Fermi polaron in two dimensions: Importance of the two-body bound state

Michael Klawunn and Alessio Recati
INO-CNR BEC Center and Dipartimento di Fisica,Università di Trento, I-38123 Povo, Italy
(Received 17 May 2011; published 6 September 2011)

We investigate a single impurity interacting with a free two-dimensional atomic Fermi gas. The interaction between the impurity and the gas is characterized by an arbitrary attractive short-range potential, which, in two dimensions, always admits a two-particle bound state. We provide analytical expressions for the energy and the effective mass of the dressed impurity by including the two-body bound state, which is crucial for strong interactions, in the integral equation for the effective interaction. Using the same method, we also give the results for the polaron parameters in one and three dimensions and find good agreement with previous results. Thus, our relations can be used as a simple way to estimate the polaron parameters once the two-body bound state of the interaction potential is known.

DOI: 10.1103/PhysRevA.84.033607 PACS number(s): 03.75.Ss, 05.30.Fk, 67.85.−d

I. INTRODUCTION

In recent years, thanks to the capability of experimentally tuning the relative population and interaction strength in atomic gases of different species, highly imbalanced gases have been extensively studied. In particular, a lot of theoretical and experimental work has been devoted to atomic Fermi gases in two different hyperfine states (see, e.g., the recent review [1] and references therein). The key is the solution of the limiting case of a single-impurity atom interacting via a short-range potential with an ideal atomic Fermi gas. Such a problem is not only relevant in the field of ultracold gases, since it is related to the more general one, the so-called impurity problem, which is present also in other area of physics. In our case, the dressed impurity is called a Fermi polaron (or polaron), in analogy with electrons dressed by the bosonic (phonon) bath in a crystal.

Important quantities characterizing the polaron are (i) its chemical potential, also called interaction energy or binding energy [i.e., the (negative) energy difference of the ground state with and without the impurity atom at rest]; and (ii) its effective mass (i.e., the dressed parabolic dispersion relation at low impurity momentum). In three dimensions (3D), these parameters have been calculated in different ways, for example, by means of variational ansatz [2,3], Monte Carlo methods [4,5], functional renormalization group [6], and experimental measurements [7].

The variational approach is known to give reasonable results also in the one-dimensional (1D) case [2,8], since it can be compared with the exact solution found by McGuire [9]. Very recently the same approach has been used to study the two-dimensional (2D) case, where it has been shown that in two dimensions its use can be questionable, at least if using the same approximations as in three dimensions [10,11].

In two dimensions, as well as in one dimension, an attractive interaction always allows for a two-body bound state. In the present work, we solve the impurity problem including such a bound state explicitly in the integral equation for the effective interaction of the impurity with the Fermi gas. Within a number of approximations, we provide analytical expressions for the polaron parameters which agree quite well with the known results in one and three dimensions. In two dimensions, we find an expression for the energy which interpolates between the correct and expected limiting values in the weakly and the strongly interacting regime. Thanks to the recent experimental advances in realizing 1D and 2D strongly interacting Fermi gases [12,13], the impurity problem in reduced dimensionality has become relevant in the context of ultracold gases.

In the next section, we introduce the formalism and give the result for the energy and the effective mass of the 2D Fermi-polaron problem. In Sec. III, we apply the method to the 1D and 3D cases.

II. FERMI POLARON IN TWO DIMENSIONS

It is known that Brueckner-Hartree-Fock theory, when applied to the Fermi-polaron problem in three dimensions, gives reasonable results [14] (see also Sec. III B). The basic equation from this theory is the Bethe-Goldstone integral equation for the reaction matrix [15], also called effective interaction (e.g., for the 2D electron gas [16]). The Bethe-Goldstone integral equation for the effective interaction between a particle in the bath with momentum $\mathbf{k}_1$ and the impurity atom with momentum $\mathbf{k}_2$ can be written as

$$g(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}) = V(\mathbf{q}) + \int \frac{d\mathbf{k}}{(2\pi)^D} V(|\mathbf{q}-\mathbf{k}|) \times \frac{\left(1-n_{\mathbf{k}_1+\mathbf{k}}\right)}{\frac{k_1^2}{2m}+\frac{k_2^2}{2m}-\frac{(\mathbf{k}_1+\mathbf{k})^2}{2m}-\frac{(\mathbf{k}_2-\mathbf{k})^2}{2m}} g(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}). \quad (1)$$

