Upper bounds on quantum uncertainty products and complexity measures

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The position-momentum Shannon and Rényi uncertainty products of general quantum systems are shown to be bounded not only from below (through the known uncertainty relations), but also from above in terms of the Heisenberg-Kennard product $\langle r^2\rangle\langle p^2\rangle$. Moreover, the Cramér-Rao, Fisher-Shannon, and López-Ruiz, Mancini, and Calbet shape measures of complexity (whose lower bounds have been recently found) are also bounded from above. The improvement of these bounds for systems subject to spherically symmetric potentials is also explicitly given. Finally, applications to hydrogenic and oscillator-like systems are done.

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I. INTRODUCTION

The uncertainty principle is a basic physico-mathematical aporia. It is not only a relevant issue in harmonic analysis [1], but also a statement of the human and technical limitations to perform measurements on a system without disturbing it [2]. Moreover, the position-momentum uncertainty principle describes a characteristic feature of quantum mechanics whose first mathematical realization is the Heisenberg-Kennard relation [2,3] based on the second-order power moment of the position and momentum probability densities $(\rho(\vec{r}),\gamma(\vec{p}))$ which characterize the quantum state of a physical system. This relation is given in atomic units $\hbar = 1$ by

$$\langle r^2\rangle\langle p^2\rangle \geqslant \tfrac{9}{4}, \tag{1}$$

valid for all quantum-mechanical states of any three-dimensional physical system. Here the symbol $\langle f(r)\rangle$ denotes the expectation value

$$\langle f(r)\rangle := \int_{\mathbb{R}^3} f(r)\rho(\vec{r})d^3r, \quad \text{with} \quad r = |\vec{r}|$$

in position space, and similarly in momentum space. Then the relation (1) was generalized for any power moments $(\langle r^a\rangle,\langle p^b\rangle)$ in the sense [4,5]

$$\langle r^a\rangle^{\frac{1}{a}}\langle p^b\rangle^{\frac{1}{b}} \geqslant \left[\frac{\pi ab}{16\Gamma\left(\frac{3}{a}\right)\Gamma\left(\frac{3}{b}\right)}\right]^{\frac{1}{3}}\left(\frac{3}{a}\right)^{\frac{1}{a}}\left(\frac{3}{b}\right)^{\frac{1}{b}} e^{1-\frac{1}{a}-\frac{1}{b}}, \tag{2}$$

valid for $a,b > 0$. See also [6] for further inequalities with moments of negative orders. For the case $a = b$ we have

$$\langle r^a\rangle\langle p^a\rangle \geqslant \left[\frac{\pi a^2}{16\Gamma^2\left(\frac{3}{a}\right)}\right]^{\frac{a}{3}}\left(\frac{3}{a}\right)^2 e^{a-2}; \quad a > 0, \tag{3}$$

which reduces to (1) when $a = 2$. However, the Heisenberg-Kennard relation is much too weak to express the uncertainty principle (see, e.g., [7,8]).

Presently it is well known that quantities based not on the position and momentum power moments $(\langle r^\alpha\rangle, \langle p^\beta\rangle)$, but instead on the frequency or entropic moments,

$$W_\alpha[\rho] := \int_{\mathbb{R}^3}[\rho(\vec{r})]^\alpha d^3r, \quad W_\beta[\gamma] := \int_{\mathbb{R}^3}[\gamma(\vec{p})]^\beta d^3p,$$

are much more appropriate and stringent uncertainty measures for quantum systems. These quantities are called information-generating functionals in other contexts [9]. The position Rényi entropy [10–12], defined as

$$R_\alpha[\rho] := \frac{1}{1-\alpha}\ln W_\alpha[\rho]; \quad 0 < \alpha < \infty; \quad \alpha \neq 1,$$

and the corresponding momentum quantity $R_\beta[\gamma]$ provide the most relevant canonical class of position and momentum uncertainty measures [13]. These quantities have been widely used in a large variety of quantum systems, phenomena, and processes, as briefly summarized in Refs. [14] and [15]. It is worth noting that the (Boltzmann-Gibbs)-Shannon entropy $S[\rho] = -\int \rho(\vec{r})\ln\rho(\vec{r})d^3r$ is the limiting case $\alpha \to 1$ of $R_\alpha[\rho]$ (see, e.g., [16]), and the Tsallis entropy [17] is a linear approximation of $R_\alpha[\rho]$ with respect to $W_\alpha[\rho]$. Moreover, uncertainty-type inequalities associated with the Shannon and Tsallis entropies have been obtained by Beckner [18] and Bialynicki-Birula–Mycielski [19], and Maassen-Uffink [20] and Rajagopal [21], respectively.

