Classical and quantum correlative capacities of quantum systems

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(Received 4 June 2011; published 31 October 2011)

How strongly can one system be correlated with another? In the classical world, this basic question concerning correlative capacity has a very satisfying answer: The "effective size" of the marginal system, as quantified by the Shannon entropy, sets a tight upper bound to the correlations, as quantified by the mutual information. Although in the quantum world bipartite correlations, like their classical counterparts, are also well quantified by mutual information, the similarity ends here: The correlations in a bipartite quantum system can be twice as large as the marginal entropy. In the paradigm of quantum discord, the correlations are split into classical and quantum components, and it was conjectured that both the classical and quantum correlations are (like the classical mutual information) bounded above by each subsystem's entropy. In this work, by exploiting the interplay between entanglement of formation, mutual information, and quantum discord, we disprove that conjecture. We further indicate a scheme to restore harmony between quantum and classical correlative capacities. The results illustrate dramatically the asymmetric nature of quantum discord and highlight some subtle and unusual features of quantum correlations.

DOI: 10.1103/PhysRevA.84.042124 PACS number(s): 03.65.Ud, 03.67.Mn

Correlations are ubiquitous and fundamental in sciences. Traditionally, correlations are quantified by covariance and correlation functionals of observables. With the development of the Shannon communication theory and the emergence of quantum information theory [1,2], various entropic quantities gained a pivotal role in the characterization and quantification of correlations.

In order for a system to establish correlations with another one, a certain correlative capacity is required. For example, a quantum system in any *pure* state cannot have any correlations with another system. This premise for correlations is mathematically synthesized by the fact that the von Neumann entropy of a pure state vanishes. For a mixed quantum system, what are its correlative capacities? In this work, we will distinguish the classical and quantum correlative capacities of a quantum system in terms of classical correlations and quantum discord and will show that while the classical correlative capacity is bounded by the system's entropy, it is not true (or more precisely, only 50% true) in general for the quantum correlative capacity.

To be precise, let us start from the classical world in which the state of a system is described by a probability distribution $p = \{p_i\}$, whose information content is well quantified by the Shannon entropy $H(p) := -\sum_i p_i \log_2 p_i$. In view of the Shannon noiseless coding theorem [1,2], $H(p)$ may be interpreted as the "effective size" or "informational capacity" of p. For a classical bivariate probability distribution $p^{ab} = \{p^{ab}_{ij}\}$ shared by two parties a and b with marginal distributions $p^a = \{p^a_i := \sum_j p^{ab}_{ij}\}$ and $p^b = \{p^b_j := \sum_i p^{ab}_{ij}\}$, it is well known that the amount of correlations (mutual information) $I(p^{ab}) := H(p^a) + H(p^b) - H(p^{ab})$ in p^{ab} is bounded above by the marginal entropies:

$$I(p^{ab}) \leqslant \min\{H(p^a), H(p^b)\}. \tag{1}$$

This upper bound can apparently be achieved, for example, by the perfectly correlated bivariate probability distribution $p^{ab}_{ij} = p_i \delta_{ij}$.

But how about the quantum world? As a rule of thumb, the quantum world is full of surprises, and indeed new phenomena arise here. The quantum analog of p^{ab} is a density operator ρ^{ab} shared by two parties a and b with marginals $\rho^a := \mathrm{tr}_b \rho^{ab}$ and $\rho^b := \mathrm{tr}_a \rho^{ab}$, and the straightforward generalizations of the Shannon entropy and the mutual information are the von Neumann entropy $S(\rho^a) := -\mathrm{tr}\rho^a \log_2 \rho^a$ and the quantum mutual information $I(\rho^{ab}) := S(\rho^a) + S(\rho^b) - S(\rho^{ab})$, respectively. According to the Schumacher quantum coding theorem [3], $S(\rho^a)$ quantifies the effective size of ρ^a as an information carrier. The (total) correlations in ρ^{ab} are usually quantified by the quantum mutual information [4,5]. Motivated by inequality (1), one might be tempted to guess that $I(\rho^{ab}) \leqslant \min\{S(\rho^a), S(\rho^b)\}$. Amazingly, due to quantum effects, which lead to stronger correlations than classically possible, the above bound does not hold in general, and instead one has the following weaker bound:

$$I(\rho^{ab}) \leqslant 2 \min\{S(\rho^a), S(\rho^b)\}, \tag{2}$$

which is actually equivalent to the celebrated Araki-Lieb inequality $|S(\rho^a) - S(\rho^b)| \leqslant S(\rho^{ab})$ [2]. The interesting point here is that now the correlations in ρ^{ab}, although they cannot be bounded above by the marginal entropies $S(\rho^a)$ and $S(\rho^b)$, *are* bounded above by $2S(\rho^a)$ and $2S(\rho^b)$. The factor 2 is apparently of a quantum origin. In particular, if $\rho^{ab} = |\Psi^{ab}\rangle\langle\Psi^{ab}|$ is a pure state, then $I(\rho^{ab}) = 2S(\rho^a) = 2S(\rho^b)$, which saturates inequality (2) and is a manifestation of the Einstein-Podolsky-Rosen correlations. Noting that the entanglement entropy of the pure state ρ^{ab} is $E(\rho^{ab}) := S(\rho^a)$ [6,7], we may regard that the total correlations in the pure state ρ^{ab}, as quantified by $I(\rho^{ab})$, consist of equal quantum (entanglement entropy) and classical (classical correlations) parts.

For a general bipartite mixed state, it is of particular significance to separate its total correlations into classical and

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quantum parts and to inquire whether they are bounded by marginal entropies. At present, we do not have a unique and universally satisfying scheme for this. A plausible and widely used method is based on the notion of quantum discord [8,9]. In this setting, the amount of classical correlations in ρ^{ab} is defined as

$$C(\rho^{ab}) := \sup_{\Pi^b} I(\Pi^b(\rho^{ab})),$$

where the sup is over all von Neumann measurements $\Pi^b := \{\Pi_j^b\}$ (family of orthogonal, one-dimensional projections summing to the identity) on party b, and

$$\Pi^b(\rho^{ab}) := \sum_j \left(\mathbf{1}^a \otimes \Pi_j^b\right)\rho^{ab}\left(\mathbf{1}^a \otimes \Pi_j^b\right) = \sum_j p_j \rho_j^a \otimes \Pi_j^b,$$

with $p_j := \mathrm{tr}\Pi_j^b \rho^b$ and $\rho_j^a := \mathrm{tr}_b(\mathbf{1}^a \otimes \Pi_j^b)\rho^{ab}(\mathbf{1}^a \otimes \Pi_j^b)/p_j$. The amount of quantum correlations in ρ^{ab}, as quantified by the quantum discord [8,9], is defined as

$$Q(\rho^{ab}) := I(\rho^{ab}) - C(\rho^{ab}).$$

This quantity is playing an increasingly interesting and significant role in the recent investigations of quantum correlations and various physical systems [10].

Now the total correlations are separated into classical correlations $C(\rho^{ab})$ and quantum correlations $Q(\rho^{ab})$, and inequality (2) can be recast into $C(\rho^{ab}) + Q(\rho^{ab}) \leqslant 2\min\{S(\rho^a), S(\rho^b)\}$. From the perspective of correlative capacities, one is naturally led to the following conjecture, which splits the preceding inequality [11]:

$$C(\rho^{ab}) \leqslant S(\rho^a), \tag{3}$$
$$C(\rho^{ab}) \leqslant S(\rho^b), \tag{4}$$
$$Q(\rho^{ab}) \leqslant S(\rho^a), \tag{5}$$
$$Q(\rho^{ab}) \leqslant S(\rho^b). \tag{6}$$

The formal meaning of this conjecture is that for a quantum system, both the classical and quantum correlations are bounded by the marginal entropies. It should be emphasized that if ρ^{ab} is pure, then the above conjecture is true, and all four inequalities become equalities. This seemingly trivial fact has the following important physical implication: In the partition "(system) + (universe − system)," both the classical and quantum correlative capacities (as well as entanglement) coincide and are given by the system's entropy.

The purpose of this work is to prove or disprove the above inequalities, to put them in an intuitive context, to reveal some fundamentally different characteristics between classical and quantum correlative capacities, and to present a scheme to restore harmony between them. It turns out that the above conjecture is 75% true in the sense that inequalities (3), (4), and (6) are true but inequality (5) fails *in general* (but might be true for low-dimensional systems). Note that while we have the above results for the asymmetrical definition of quantum discord, we will also show that for two symmetrical approaches, both (5) and (6) are (symmetrically) violated in one case and satisfied in another.

Explicit proofs of inequalities (3), (4), and (6) were given in Ref. [12], in which some sufficient conditions for inequality (5) were also given. Inequalities (3) and (4) can also be derived from Ref. [13]. Our main results are the (surprising) disproof of inequality (5) based on a "peculiar" property of entanglement of formation [5] and the (completely different) solutions for two natural ramifications of the conjecture concerning symmetric versions of quantum discord. The method of purification plays an instrumental role here. For completeness and preparation of notation, we also give alternative or simplified proofs of inequalities (3), (4), and (6).