In Eq. (1), $\mathbf{q}$ is a transfer momentum, $V(\mathbf{q})$ is the Fourier transform of the two-particle interaction potential, and $n_{\mathbf{k}}$ is the Fermi distribution function at zero temperature. The interaction energy or correlation energy follows then from the mean value of the effective interaction $\epsilon_p = \langle g(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q})\rangle$.

For $\mathbf{k}_2 = 0$, one gets the rest correlation energy ϵ_p^0 of the polaron, and by expanding this solution in k_2^2 one get its effective mass m^* as usual by the relation $E = \epsilon_p^0 + k_2^2/2m^*$. We note that in Eq. (1) for the effective interaction only ladder diagrams are summed and the Fermi sea limits the momenta in the intermediate states.

We consider an interaction characterized by an attractive short-range potential of arbitrary shape. In three dimensions, this potential can be approximated by a δ function, and Eq. (1) coincides with the self-consistent equation obtained via single

particle-hole variational ansatz [2,14]. In two dimensions, it is not clear whether one can use a δ function pseudopotential (see, e.g., [17]); hence, a solution of Eq. (1) obtained in the same way as in three dimensions is questionable. As usual, in order to treat properly the two-body problem, we write Eq. (1) by expressing V in terms of the two-particle scattering amplitude f [18]:

$$g(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}) = f(\mathbf{k}_1-\mathbf{k}_2,\mathbf{q}) + \int \frac{d\mathbf{k}}{(2\pi)^D} \frac{f(\mathbf{k}_1-\mathbf{k}_2,\mathbf{q}-\mathbf{k})(-n_{\mathbf{k}_1+\mathbf{k}})}{\frac{k_1^2}{2m}+\frac{k_2^2}{2m}-\frac{(\mathbf{k}_1+\mathbf{k})^2}{2m}-\frac{(\mathbf{k}_2-\mathbf{k})^2}{2m}} \times g(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}), \quad (2)$$

where $f(\mathbf{k}_1-\mathbf{k}_2,\mathbf{q})$ is the off-shell scattering amplitude [19]. Note that this equation is already renormalized with respect to ultraviolet divergencies.

For short-range potentials, the exchange momentum $\mathbf{q}$ is small and the main contribution from the integral comes from small values of $\mathbf{k}$; thus we can approximate the off-shell scattering amplitude by the on-shell scattering amplitude $f(\mathbf{k}_1-\mathbf{k}_2)$. Then the effective interaction does not depend on the exchange momentum, and for the impurity at rest ($k_2=0$) Eq. (2) reduces to

$$\frac{1}{g(\mathbf{k}_1)} \approx \left[\frac{1}{f(\mathbf{k}_1)} - \int_{p<k_F} \frac{d\mathbf{p}}{(2\pi)^2} \frac{m}{\mathbf{p}\cdot(\mathbf{p}-\mathbf{k}_1)}\right]. \quad (3)$$

We assume that the finite range R of the attractive interaction potential is the shortest length scale in the system. In particular, for $k_F R \ll 1$ the s-wave scattering amplitude reads (see, e.g., [20])

$$f(k)^{-1} = -[\ln(k^2/m|\epsilon_b|) - i\pi]m/4\pi, \quad (4)$$

where ϵ_b is the binding energy of the two-particle bound state, which in two dimensions is always present (but not in three dimensions).[1]

Solving Eq. (3) for the weakly interacting case $|\epsilon_b| \ll \epsilon_F$, where $\epsilon_F = \hbar^2 k_F^2/(2m)$ is the Fermi energy, we find the interaction $g(k_1) = -4\pi/[m\ln(2\epsilon_F/|\epsilon_b|)]$ from which we get the mean field energy $\epsilon_p^0 = -2\epsilon_F/[m\ln(2\epsilon_F/|\epsilon_b|)]$, which is obviously in agreement with the result found in the weakly interacting regime using the single particle-hole variational ansatz [10].