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Bialynicki-Birula [14] has shown in 2006 that the Renyi entropies satisfy the uncertainty relation

$$R_\alpha[\rho] + R_\beta[\gamma] \geqslant -\frac{\ln\left(\frac{\alpha}{\pi}\right)}{2(1-\alpha)} - \frac{\ln\left(\frac{\beta}{\pi}\right)}{2(1-\beta)}; \quad \text{with } \frac{1}{\alpha}+\frac{1}{\beta}=2,$$

which has been extended and rewritten in 2009 by Zozor *et al.* [22] in the form

$$N_\alpha[\rho]N_\beta[\gamma] \geqslant C(\alpha,\beta), \tag{4}$$

where $N_\alpha[\rho]$ denotes the position Renyi α-entropy power [22,23]

$$N_\alpha[\rho] := \exp\left(\frac{1}{3}R_\alpha[\rho]\right) = \{W_\alpha[\rho]\}^{\frac{1}{3(1-\alpha)}} \tag{5}$$

in position space, and $N_\beta[\gamma]$ the corresponding momentum quantity. The constant $C(\alpha,\beta)$ has the value

$$C(\alpha,\beta) = 2\pi\,(2\alpha)^{\frac{1}{2(\alpha-1)}}\,(2\beta)^{\frac{1}{2(\beta-1)}}. \tag{6}$$

On the other hand, the translationally invariant Fisher information [24], defined by

$$F[\rho] := \int_{\mathbb{R}^3} \frac{[\vec{\nabla}\rho(\vec{r})]^2}{\rho(\vec{r})}\, d^3\vec{r},$$

has been shown to be a particularly useful uncertainty measure. This is because, contrary to the previous measures, it has a locality property: it is very sensitive to the fluctuations or irregularities of the position probability density of the stationary states of the quantum systems. Moreover, it has the bounds

$$\frac{81}{\langle r^2\rangle\langle p^2\rangle} \leqslant F[\rho]F[\gamma] \leqslant 16\langle r^2\rangle\langle p^2\rangle,$$

where we have used the Cramér-Rao inequalities [23]

$$F[\rho] \geqslant \frac{9}{\langle r^2\rangle}, \quad F[\gamma] \geqslant \frac{9}{\langle p^2\rangle},$$

and the Stam relation [25]

$$F[\rho] \leqslant 4\,\langle p^2\rangle, \quad F[\gamma] \leqslant 4\,\langle r^2\rangle.$$

Let us also mention, for the sake of completeness, that the Fisher information also satisfies the uncertainty relation

$$F[\rho]F[\gamma] \geqslant 36$$

for quantum systems when either the position wave function or the momentum wave function is real [26].

Furthermore, there exist some products of two single information-theoretic measures which have been shown to be most appropriate to grasp various facets of the internal disorder of quantum systems and to disentangle among their rich three-dimensional geometries: the Cramér-Rao [16,27,28], Fisher-Shannon [29–31], and López-Ruiz, Mancini, and Calbet (LMC) [32,33] complexities. These composite quantities are defined as

$$C_{CR}[\rho] = F[\rho]V[\rho] \tag{7}$$

for the Cramér-Rao complexity; as

$$C_{FS}[\rho] = F[\rho]J[\rho] = \frac{1}{2\pi e}F[\rho]N_1^2[\rho] \tag{8}$$

for the Fisher-Shannon complexity; and as

$$C_{\text{LMC}}[\rho] = D[\rho]\exp(S[\rho]) = D[\rho]N_1^3[\rho] = \left[\frac{N_1[\rho]}{N_2[\rho]}\right]^3 \tag{9}$$

for the LMC shape complexity. The symbols $V[\rho]$ and $J[\rho]$ denote the variance $V[\rho] = \langle r^2\rangle - |\langle\vec{r}\rangle|^2$ and the Shannon quantity $J[\rho] = (2\pi e)^{-1}\exp(\frac{2}{3}S[\rho]) = (2\pi e)^{-1}N_1^2[\rho]$, where $N_1[\rho] = \exp(\frac{1}{3}S[\rho])$ gives the Shannon entropy power, and the disequilibrium $D[\rho] = \langle\rho\rangle = W_2[\rho] = N_2^{-3}[\rho]$ of the system. These three dimensionless two-ingredient measures of complexity, which quantify how easily a quantum system may be modeled, differ from the remaining complexities in the following properties: (i) mathematical simplicity, (ii) invariance under replication, translation, and scaling transformations, and (iii) minimal values in the two extreme cases: perfect order (i.e., for completely ordered systems, that is, when the density denotes a Dirac δ function) and perfect disorder (i.e., for completely disordered systems which have a uniform or highly spread density and an ideal gas in one and three dimensions, respectively). Moreover, they are bounded from below as

$$C_{CR}[\rho] \geqslant 9, \quad C_{FS}[\rho] \geqslant 3, \quad \text{and} \quad C_{\text{LMC}}[\rho] \geqslant 1 \tag{10}$$

for general three-dimensional systems.

In this work we first highlight (see Sec. II) the connection between the Rényi- and Shannon-entropy-based uncertainty products of general quantum systems, as well as the two-ingredient measures of complexity mentioned above, with the Heisenberg-like uncertainty products $\langle r^a\rangle\langle p^b\rangle$. Then in Sec. III we show that the resulting upper bounds can be improved for systems subject to central potentials of arbitrary analytical form. Finally, these quantities are explicitly given and numerically discussed for the hydrogen atom and harmonic oscillator quantum systems in Sec. IV. Some conclusions and open problems are also given.

