$ as being discarded. The vacuum operators $\hat{V}_k$ for $k \neq j$ are not orthogonal to $\hat{V}_j$; however, since those modes are discarded, the addition of vacuum $\hat{V}_j$ is equivalent to loss. Therefore, the efficiency of the state in mode $\hat{a}_j$ is no greater than

$$p'_j := [\hat{B}_j, \hat{B}_j^\dagger] = \sum_{\ell=1}^{N} |W_{j\ell}|^2 p_\ell. \tag{35}$$

The sum of the K largest values of p'_j upper bounds the s-efficiency; that is,

$$E_s(\hat{\rho}, K) \leqslant \sum_{j=1}^{K} p'^\downarrow_j. \tag{36}$$

Because W is unitary, $|W_{j\ell}|^2$ is a doubly stochastic matrix, and the vector of values $\vec{p}$ majorizes $\vec{p}'$. That means that

$$\sum_{j=1}^{K} p_j'^{\downarrow} \leqslant \sum_{j=1}^{K} p_j^{\downarrow} \leqslant E_{\mathrm{u}}(\hat{\rho}, K) + \epsilon. \tag{37}$$

Because this holds for all $\epsilon > 0$, we have $E_{\mathrm{s}}(\hat{\rho}, K) \leqslant E_{\mathrm{u}}(\hat{\rho}, K)$.

The second inequality in Eq. (32) is because the definition of $E_{\mathrm{u}}(\hat{\rho}, K)$ in Eq. (9) looks for the minimum in a larger set of states than that of $E_{\mathrm{d}}(\hat{\rho}, K)$ in Eq. (7).

For tensor product states, the s- and d-efficiencies are the same. To see this, use the definition (8) on the tensor product state

$$\hat{\rho} = \bigotimes_{j=1}^{N} \hat{\rho}_j. \tag{38}$$

One obtains

$$E_{\mathrm{s}}(\hat{\rho}, K) = \sum_{\ell=1}^{K} E(\hat{\rho}_\ell)^{\downarrow}. \tag{39}$$

Therefore there exists a set of states $\hat{\rho}_j^0$ and transmissivities p_j such that the sum of the K largest values of p_j is no more than $E_{\mathrm{s}}(\hat{\rho}, K) + \epsilon$, and the final states $\hat{\rho}_j$ may be obtained via loss channels with transmissivities p_j from initial states $\hat{\rho}_j^0$. This would also provide a scheme for producing $\hat{\rho}$ for the definition of $E_{\mathrm{d}}(\hat{\rho}, K)$, and therefore

$$E_{\mathrm{d}}(\hat{\rho}, K) \leqslant \sum_{j=1}^{K} p_j^{\downarrow} \leqslant E_{\mathrm{s}}(\hat{\rho}, K) + \epsilon. \tag{40}$$

Because this is true for all $\epsilon > 0$, we obtain $E_{\mathrm{d}}(\hat{\rho}, K) \leqslant E_{\mathrm{s}}(\hat{\rho}, K)$. Combining this with Eq. (32), we find that $E_{\mathrm{d}}(\hat{\rho}, K) = E_{\mathrm{s}}(\hat{\rho}, K)$ for tensor product states, and all inequalities in (32) saturate.

This result leads us to an important conclusion. Suppose we start with N separable states [for example, imperfect single photons as in Eq. (2)] with efficiencies p_ℓ, which we subject to LO processing, resulting in a set of modes in which the efficiencies, when analyzed separately, are given by p_ℓ'. Using

the result that LO processing cannot increase the u-efficiency, and Eq. (32), we have for any integer K,

$$\sum_{\ell=1}^{K} p_\ell' \leqslant \sum_{\ell=1}^{K} p_\ell^{\downarrow}. \tag{41}$$

In other words, the LO processing can act to average the efficiencies, but not to concentrate them. One consequence is the exclusion of any possibility for "catalytic" efficiency improvement, in which some highly efficient sources are used to increase the efficiency in other optical modes, without themselves suffering from loss.

These results do not rule out increases in the individual efficiencies; for example, if the largest efficiency is decreased, it is possible for the second largest efficiency to be increased.

VI. SUMMARY

We have introduced a number of measures that enable us to quantify the efficiency in multimode systems. The u-efficiency is a powerful general measure that takes account of how loss may have been mixed between the different modes. It is unchanged under linear interferometers, and cannot increase under more general LO processing with destructive measurements. We have used this result to show that catalytic improvement of photon sources is not possible with LO processing. If one starts with independent optical sources (which produce a tensor product of states), then the efficiencies in the individual output modes are weakly majorized by the efficiencies in the input. This means that it is not possible to concentrate the efficiencies, such that the sum of the highest K output efficiencies is greater than the sum of the highest K input efficiencies.

It is clearly possible to increase the u-efficiency if one uses *nonlinear* optical elements. For example, a standard method of producing single photons is via parametric down-conversion (a nonlinear process), and postselection on detection of a photon in one of the output modes. The initial beam is coherent, with efficiency zero, but the final output (ideally) has unit efficiency.

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[16] We will call E_d the d-efficiency, E_s the s-efficiency, and so on.

[17] Consider, for example, a single photon in mode $\hat{a}$ overlapped with vacuum $\hat{v}$ on a beam splitter. One of the beam splitter output modes $(\hat{a} - \hat{v})/\sqrt{2}$ is subjected to a photon number measurement, and the state of the second output mode is analyzed conditioned on the result of the measurement. The annihilation operator of the latter mode is given by $(\hat{a} + \hat{v})/\sqrt{2}$,

so one may argue that its efficiency cannot exceed 1/2 due to the vacuum contribution. However, if we condition on detecting the vacuum state in the first mode, the state of the second mode is a single photon; that is, it has efficiency 1, in spite of the vacuum contribution. In this context we say that the vacuum contribution of the second mode is "compromised" by the conditional measurement in the first one.

[18] If p_m'' equals 0 or 1, modes $\hat{b}_m''$ and $\hat{v}_m''$, respectively, are not defined. In these cases we have, respectively, $\hat{a}_m'' = \hat{v}_m''$ or $\hat{a}_m'' = \hat{b}_m''$.

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