Until now, deriving Eq. (3) from the Bethe-Goldstone equation, we have neglected the two-particle bound state. In order to take it into account, we go back to Eq. (2). We rewrite the initial energy of excitation processes appearing in the denominator as $\frac{k_1^2}{2m}+\frac{k_2^2}{2m} = \frac{k_r^2}{m}+\frac{P^2}{4m}$, with relative momentum $\mathbf{k}_r$ and center-of-mass momentum $\mathbf{P}$. Further, we note that the scattering amplitude $f(\mathbf{k}_r)$ depends on the relative momentum only. It is well known that in the case where the majority particle with k_1 and the impurity with k_2 form a two-body bound state, their relative momentum k_r is purely imaginary with $\frac{k_r^2}{m} = \epsilon_b < 0$ being the binding energy. For the calculation of the interaction energy, we assume the impurity to be at rest and $P=0$. Then the effective interaction in ladder approximation obeys, instead of Eq. (3),

$$\frac{1}{g(\epsilon_b)} \approx \frac{1}{f(k_r)} - \int_{k<k_F} \frac{d\mathbf{k}}{(2\pi)^2} \left(|\epsilon_b| + \frac{k^2}{m}\right)^{-1}. \quad (5)$$

Let us notice that a very similar equation is found in [22] for the vertex function in the presence of a two-particle bound state, where the molecular propagator is expressed by the two-particle scattering amplitude at the vacuum energy of the molecule.

At the momentum corresponding to the bound state, the scattering amplitude has a pole $f(k_r = \sqrt{m\epsilon_b})^{-1} = 0$ and the interaction energy is given by

$$\epsilon_p^0 \approx n g(\epsilon_b) = \frac{-2\epsilon_F}{\ln\left[1+\frac{2\epsilon_F}{|\epsilon_b|}\right]}, \quad (6)$$

shown in Fig. 1.

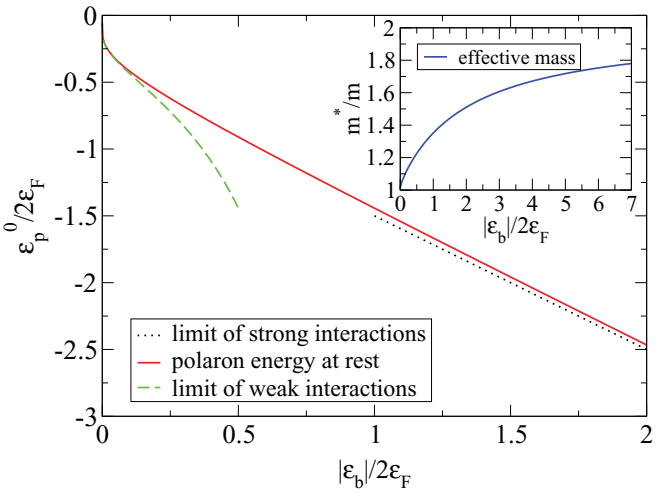

FIG. 1. (Color online) Main: polaron energy as a function of the two-body binding energy ϵ_b as given by Eq. (6) (solid red line) in two dimensions. For completeness, we report also the weakly interacting (dashed green line) and the strongly interacting (dotted black line) results (see text). Inset: ratio m^*/m between the effective and the bare mass as given by Eq. (9).