We first recall the entanglement of formation $E(\rho^{ab}) := \inf \sum_i p_i E(|\Psi_i^{ab}\rangle)$ of a bipartite state ρ^{ab} [6,7], which will play a crucial role in our approach. Here the inf is over all pure state decompositions $\rho^{ab} = \sum_i p_i |\Psi_i^{ab}\rangle\langle\Psi_i^{ab}|$, and $E(|\Psi_i^{ab}\rangle) := S(\mathrm{tr}_b|\Psi_i^{ab}\rangle\langle\Psi_i^{ab}|)$. The entanglement of formation is usually regarded as a measure of entanglement, which in turn is understood as some kind of quantum correlation. In particular, $E(|\Psi^{ab}\rangle) = \frac{1}{2}I(|\Psi^{ab}\rangle)$. Inspired by the preceding identity and the fact that $Q(\rho^{ab}) \leqslant I(\rho^{ab})$, it is tempting to assume that the entanglement of formation is dominated by the total correlations, namely, $E(\rho^{ab}) \leqslant I(\rho^{ab})$. However, the above relation fails in general [5]. This fact will be one of the key points in our disproof of inequality (5).

To establish inequality (3), it suffices to show that $I(\Pi^b(\rho^{ab})) \leqslant S(\rho^a)$ for any von Neumann measurement $\Pi^b = \{\Pi_j^b\}$ on party b, which follows readily from

$$\begin{aligned} I(\Pi^b(\rho^{ab})) &= S(\rho^a) + S(\Pi^b(\rho^b)) - S(\Pi^b(\rho^{ab})) \\ &= S(\rho^a) - \sum_j p_j S(\rho_j^a) \leqslant S(\rho^a). \end{aligned}$$

To prove inequality (4), let $|\Psi^{abc}\rangle$ be a purification of ρ^{ab} and put $\rho^{abc} := |\Psi^{abc}\rangle\langle\Psi^{abc}|$, $\rho^{bc} := \mathrm{tr}_a\rho^{abc}$, etc.; then,

$$\begin{aligned} \Pi^b(\rho^{abc}) &= \sum_j \left(\mathbf{1}^a \otimes \Pi_j^b \otimes \mathbf{1}^c\right)\rho^{abc}\left(\mathbf{1}^a \otimes \Pi_j^b \otimes \mathbf{1}^c\right) \\ &= \sum_j p_j \rho_j^{ac} \otimes \Pi_j^b, \end{aligned} \tag{7}$$

where $\rho_j^{ac} := \mathrm{tr}_b(\mathbf{1}^a \otimes \Pi_j^b \otimes \mathbf{1}^c)\rho^{abc}(\mathbf{1}^a \otimes \Pi_j^b \otimes \mathbf{1}^c)/p_j$ and $p_j := \mathrm{tr}\Pi_j^b\rho^b$. Since ρ^{abc} is pure, it follows that ρ_j^{ac} is pure, and we may denote $\rho_j^{ac} = |\Psi_j^{ac}\rangle\langle\Psi_j^{ac}|$. Moreover, $S(\rho_j^a) = S(\rho_j^c)$. Here $\rho_j^a := \mathrm{tr}_c\rho_j^{ac}$ and $\rho_j^c := \mathrm{tr}_a\rho_j^{ac}$. Let $\sigma^{abc} := \Pi^b(\rho^{abc})$, $\sigma^{ab} := \mathrm{tr}_c\sigma^{abc}$, and $\sigma^{bc} := \mathrm{tr}_a\sigma^{abc}$, then

$$\sigma^{ab} = \Pi^b(\rho^{ab}) = \sum_j p_j \rho_j^a \otimes \Pi_j^b,$$

$$\sigma^{bc} = \Pi^b(\rho^{bc}) = \sum_j p_j \Pi_j^b \otimes \rho_j^c.$$

By the monotonicity of quantum relative entropy [2], we know that quantum mutual information decreases under local partial trace, that is,

$$I(\sigma^{ab}) \leqslant I(\sigma^{ac:b}). \tag{8}$$

Here $\sigma^{ac:b}$ means that σ^{abc} is considered as a bipartite state with the partition $ac : b$. Now from Eq. (7) and noting that ρ_j^{ac} and ρ^{abc} are pure, we have

$$I(\sigma^{ac:b}) = S(\rho^{ac}) - \sum_j p_j S\left(\rho_j^{ac}\right) = S(\rho^{ac}) = S(\rho^b).$$