II. UNCERTAINTY PRODUCTS AND COMPLEXITY MEASURES FOR GENERAL SYSTEMS: UPPER BOUNDS

In this section we first study the relation of the Rényi- and Shannon-entropy-based uncertainty products with the Heisenberg-like products in general quantum systems. Then, we derive upper bounds on the Cramér-Rao, Fisher-Shannon, and LMC shape complexities in terms of Heisenberg-like products.

A. Uncertainty products

We will first show the uncertainty character of the product $N_\alpha[\rho]N_\beta[\gamma]$ of the Rényi entropy powers in position and momentum spaces via its connection with the Heisenberg-like products (1), (2), and (3). To do this we use the variational optimization procedure described in Ref. [34], subject to the constraint $\langle r^a\rangle$, $a = 1,2,\ldots$, to obtain the following lower bound to the position entropic moment $W_\alpha[\rho]$:

$$W_\alpha[\rho] \geqslant A_1(\alpha,a)\langle r^a\rangle^{-[3(\alpha-1)/a]}; \quad \alpha > 1,$$

with the constant

$$A_1(\alpha,a) = \frac{a\alpha}{(a+3)\alpha - 3} \times \left\{ \frac{a}{4\pi B\left(\frac{\alpha}{\alpha-1},\frac{3}{a}\right)} \left[\frac{3\alpha - 3}{(a+3)\alpha - 3} \right]^{\frac{3}{a}} \right\}^{\alpha-1},$$

where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ denotes the β function. Similarly, we find the lower bound

$$W_\beta[\gamma] \geqslant A_1(\beta,b)\langle p^b\rangle^{-[3(\beta-1)/b]}; \quad b = 1,2,\ldots; \quad \beta > 1,$$

for the entropic moment $W_\beta[\gamma]$ of the momentum density $\gamma(\vec{p})$ in terms of the momentum expectation value $\langle p^b\rangle$.

For $\alpha < 1$ we have the following upper bound [35]:

$$W_\alpha[\rho] \leqslant \tilde{A}_1(\alpha,a)\langle r^a\rangle^{[3(1-\alpha)/a]}; \quad \alpha < 1, \quad a > 3\tfrac{1-\alpha}{\alpha},$$

with the constant

$$\tilde{A}_1(\alpha,a) = \frac{\alpha a}{[(a+3)\alpha - 3]^\alpha} \times \left\{ \frac{\alpha a^2}{4\pi B\left(\frac{\alpha}{1-\alpha} - \frac{3}{a},\frac{3}{a}\right)} \left[\frac{3 - 3\alpha)}{(a+3)\alpha - 3} \right]^{\frac{3}{a}} \right\}^{\alpha-1}.$$

Then, taking into account Eq. (5) we have the following upper bounds:

$$N_\alpha[\rho] \leqslant A_2(\alpha,a)\langle r^a\rangle^{1/a} \tag{11}$$

and

$$N_\beta[\gamma] \leqslant A_2(\beta,b)\langle p^b\rangle^{1/b}, \tag{12}$$

for the position and momentum Rényi entropy power, where the constant A_2 is given by

$$A_2(\alpha,a) = \begin{cases} [A_1(\alpha,a)]^{1/3(1-\alpha)}, & \alpha > 1, \\ [\tilde{A}_1(\alpha,a)]^{1/3(1-\alpha)}, & \alpha < 1. \end{cases}$$

The expression (11) extends and generalizes various inequalities of similar type obtained differently by various authors [36,37]. Then, from (11) and (12) we obtain the inequality

$$N_\alpha[\rho]N_\beta[\gamma] \leqslant A_2(\alpha,a)A_2(\beta,b)\langle r^a\rangle^{\frac{1}{a}}\langle p^b\rangle^{\frac{1}{b}}, \tag{13}$$

for $\alpha > 1, \beta > 1$, for the position-momentum Rényi products. Remark that a and $b = 1,2,3,\ldots$ In case that $a = b$, one has

$$N_\alpha[\rho]N_\beta[\gamma] \leqslant A_2(\alpha,a)A_2(\beta,a)[\langle r^a\rangle\langle p^a\rangle]^{1/a}, \tag{14}$$

for $a = 1,2,\ldots; \alpha > 1, \beta > 1$, which connects the Rényi products with the Heisenberg-like uncertainty products $\langle r^a\rangle\langle p^a\rangle$. Moreover, from Eqs. (4)–(6) and Eq. (14) with $a = 2$ one finds

$$N_\alpha[\rho]N_\beta[\gamma] \leqslant A_2(\alpha,2)A_2(\beta,2)(\langle r^2\rangle\langle p^2\rangle)^{1/2}. \tag{15}$$

To obtain the corresponding Shannon-entropy-based uncertainty products, either we carefully take the limits $\alpha \to 1$ and $\beta \to 1$ in the expressions (11), (12), (13), (14), and (15), or we start from the variational bound [38,39] on the Shannon entropy $S[\rho]$ given by

$$S[\rho] \leqslant A_3(a) + \frac{3}{a}\ln\langle r^a\rangle, \quad \forall a > 0,$$

with

$$A_3(a) = \ln\left[\frac{4\pi}{a}\Gamma\left(\frac{3}{a}\right)\left(\frac{ae}{3}\right)^{3/a}\right].$$