In the limit of weak interactions (i.e., $|\epsilon_b| \ll \epsilon_F$), the polaron energy reduces to $\epsilon_p \approx -2\epsilon_F/\ln(2\epsilon_F/|\epsilon_b|)$, which coincides with the one obtained from Eq. (3) or the variational ansatz [10], where the two-particle bound state is not included. In the opposite limit, $|\epsilon_b| \gg \epsilon_F$, Eq. (6) yields $\epsilon_p = -|\epsilon_b| - \epsilon_F + o(\epsilon_F/|\epsilon_b|)$. This is the expected result, because in this regime one atom of the majority is strongly bound to the impurity with $-|\epsilon_b|$ and thus it has to be removed from the Fermi sea, leading to the first correction $-\epsilon_F$. Thus, Eq. (6) smoothly interpolates between these two limits and it provides a good approximation for the interaction energy of the 2D polaron in all regimes. Our results are in good agreement with recent preliminary Monte Carlo calculations [23]. We note that Eq. (6) is valid for all attractive potentials with s-wave scattering amplitude of logarithmic form [Eq. (4)].

[1]The relation between the binding energy and the potential $V(\mathbf{r})$ in two dimensions has been recently discussed in detail in [21].

A. Effective mass

In this section, we study the effect of the interaction on the motion of the impurity. As already mentioned, its energy can be expanded for small momentum k_2 as $E = \epsilon_p^0 + k_2^2/2m^*$, where we define the effective mass of the impurity as

$$\frac{1}{m^*} = \frac{1}{m} + \frac{1}{m}\frac{d\epsilon_p(\mathbf{k}_2)}{d\left(k_2^2/2\right)}|_{k_2=0} \tag{7}$$

and $\epsilon_p(k_2) = ng(\epsilon_b,k_2)$, with the effective interaction g calculated again including the two-body bound state. In particular, the initial energy of excitation processes is $-|\epsilon_b| + k_2^2/4m$ and instead of Eq. (5) we obtain

$$\frac{1}{g(\epsilon_b,\mathbf{k}_2)} \approx \frac{1}{f\,(k_r)} - \int_0^{k_F} \frac{d\mathbf{k}}{(2\pi)^2}\frac{1}{|\epsilon_b| + e(k_2,k)}, \tag{8}$$

where $e(k_2,k) = \frac{k_2^2}{4m} + \frac{k_2 k\cos\phi}{m} + \frac{k^2}{m}$. The ratio between the bare and the effective mass of the impurity atom in two dimensions within our approximation reads

$$\frac{m}{m^*} = 1 - \frac{1}{2}\left(\frac{\epsilon_p^0}{2\epsilon_F}\right)^2\left(1 + \frac{|\epsilon_b|}{2\epsilon_F}\right)^{-2}. \tag{9}$$

As shown in Fig. 1, the effective mass m^* obtained from the previous equation has the expected behavior: For small interactions it is close to the bare mass value m and for large interactions it approaches the molecular mass value $2m$.

III. FERMI-POLARON PARAMETERS IN ONE AND THREE DIMENSIONS

For a 2D system, our approach seems to give quite reasonable results and provides analytical expressions for the polaron's parameters. In the present section, we apply our approach to the 1D and 3D case. The simple expressions we find are in reasonable agreement with the known results.

A. The Fermi-polaron in one dimension

In one dimension, the Fermi-polaron problem admits an exact solution [9] and the interaction energy reads

$$\frac{\epsilon_p^0}{2\epsilon_F} = -\frac{1}{\pi}\left[y + \frac{\pi}{2}y^2 + (1+y^2)\arctan(y)\right], \tag{10}$$

where $y = \sqrt{|\epsilon_b|/(2\epsilon_F)}$. Again ϵ_b is the binding energy of the lowest two-body bound state.

