Quantum entanglement in multiparticle systems of two-level atoms

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We propose the necessary and sufficient condition for the presence of quantum entanglement in arbitrary symmetric pure states of two-level atomic systems. We introduce a parameter to quantify quantum entanglement in such systems. We express the inherent quantum fluctuations of a composite system of two-level atoms as a sum of the quantum fluctuations of the individual constituent atoms and their correlation terms. This helps to separate out and study solely the quantum correlations among the atoms and obtain the criterion for the presence of entanglement in such multiatomic systems.

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I. INTRODUCTION

Over the past few years there has been a growing interest in quantum mechanically correlated multiatomic systems [1–4]. Quantum entanglement, which is the basic ingredient of quantum information theory, is yet to be understood completely in the context of such systems. The proposal of the criterion made by Peres and Horodecki in Refs. [5] and [6], regarding the presence of quantum entanglement in a quantum system, forms an important step toward the understanding of quantum entanglement in the context of bipartite states. It has also been found that spin squeezing has a close relationship with quantum entanglement and a lot of work has been done in this direction [7–13]. A system that is in a spin squeezed state is also quantum mechanically entangled. But, quantum entanglement does not ensure spin squeezing. A system that is in a quantum mechanically entangled state may not always show spin squeezing. Therefore, spin squeezing cannot always be used to detect and quantify quantum entanglement.

In this paper, we introduce the necessary and sufficient condition for the presence of quantum entanglement in multiatomic systems and also introduce a parameter to quantify quantum entanglement.

An atom has many energy levels, but when it is interacting with an external monochromatic electromagnetic field, we concentrate mainly on two of its energy levels, among which the transition of the atom takes place. Hence, the atom is called a two-level atom.

We consider a system of N such two-level atoms. If among the two energy levels of the n-th atom in the assembly, the upper and lower energy levels are denoted as $|u_n\rangle$ and $|l_n\rangle$, respectively, then we can construct a vector operator $\hat{\mathbf{J}}_n$ whose components are

$$\hat{J}_{n_x} = (1/2)(|u_n\rangle\langle l_n| + |l_n\rangle\langle u_n|), \tag{1}$$

$$\hat{J}_{n_y} = (-i/2)(|u_n\rangle\langle l_n| - |l_n\rangle\langle u_n|), \tag{2}$$

$$\hat{J}_{n_z} = (1/2)(|u_n\rangle\langle u_n| - |l_n\rangle\langle l_n|), \tag{3}$$

such that

$$\left[\hat{J}_{n_x}, \hat{J}_{n_y}\right] = i\hat{J}_{n_z} \tag{4}$$

and two more relations with cyclic changes in x, y, and z. Since the operators $\hat{J}_{n_x}$, $\hat{J}_{n_y}$, and $\hat{J}_{n_z}$ obey the same commutation relations as the spin operators, these are called pseudo-spin operators.

For the entire system of N two-level atoms, we construct collective pseudo-spin operators

$$\hat{J}_x = \sum_{i=1}^{N} \hat{J}_{i_x}, \quad \hat{J}_y = \sum_{i=1}^{N} \hat{J}_{i_y}, \quad \hat{J}_z = \sum_{i=1}^{N} \hat{J}_{i_z}, \tag{5}$$

where it is implicitly assumed that each term in the above summations is in direct product with the identity operators of all other atoms.

The individual atomic operators satisfy

$$\left[\hat{J}_{1_x}, \hat{J}_{2_y}\right] = 0, \quad \left[\hat{J}_{1_x}, \hat{J}_{1_y}\right] = i\hat{J}_{1_z}, \quad \left[\hat{J}_{2_x}, \hat{J}_{2_y}\right] = i\hat{J}_{2_z}, \ldots \tag{6}$$

As a direct consequence of these commutation relations we have

$$[\hat{J}_x, \hat{J}_y] = i\hat{J}_z \tag{7}$$

and two more relations with cyclic changes in x, y, and z.

The simultaneous eigenvectors of $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$ and $\hat{J}_z$ are denoted as $|j,m\rangle$, where

$$\hat{J}^2|j,m\rangle = j(j+1)|j,m\rangle \tag{8}$$

and

$$\hat{J}_z|j,m\rangle = m|j,m\rangle. \tag{9}$$

Here $j = N/2$ and $m = -j, -j+1, \ldots, (j-1), j$. The collective quantum state vector for a system of N two-level atoms can be expressed as a linear superposition of $|j,m\rangle$ as

$$|\psi_j\rangle = \sum_{m=-j}^{j} c_m |j,m\rangle. \tag{10}$$

To find out whether a quantum state $|\psi_j\rangle$ of the system is an atomic coherent state [16] or an atomic squeezed state [14], [15] we calculate the mean pseudo-spin vector

$$\langle\hat{\mathbf{J}}\rangle = \langle\hat{J}_x\rangle\hat{i} + \langle\hat{J}_y\rangle\hat{j} + \langle\hat{J}_z\rangle\hat{k} \tag{11}$$

for the quantum state $|\psi_j\rangle$. The vector $\langle\hat{\mathbf{J}}\rangle$ may have arbitrary direction in space. We calculate the variances