Then one easily obtains the following lower bound on the Shannon entropy power $N_1[\rho] = \exp(\frac{1}{3}S[\rho])$ in position space:

$$N_1[\rho] \leqslant A_4(a)\,\langle r^a\rangle^{1/a}, \tag{16}$$

with

$$A_4(a) = \left[\frac{4\pi}{|a|}\Gamma\left(\frac{3}{a}\right)\right]^{\frac{1}{3}}\left(\frac{ae}{3}\right)^{1/a}.$$

A similar result can be obtained for the upper bound on the momentum Shannon entropy power $N_1[\gamma]$ in terms of an arbitrary expectation value $\langle p^b\rangle$, $b > 0$. So the uncertainty product $N_1[\rho]N_1[\gamma]$ is bounded from above as

$$N_1[\rho]N_1[\gamma] \leqslant A_4(a)\,A_4(b)\,\langle r^a\rangle^{\frac{1}{a}}\,\langle p^b\rangle^{1/b}.$$

In case that $a = b$, one has that

$$N_1[\rho]N_1[\gamma] \leqslant [A_4(a)]^2\,(\langle r^a\rangle\langle p^a\rangle)^{1/a} \tag{17}$$

for $a = 1,2,\ldots$, which connects the Shannon-entropy-power-based products with the Heisenberg-like uncertainty products $\langle r^a\rangle\langle p^a\rangle$. Moreover, from Eq. (15) with ($\alpha \to 1$, $\beta \to 1$) or from Eqs. (4)–(6) with $\alpha \to 1$ and (17) with $a = 2$ we have the Shannon product

$$N_1[\rho]N_1[\gamma] \leqslant \frac{2\pi e}{3}(\langle r^2\rangle\langle p^2\rangle)^{1/2} \tag{18}$$

in terms of the Heisenberg-Kennard uncertainty relation $\langle r^2\rangle\langle p^2\rangle$.

B. Complexity measures

In Eq. (10) of Sec. I we have pointed out the lower bounds of the Cramer-Rao, Fisher-Shannon, and LMC measures of complexity. Let us now explore the upper bounds on these three measures.

1. Cramér-Rao complexity $C_{CR}[\rho]$

Taking into account its definition (7) and the Cramér-Rao inequality [i.e., the first expression in (10)], the Stam relation $F[\rho] \leqslant 4\langle p^2\rangle$, and that $V[\rho] \leqslant \langle r^2\rangle$, one obtains that the Cramér-Rao complexity is bounded from both sides as

$$9 \leqslant C_{CR}[\rho] \leqslant 4\langle r^2\rangle\langle p^2\rangle. \tag{19}$$

2. Fisher-Shannon complexity $C_{FS}[\rho]$

Let us start with its definition (8). Taking into account the Stam relation $F[\rho] \leqslant 4\langle p^2\rangle$ and the upper bound $N_1[\rho] \leqslant (\frac{2\pi e}{3}\langle r^2\rangle)^{1/2}$ on the Shannon power entropy previously mentioned, we obtain the upper bound $C_{FS}[\rho] \leqslant \frac{4}{3}\langle r^2\rangle\langle p^2\rangle$ on the Fisher-Shannon complexity. This result together with the lower bound (10) allows us to write the following chain of inequalities:

$$3 \leqslant C_{FS}[\rho] \leqslant \tfrac{4}{3}\langle r^2\rangle\langle p^2\rangle. \tag{20}$$

3. LMC shape complexity $C_{\mathrm{LMC}}[\rho]$

From its definition (9) one has that

$$C_{\mathrm{LMC}}[\rho] = D[\rho] N_1^3[\rho].$$

Now, taking into account that $N_1^3[\rho] \leqslant (\frac{2\pi e}{3}\langle r^2\rangle)^{3/2}$ because of Eq. (16), and the Gadre-Chakraborty inequality for the disequilibrium [40]

$$D[\rho] \leqslant \frac{4}{3\sqrt{3}\pi^2}\langle p^2\rangle^{3/2}, \tag{21}$$

one obtains the upper bound

$$1 \leqslant C_{\mathrm{LMC}}[\rho] \leqslant \frac{2^{7/2}}{3^3\sqrt{\pi}} e^{3/2}(\langle r^2\rangle\langle p^2\rangle)^{3/2}. \tag{22}$$

We know from the very beginning that the upper bound (21) on the disequilibrium is not so accurate because it is the result of two concatenated general inequalities (namely, Cauchy-Schwarz and Sobolev) [40,41]. Consequently, the upper bound in Eq. (22) is poor; nevertheless, it is the only existing one to the best of our knowledge.

III. UNCERTAINTY PRODUCTS AND COMPLEXITY MEASURES FOR CENTRAL POTENTIALS: UPPER BOUNDS

In this section we study the improvement of the upper bounds on the uncertainty products and complexity measures found in the previous section, when the potential of the quantum system is spherically symmetric.

A. Uncertainty products

Let us first improve the inequality (15) between the Rényi-entropy-based product $N_\alpha[\rho]N_\beta[\gamma]$ and the Heisenberg product $\langle r^2\rangle\langle p^2\rangle$ for central potentials.