When applied to one dimension, Eq. (5) gives for the polaron energy

$$\frac{\epsilon_p^0}{2\epsilon_F} \approx -\frac{y}{\arctan\left(\frac{1}{y}\right)}, \tag{11}$$

which we compare against the exact result [Eq. (10)] in Fig. 2. The agreement looks pretty good, although in the strongly interacting case, we get $-|\epsilon_b| - 2/3\epsilon_F$ instead of $-|\epsilon_b| - \epsilon_F$. However, our results are closer to the exact solution than the one obtained with the single particle-hole variational ansatz (see, e.g., [8]).

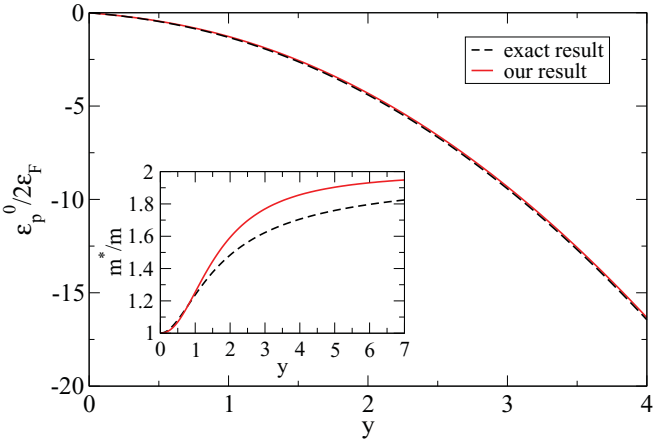

FIG. 2. (Color online) Interaction energy (main panel) and effective mass (inset) as a function of the two-body binding energy ϵ_b in one dimension. The approximate result [Eq. (11)] (solid red line) is compared with the exact expression [Eq. (10)] given by McGuire [9].

B. The Fermi polaron in three dimensions

In a 3D geometry when no two-body bound state is present [i.e., for negative s-wave scattering length ($a < 0$)], we can use Eq. (3)—the usual Brueckner-Hartree-Fock theory—with scattering amplitude $4\pi f(\mathbf{k}_1)^{-1}/m = a^{-1} + i|\mathbf{k}_1|$. The main contribution to the effective interaction is $g(k_1 = 0)$, and thus we can write an approximated expression for the polaron energy as

$$\frac{\epsilon_p^0}{\epsilon_F} \approx -\frac{2}{3\left(1 - \frac{\pi}{2}\frac{1}{k_F a}\right)}. \tag{12}$$

Note that adding self-consistency to the Bethe-Goldstone equation by changing the initial energy of excitation processes $\frac{k_1^2}{2m} + \frac{k_2^2}{2m} \to \epsilon_p^0 + \frac{k_1^2}{2m} + \frac{k_2^2}{2m}$ [14] increases the accuracy of the results in the polaron regime and yields the same equations as in [2].

In the molecular regime, the potential admits a two-body bound state ($a > 0$) with binding energy $\epsilon_b = -1/(ma^2)$. From the 3D version of Eq. (5), we obtain the interaction energy of the impurity, which reads

$$\frac{\epsilon_p^0}{\epsilon_F} \approx -\frac{2}{3\left[1 - \sqrt{\frac{|\epsilon_b|}{2\epsilon_F}}\arctan\left(\sqrt{\frac{2\epsilon_F}{|\epsilon_b|}}\right)\right]}, \tag{13}$$

with $\sqrt{|\epsilon_b|/(2\epsilon_F)} = 1/(k_F a)$. For large binding energy $|\epsilon_b| \gg \epsilon_F$ (or $k_F a \ll 1$), one gets $\epsilon_p^0 = -|\epsilon_b| - 6\epsilon_F/5$, which is larger than the expected result by $-1/5\ \epsilon_F$. In Fig. 3, we show the comparison between the previous simple expression and the results obtained from Monte Carlo calculations [5]. Although the agreement is good, it is worse (as already mentioned) than the results obtained from the variational approach in both the polaron [2] and the molecular regime [3] as well as the results obtained from the functional renormalization group [6].