Substituting the above equation into inequality (8) and recalling that $\sigma^{ab} = \Pi^b(\rho^{ab})$, we obtain $I(\Pi^b(\rho^{ab})) \leqslant S(\rho^b)$, which in turn implies that

$$C(\rho^{ab}) = \sup_{\Pi^b} I(\Pi^b(\rho^{ab})) \leqslant S(\rho^b).$$

Next, we proceed to prove inequality (6). Noting that ρ_j^{ac} and ρ^{abc} are pure, we have

$$\begin{aligned} I(\sigma^{ab}) - I(\sigma^{bc}) &= \left[S(\rho^a) - \sum_j p_j S(\rho_j^a)\right] - \left[S(\rho^c) - \sum_j p_j S(\rho_j^c)\right] \\ &= [S(\rho^a) - S(\rho^c)] - \sum_j p_j [S(\rho_j^a) - S(\rho_j^c)] \\ &= S(\rho^a) - S(\rho^c) = S(\rho^a) - S(\rho^{ab}), \end{aligned}$$

from which we obtain

$$\begin{aligned} C(\rho^{ab}) = \sup_{\Pi^b} I(\Pi^b(\rho^{ab})) &\geqslant I(\sigma^{ab}) \geqslant I(\sigma^{ab}) - I(\sigma^{bc}) \\ &= S(\rho^a) - S(\rho^{ab}). \end{aligned}$$

Consequently,

$$\begin{aligned} Q(\rho^{ab}) = I(\rho^{ab}) - C(\rho^{ab}) &\leqslant I(\rho^{ab}) - [S(\rho^a) - S(\rho^{ab})] \\ &= S(\rho^b), \end{aligned}$$

which establishes inequality (6).

Finally, we disprove inequality (5). Noting that ρ^{abc} is pure, we have

$$\begin{aligned} Q(\rho^{ab}) - S(\rho^a) & \qquad (9) \\ &= \left[I(\rho^{ab}) - \sup_{\Pi^b} I(\Pi^b(\rho^{ab}))\right] - S(\rho^a) \\ &= \inf_{\Pi^b} \sum_j p_j S(\rho_j^a) - [S(\rho^a) + S(\rho^{ab}) - S(\rho^b)] \\ &= \inf_{\Pi^b} \sum_j p_j S(\rho_j^a) - [S(\rho^a) + S(\rho^c) - S(\rho^{ac})] \\ &= \inf_{\Pi^b} \sum_j p_j E(\rho_j^{ac}) - I(\rho^{ac}) \\ &\geqslant E(\rho^{ac}) - I(\rho^{ac}). \qquad (10) \end{aligned}$$

Here $E(\rho_j^{ac})$ is the entanglement entropy of the pure state $\rho_j^{ac} = |\Psi_j^{ac}\rangle\langle\Psi_j^{ac}|$, and $E(\rho^{ac})$ is the entanglement of formation of $\rho^{ac} = \sum_j p_j |\Psi_j^{ac}\rangle\langle\Psi_j^{ac}|$.

Now following the result in Ref. [5], we know that there exist some bipartite states ρ^{ac} such that

$$E(\rho^{ac}) - I(\rho^{ac}) > 0,$$

and in such a situation, we will have

$$Q(\rho^{ab}) - S(\rho^a) > 0$$

in view of inequality (10). In particular, if we let $|\Psi^{abc}\rangle$ be a purification of such a state ρ^{ac}, then for $\rho^{ab} := \mathrm{tr}_c|\Psi^{abc}\rangle\langle\Psi^{abc}|$, inequality (5) is reversed.

The classical correlations and quantum correlations, as quantified by $C(\rho^{ab})$ and the quantum discord $Q(\rho^{ab})$, are based on one-sided measurements. If we consider two-sided measurements, then we have an alternative measure of classical correlations:

$$C_2(\rho^{ab}) := \sup_{\Pi^a,\Pi^b} I((\Pi^a \otimes \Pi^b)(\rho^{ab}))$$

and the corresponding measure of quantum correlations $Q_2(\rho^{ab}) := I(\rho^{ab}) - C_2(\rho^{ab})$. Here $(\Pi^a \otimes \Pi^b)(\rho^{ab}) = \sum_{ij}(\Pi_i^a \otimes \Pi_j^b)\rho^{ab}(\Pi_i^a \otimes \Pi_j^b)$, and the above sup is over all von Neumann measurements $\Pi^a = \{\Pi_i^a\}$ on party a and $\Pi^b = \{\Pi_j^b\}$ on party b. The Lindblad conjecture $C_2(\rho^{ab}) \geqslant Q_2(\rho^{ab})$ was disproved in Ref. [14].