The stationary states of a single-particle system in a spherically symmetric potential $V(r)$ are known to be described by the wave functions

$$\psi_{nlm}(\vec{r}) = R_{nl}(r)\, Y_{lm}(\theta,\phi),$$

which are characterized through the quantum numbers (n,l,m), where the principal quantum number $n = 0,1,\ldots$, the orbital quantum number $l = 0,1,\ldots$, and the magnetic quantum number $m = -l,-l+1,\ldots,l-1,l$. The angular part is given by the well-known spherical harmonics $Y_{lm}(\theta,\phi)$, and the radial part $R_{nl}(r)$ depends on the analytical form of the potential.

Recently, Sánchez-Moreno *et al.* [42] have used a variational method to bound the Rényi entropy $N_\alpha[\rho]$ with the covariance matrix as a constraint. They found that

$$N_\alpha[\rho] \leqslant B(l,m)\, A_2(\alpha,2)\, \langle r^2\rangle^{1/2} \tag{23}$$

with

$$B(l,m) = \sqrt{3}\left(\frac{2l(l+1)-2m^2-1}{4l(l+1)-3}\right)^{1/6} \times \left(\frac{l(l+1)+m^2-1}{4l(l+1)-3}\right)^{1/3}.$$

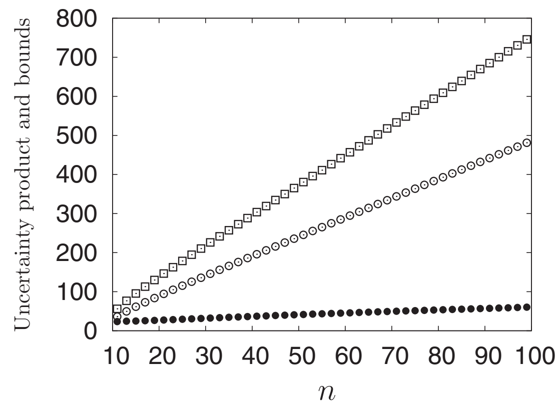

FIG. 1. Uncertainty product $N_2[\rho]N_2[\gamma]$ (•), central upper bound (⊙), and general upper bound (⊡), with $\alpha = \beta = 2$, for the states of the hydrogen atom with quantum numbers $l = m = 10$, as a function of the principal quantum number n, for $n = 11$–100.

It is worth noting that $B(0,0) = 1$. Moreover, $B(l,m) \leqslant 1$ so that the bound (23) is lower (so, better) than the bound given by Eq. (11) with $a = 2$.

By working in momentum space we can obtain a similar bound for the momentum Rényi entropy $N_\beta[\gamma]$ in terms of the expectation value $\langle p^2\rangle$. Then, we can obtain in a straightforward manner the following inequality between the Rényi-entropy-based and Heisenberg uncertainty products:

$$N_\alpha[\rho]N_\beta[\gamma] \leqslant [B(l,m)]^2\, A_2(\alpha,2)\, A_2(\beta,2)\, (\langle r^2\rangle\langle p^2\rangle)^{1/2}. \tag{24}$$

Now, let us show the improvement of the upper bound (18) on the Shannon-entropy-based uncertainty product $N_1[\rho]N_1[\gamma]$ for the central potentials. This is obtained either by taking the limit $\alpha \to 1$ in expressions (23) and (24) or by using the corresponding variational result [42]. The former one yields the value

$$N_1[\rho] \leqslant \left(\frac{2\pi e}{3}\right)^{1/2} B(l,m)\, \langle r^2\rangle^{1/2} \tag{25}$$

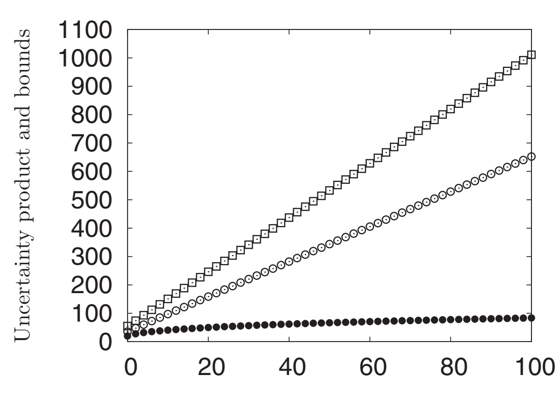

FIG. 2. Uncertainty product $N_2[\rho]N_2[\gamma]$ (•), central upper bound (⊙), and general upper bound (⊡), with $\alpha = \beta = 2$, for the states of the harmonic oscillator with quantum numbers $l = m = 10$, as a function of the principal quantum number n, for $n = 0$ to 100.

for the upper bound on the Shannon entropy power $N_1[\rho]$. And from Eq. (24) one obtains the inequality

$$N_1[\rho]N_1[\gamma] \leqslant \frac{2\pi e}{3}[B(l,m)]^2(\langle r^2\rangle\langle p^2\rangle)^{1/2},$$

which improves the general upper bound (18) because $B(l,m) \leqslant 1$.

B. Complexity measures

Here we improve for central potentials the upper bounds on the Cramer-Rao, Fisher-Shannon, and LMC measures of complexity given in Eqs. (19), (20), and (22), respectively.