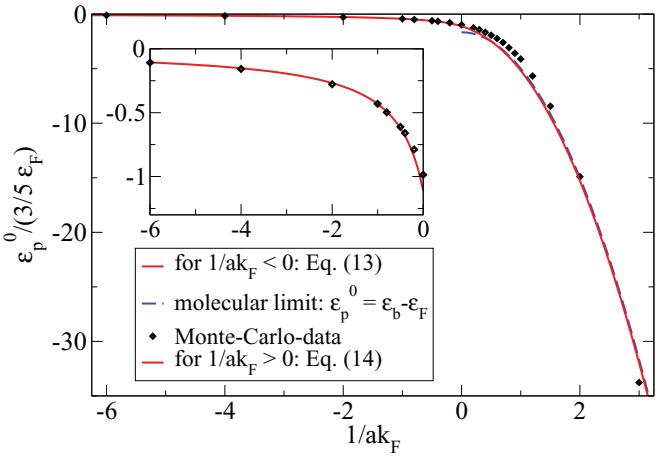

FIG. 3. (Color online) Interaction energy as a function of the inverse 3D scattering length $1/(k_F a)$ (red line) in comparison with the results obtained from Monte Carlo calculations [5] in the polaron regime and at unitarity (black diamonds). In the molecule regime ($|\epsilon_b| \gg \epsilon_F$), we compare the expected result $\epsilon_p^0 = -|\epsilon_b| - \epsilon_F$ (blue line). Inset: zoom on the negative axis.

C. Remarks on the effective mass in one and three dimensions

The effective mass of the impurity is given by Eq. (9) in any dimension.

In the inset of Fig. 2, we compare our result [Eq. (9)] with the exact one found by McGuire in [9] which reads

$$\frac{m^*}{m} = \frac{\left(1+\frac{2}{\pi}\arctan y\right)^2}{1+\frac{2}{\pi}\left(\arctan y + \frac{y}{1+y^2}\right)}, \tag{14}$$

where $y = \sqrt{|\epsilon_b|/(2\epsilon_F)}$ is defined as in Eq. (10). Again the agreement is reasonable and better than the single particle-hole variational ansatz [2,8].

In three dimensions, the situation is more involved since a maximum (quite larger than $2m$) for the effective mass has been found when the nature of the impurity changes from a fermionic quasiparticle to a bosonic quasiparticle [4]. In our approximation, this maximum cannot be found. In order to find this maximum, one has to take three-particle scattering into account, which is beyond the scope of this paper.

IV. CONCLUSION

In conclusion, we have investigated the problem of an impurity atom interacting with a noninteracting Fermi gas in a 2D geometry. We consider a short-range, attractive potential, which implies the presence of a two-body bound state for any interaction strength. We have calculated the interaction energy and the effective mass of the impurity by including the bound state in the Bethe-Goldstone integral equation for the effective interaction. We were able to obtain simple analytical expressions which give reasonable results in two dimensions, interpolating between the weakly and the strongly interacting regime (see Fig. 1). Moreover, when applied to the 3D and 1D cases, our polaron parameters compare well with most of the known results (see Sec. III). Thus our expressions can be used to estimate the dressed impurity's parameters in a simple way once the two-body bound state is known. Our analysis shows how important the two-body bound state is for the polaron problem.

Finally, let us stress that we do not address the question of whether there exists a polaron-to-molecule transition in two dimensions as debated in [10,11]. The problem is still open and it could happen that the system behaves similarly to the 1D case where there is not such a transition. Such a question is clearly relevant for the possible low-temperature phases of a 2D polarized Fermi gas, whose balanced version has been recently experimentally realized [13].

ACKNOWLEDGMENTS

Useful discussions with Georg Bruun and Alexander Pikovski are acknowledged. We also thank G. Bertaina and S. Giorgini for the opportunity to compare our calculations with the unpublished results of a Monte Carlo calculation. This work has been supported by the ERC through a QGBE grant.

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