By the monotonicity of relative entropy, we know that $C_2(\rho^{ab}) \leqslant C(\rho^{ab})$ and thus $Q_2(\rho^{ab}) \geqslant Q(\rho^{ab})$. Consequently, in view of inequalities (3) and (4) and the violation of inequality (5), if we consider correlative capacities in terms of $C_2(\rho^{ab})$ and $Q_2(\rho^{ab})$, then

$$C_2(\rho^{ab}) \leqslant \min\{S(\rho^a), S(\rho^b)\}$$

for the classical correlative capacity, but neither $Q_2(\rho^{ab}) \leqslant S(\rho^a)$ nor $Q_2(\rho^{ab}) \leqslant S(\rho^b)$ is true in general for the quantum correlative capacity in this context.

The definition of $C_2(\rho^{ab})$ is one way to symmetrize the classical correlations and the quantum discord, but not the only way. It is also intuitive and reasonable to define

$$C_{\mathrm{m}}(\rho^{ab}) := \max\{C_a(\rho^{ab}), C_b(\rho^{ab})\}$$

as a measure of classical correlations. Here $C_a(\rho^{ab}) := \sup_{\Pi^a} I(\Pi^a(\rho^{ab}))$ and $C_b(\rho^{ab}) := \sup_{\Pi^b} I(\Pi^b(\rho^{ab}))$. The corresponding amount of quantum correlations is $Q_{\mathrm{m}}(\rho^{ab}) := I(\rho^{ab}) - C_{\mathrm{m}}(\rho^{ab})$. Clearly, $C_{\mathrm{m}}(\rho^{ab}) \geqslant C(\rho^{ab})$ and $Q_{\mathrm{m}}(\rho^{ab}) \leqslant Q(\rho^{ab})$. Now, inequalities (3), (4), (5), and (6) are *all* satisfied if C and Q are replaced by C_{m} and Q_{m} there. In fact, denote $Q_a(\rho^{ab}) = I(\rho^{ab}) - C_a(\rho^{ab})$ and $Q_b(\rho^{ab}) = I(\rho^{ab}) - C_b(\rho^{ab})$; then, by two applications of inequality (5) (interchanging the roles of a and b for the second application), we have

$$Q_{\mathrm{m}}(\rho^{ab}) = \min\{Q_a(\rho^{ab}), Q_b(\rho^{ab})\} \leqslant \min\{S(\rho^a), S(\rho^b)\}.$$

This stands in sharp contrast to the asymmetric case and the two-sided measurement case.

We make some remarks on the three measures C, C_2, and C_{m} of classical correlations from the measurement perspective with the measurement order in mind. All quantum information quantities are, inevitably, about measurements of some sort. This includes the von Neumann entropy, which is the entropy of the most predictable measurement. Similarly, C, as defined, is not precisely about "one-sided measurements"; it does correspond to an experiment in which both sides are measured: Party b first measures $\{\Pi_j^b\}$ and then communicates his result to party a, who then gets to pick her most predictable measurement (i.e., the diagonal basis of her conditional density operator given b's result). In contrast, C_2 represents a situation where party a must choose her measurement in advance, without knowing what result party b got. This description demystifies quantum discord a bit and more accurately explains the difference between C and C_2. It also motivates the introduction of C_{m}, which corresponds to a case where parties a and b want to maximize their correlations, and they get to choose which of them will measure first and which will measure second. It is remarkable that these subtle differences manifest themselves dramatically in the correlative capacities and also imply immediately that $C_2 \leqslant C \leqslant C_{\mathrm{m}}$ and thus $Q_2 \geqslant Q \geqslant Q_{\mathrm{m}}$.

To summarize, by investigating the correlative capacities of quantum systems in three scenarios, we have revealed some "peculiar" and extremely subtle features of quantum correlations: Depending on the quantification of quantum correlations, the quantum correlative capacity of a system may or may not exceed the system's entropy, while the classical correlative capacity is always limited by the entropy. The key point here is that these three scenarios are highly similar, yet they illustrate dramatically different characteristics. In particular, by exploiting an intimate link between quantum discord and entanglement of formation, we have resolved completely Conjecture 1 in Ref. [11]. The results, apart from their own fundamental significance, may be useful in an informational approach to quantum measurement theory and decoherence theory.

Finally, since quantum correlations have many facets, it will be interesting to further study the correlative capacities of quantum systems in terms of other measures of quantum correlations and their applications in physical systems.

The authors are very grateful to the referees for helpful comments and criticisms and, in particular, for suggesting the results about C_{m} and Q_{m} and for proffering the remarks about the intrinsic relations between C, C_2, and C_{m}. This work was supported by the National Center for Mathematics and Interdisciplinary Sciences, Grant No. Y029152K51.

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