1. Cramer-Rao complexity $C_{CR}[\rho]$

Since, for central potentials, the variance is $V[\rho] = \langle r^2\rangle$ and the Fisher information is given [43] as

$$F[\rho] = 4\langle p^2\rangle - 2(2l+1)|m|\langle r^{-2}\rangle, \tag{26}$$

one has the following exact value,

$$C_{CR}[\rho] = 4\langle p^2\rangle\langle r^2\rangle - 2(2l+1)|m|\langle r^{-2}\rangle\langle r^2\rangle, \tag{27}$$

for the Cramer-Rao complexity of physical systems with central potentials, as a function of the expectation values $\langle p^2\rangle$, $\langle r^2\rangle$, and $\langle r^{-2}\rangle$. Notice that for $m = 0$, this expression reduces to $C_{CR}[\rho] = 4\langle p^2\rangle\langle r^2\rangle$. Thus these states saturate the previous general upper bound (19).

2. Fisher-Shannon complexity $C_{FS}[\rho]$

The combination of its definition (8) with the inequality (25) for the Shannon entropy power and the exact expression (26) of the Fisher information for central potentials, yields the upper bound

$$C_{FS} \leqslant \tfrac{1}{3}[4\langle p^2\rangle\langle r^2\rangle - 2(2l+1)|m|\langle r^{-2}\rangle\langle r^2\rangle]\,[B(l,m)]^2 \tag{28}$$

on the Fisher-Shannon complexity of the central potentials, which clearly improves the general upper bound $\frac{4}{3}\langle r^2\rangle\langle p^2\rangle$ of Eq. (20).

3. LMC shape complexity $C_{\rm LMC}[\rho]$

From the definition (9) and the inequality (25) for the Shannon entropy power $N_1[\rho]$ of central potentials one has that

$$C_{\rm LMC}[\rho] = D[\rho]N_1^3[\rho] \leqslant \left(\frac{2\pi e}{3}\right)^{3/2}[B(l,m)]^3\, D[\rho]\,\langle r^2\rangle^{3/2}.$$

On the other hand, we have the general inequality (21) that allows us to find the upper bound

$$C_{\rm LMC}[\rho] \leqslant \frac{1}{3^3}\left(\frac{2^7 e^3}{\pi}\right)^{1/2}[B(l,m)]^3\,(\langle r^2\rangle\langle p^2\rangle)^{3/2} \tag{29}$$

on the LMC complexity of central potentials. It is worth emphasizing that this inequality, which certainly improves the generalized upper bound, can be more refined, provided that we improve for central potentials the inequality (21) of the disequilibrium $D[\rho]$.

IV. APPLICATION TO HYDROGENIC AND OSCILLATOR-LIKE SYSTEMS

In this section we examine the improvement of the general upper bounds on the uncertainty Rényi product and the complexity measures studied in Sec. II by the inclusion of the spherical symmetry (see Sec. III). This is done by comparing the bounds on the uncertainty product and on the complexity measures found in Sec. III for central systems, with respect to the corresponding ones described in Sec. II for general systems. We study these bounds in the two main prototype systems in quantum physics [44]: the hydrogen atom, characterized by the Coulomb potential $V(r) = -\frac{1}{r}$, and the harmonic oscillator, characterized by the potential $V(r) = \frac{1}{2}r^2$. Specifically, we consider the bounds described by the inequalities (15), (19), (20), and (22) for general systems, and (24), (27), (28), and (29) for central systems. Notice that all these bounds are expressed in terms of the expectation values $\langle r^2\rangle$, $\langle p^2\rangle$, and $\langle r^{-2}\rangle$. These expectation values have the expressions

$$\langle r^2\rangle = \frac{n^2}{2}[5n^2 - 3l(l+1) + 1],$$
$$\langle p^2\rangle = \frac{1}{n^2}, \quad \langle r^{-2}\rangle = \frac{2}{n^3}\frac{1}{2l+1},$$

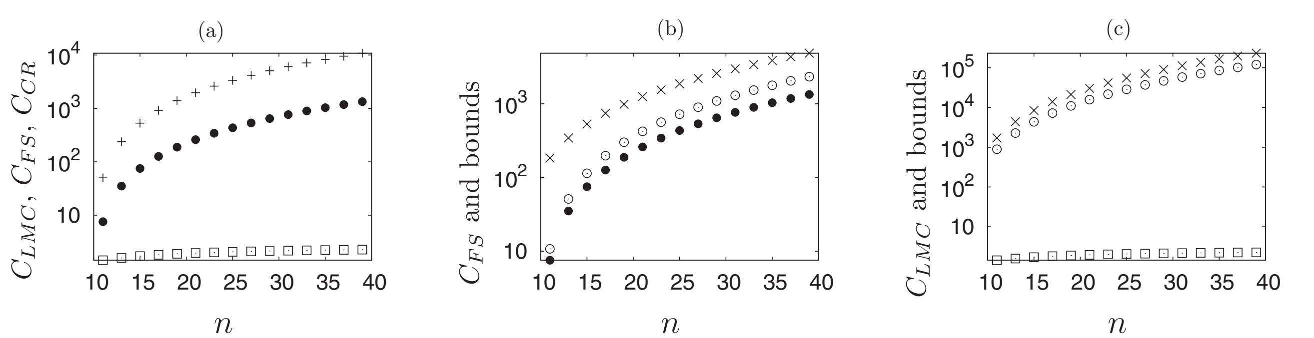

FIG. 3. (a) Cramer-Rao (+), Fisher-Shannon (•), and LMC (⊡) complexity measures. (b) Fisher-Shannon complexity measure (•), central upper bound (⊙), and general upper bound (×). (c) LMC complexity measure (⊡), central upper bound (⊙), and general upper bound (×). All the quantities are plotted for the states of the hydrogen atom with $l = m = 10$ from $n = 11$ to 40.

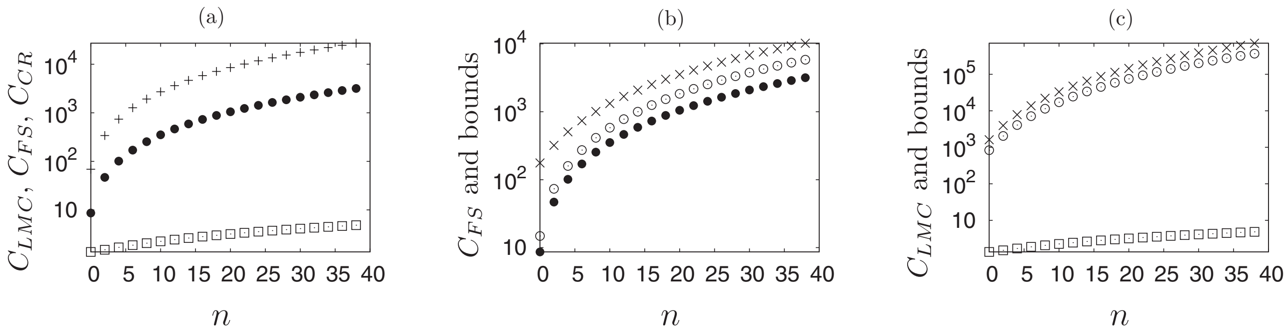

FIG. 4. (a) Cramer-Rao (+), Fisher-Shannon (•), and LMC (⊡) complexity measures. (b) Fisher-Shannon complexity measure (•), central upper bound (⊙), and general upper bound (×). (c) LMC complexity measure (⊡), upper bound for central systems (⊙), and general upper bound (×). All the quantities are plotted for the states of the harmonic oscillator with $l = m = 10$ from $n = 0$ to 40.

for the hydrogen atom, and

$$\langle r^2\rangle = \langle p^2\rangle = 2n + l + \frac{3}{2}, \quad \langle r^{-2}\rangle = \frac{2}{2l+1},$$

for the harmonic oscillator. With these expressions we can calculate the upper bounds of the hydrogen and the oscillator systems on the uncertainty Rényi product and the Cramer-Rao, Fisher-Shannon, and LMC complexities, valid for general and central systems.

First, we study the dependence on n with (l,m) fixed. Figure 1 shows the exact value of the uncertainty product $N_2[\rho]N_2[\gamma]$ (•), the upper bound for central systems given by (24) (⊙), and the general upper bound described in (15) (⊡), with $\alpha = \beta = 2$ for the states of the hydrogen atom, and with quantum numbers $l = m = 10$ as a function of the principal quantum number n for $n = 11 - 100$. Figure 2 shows the same quantities for the harmonic oscillator states with quantum numbers $l = m = 10$ as a function of n for $n = 0 - 100$. In both figures we can see how the upper bound for central systems represents a significant improvement with respect to the general upper bound. Nevertheless, notice that there is still room for much sharper bounds.

Figure 3(a) shows the exact value of the hydrogenic Cramer-Rao (+), Fisher-Shannon (•), and LMC (⊡) complexity measures as a function of the principal quantum number n when $l = m = 10$. Figures 3(b) and 3(c) show the exact values of the hydrogenic Fisher-Shannon (•) and LMC (⊡) complexity measures, respectively, together with their upper bounds for central systems (⊙), and their general upper bounds (×) as a function of the principal quantum number n when $l = m = 10$. Figures 4(a)–4(c) represent the same quantities for the harmonic oscillator states with $l = m = 10$.

The three complexity measures increase with n, since the spreading and the oscillatory behavior of these densities grows with n, both for the hydrogen atom and the harmonic oscillator. Furthermore, in Figs. 3(b) and 4(b) we see that for the Fisher-Shannon complexity, the central upper bound is much sharper than the general bound. This is not the case for the LMC complexity measure, whose bounds, as seen in Figs. 3(c) and 4(c), are relatively far from the exact value, mainly because the upper bound of one of its ingredients (the disequilibrium) has not yet been improved for central potentials.

Second, we study the dependence on l with (n,m) fixed. Figures 5 and 6 represent the same quantities as in Figs. 3 and 4 for the hydrogen atom and the harmonic oscillator, respectively. In Fig. 5 the quantities are given for the hydrogen atom states with $n = 20$ and $m = 0$ as a function of l. Figure 6 shows these quantities for the harmonic oscillator states with $n = m = 0$ as a function of l. In the hydrogen atom case, as shown in Fig. 5(a), the Cramer-Rao and the Fisher-Shannon complexities have a decreasing behavior as l increases. This indicates that the complexity of the density is lower for the states with higher values of l, from the point of view of these measures. However, the LMC complexity

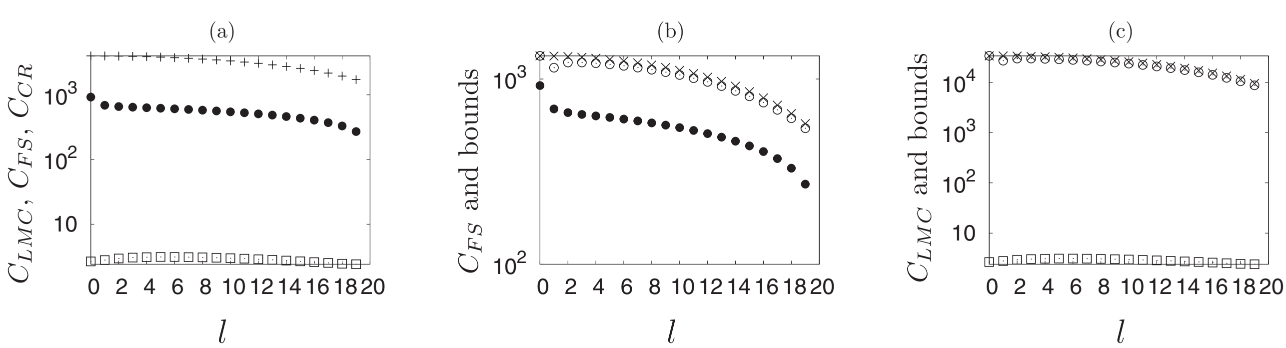

FIG. 5. (a) Cramer-Rao (+), Fisher-Shannon (•), and LMC (⊡) complexity measures. (b) Fisher-Shannon complexity measure (•), central upper bound (⊙), and general upper bound (×). (c) LMC complexity measure (⊡), upper bound for central systems (⊙), and general upper bound (×). All the quantities are plotted for the states of the hydrogen atom with $n = 20$ and $m = 0$ from $l = 0$ to 19.

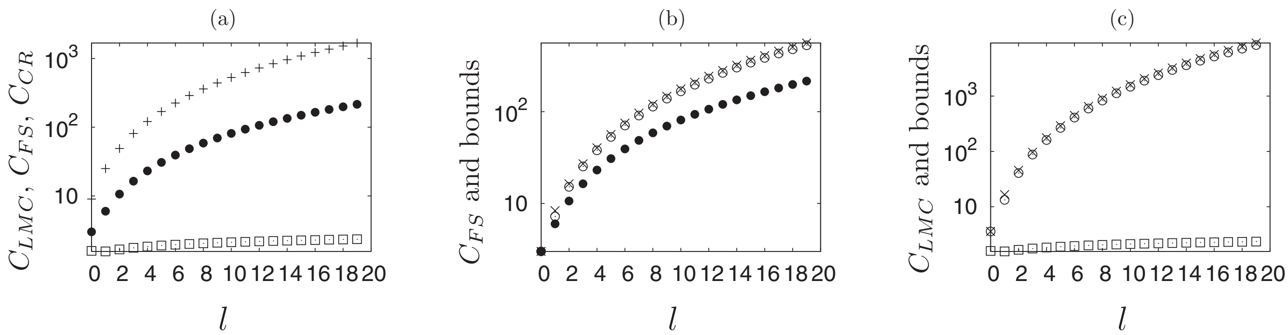

FIG. 6. (a) Cramer-Rao (+), Fisher-Shannon (•), and LMC (⊡) complexity measures. (b) Fisher-Shannon complexity measure (•), central upper bound (⊙), and general upper bound (×). (c) LMC complexity measure (⊡), upper bound for central systems (⊙), and general upper bound (×). All the quantities are plotted for the states of the harmonic oscillator with $n = 0$ and $m = 0$ from $l = 0$ to 19.

shows an increasing behavior for low values of l, which indicates a lower complexity for values of l near 0 for this complexity measure. For the harmonic oscillator, all the complexity measures have an increasing behaviour with l. The reason is that the oscillatory behavior of the density, and hence its complexity, increases with l, as well as with n, for this system. Figures 5(b) and 5(c), and 4(b) and 4(c), show the Fisher-Shannon and LMC complexity measures for the hydrogen atom and harmonic oscillator, respectively, together with the central and general upper bounds. We can clearly see in these figures the improvement of the bound for central potentials. However, this improvement is very small, especially for the LMC complexity for the aforementioned reason.

V. CONCLUSIONS

In this paper we first highlight the uncertainty character of the product of the Rényi entropy powers of position and momentum spaces by showing its inequality-based relationship with the Heisenberg-like uncertainty products. Then the Cramer-Rao, Fisher-Shannon, and LMC complexity measures are shown to be upper-bounded by the Heisenberg-Kennard product. Later on, the resulting bounds are improved for arbitrary spherically symmetric (i.e., central) potentials. Finally, the accuracy of all these bounds is studied in various states of the hydrogenic and oscillator-like systems. In summary, we observe that the inclusion of spherical symmetry considerably improves the general upper bounds found on the Shannon and Rényi uncertainty product as well as on the three complexity measures.

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