$$\Delta J_{1,2}^2 = \left\langle\hat{J}_{1,2}^2\right\rangle - \langle\hat{J}_{1,2}\rangle^2, \tag{12}$$

where $\hat{J}_1$ and $\hat{J}_2$ are the components of $\hat{\mathbf{J}}$ along two mutually perpendicular directions in a plane perpendicular to $\langle\hat{\mathbf{J}}\rangle$. If these variances satisfy

$$\Delta J_1^2 = \Delta J_2^2 = j/2 = N/4, \tag{13}$$

then the state $|\psi_j\rangle$ is called an atomic coherent state. If

$$\Delta J_1^2 \quad \text{or} \quad \Delta J_2^2 < j/2 = N/4, \tag{14}$$

the state $|\psi_j\rangle$ is said to be an atomic squeezed state or spin squeezed state [14]. This definition of spin squeezing is free from the coordinate dependency and includes quantum correlation among the atoms in the notion of squeezing. We can now define quantities

$$Q_1 = \sqrt{\frac{2}{j}}\Delta J_1 \tag{15}$$

and

$$Q_2 = \sqrt{\frac{2}{j}}\Delta J_2 \tag{16}$$

such that if Q_1 and Q_2 are equal to 1, then $|\psi_j\rangle$ is called an atomic coherent state. If

$$Q_1 \quad \text{or} \quad Q_2 < 1, \tag{17}$$

the state $|\psi_j\rangle$ is said to be an atomic squeezed state [15].

It is to be mentioned here that

$$Q_1 Q_2 \geqslant 1, \tag{18}$$

which is Heisenberg's uncertainty principle.

Now, normally, to perform the above calculations, we rotate the coordinate system $\{x,y,z\}$ to $\{x',y',z'\}$, such that the mean pseudo-spin vector $\langle\hat{\mathbf{J}}\rangle$ points along the z' axis. We then calculate the variances

$$\Delta J_{x',y'}^2 = \langle \hat{J}_{x',y'}^2\rangle - \langle \hat{J}_{x',y'}\rangle^2 \tag{19}$$

and investigate the behavior of

$$Q_x = \sqrt{\frac{2}{j}}\Delta J_{x'} \tag{20}$$

and

$$Q_y = \sqrt{\frac{2}{j}}\Delta J_{y'}. \tag{21}$$

If

$$Q_x = Q_y = 1, \tag{22}$$

the corresponding quantum state of the system is a coherent state. If

$$Q_x \quad \text{or} \quad Q_y < 1, \tag{23}$$

the quantum state is a spin squeezed state. Here, Q_x and Q_y satisfy

$$Q_x Q_y \geqslant 1. \tag{24}$$

A collective state vector $|\alpha\rangle$ for a system of two atoms is said to be quantum mechanically entangled if $|\alpha\rangle$ cannot be expressed as a direct product of the two individual atomic state vectors, i.e.,

$$|\alpha\rangle \neq |\alpha_1\rangle \otimes |\alpha_2\rangle, \tag{25}$$

where $|\alpha_1\rangle$ and $|\alpha_2\rangle$ are the state vectors of the two individual atoms [17].

This paper is organized as follows. In Sec. II we formulate the necessary and sufficient condition for the presence of quantum entanglement in arbitrary symmetric pure states of two two-level atoms. We also construct a parameter, called quantum entanglement parameter, to quantify quantum entanglement in such systems. In Sec. III we establish the relationship between quantum entanglement parameter and experimentally measurable quantities. That is, we show how the quantum entanglement parameter can be measured experimentally. In Sec. IV, we generalize these ideas in case of systems containing N number of two-level atoms.

II. QUANTUM ENTANGLEMENT IN A SYSTEM OF TWO TWO-LEVEL ATOMS

We consider a system of two two-level atoms. To formulate the necessary and sufficient condition for the presence of quantum entanglement in arbitrary symmetric pure states of this system, we first find out the quantum fluctuations of the composite system in terms of the components of $\hat{\mathbf{J}}$ along two mutually orthogonal directions in a plane perpendicular to $\langle\hat{\mathbf{J}}\rangle$. We then express these fluctuations as an algebraic sum of the quantum fluctuations of the individual constituent atoms and their correlation terms. This helps us to isolate and study solely the quantum correlation terms among the two atoms and obtain the necessary and sufficient condition for the presence of quantum entanglement. From there we also construct a parameter to quantify quantum entanglement.

A normalized collective quantum state $|\psi\rangle$ for this system can be expressed as

$$|\psi\rangle = C_1|j=1,m=1\rangle + C_2|j=1,m=0\rangle + C_3|j=1,m=-1\rangle, \tag{26}$$

where C_1, C_2, and C_3 are constants satisfying

$$|C_1|^2 + |C_2|^2 + |C_3|^2 = 1. \tag{27}$$

The state $|j=1,m=1\rangle$ corresponds to the case when both the atoms are in their respective upper states. $|j=1,m=-1\rangle$ means both the atoms are in their lower states and $|j=1,m=0\rangle$ implies one atom in the upper and the other in its lower state.

In $\{m_1,m_2\}$ representation, if $|m_1,m_2\rangle$ is the simultaneous eigenvector of $\hat{J}_{1_z}$ and $\hat{J}_{2_z}$ with eigenvalues m_1 and m_2, respectively, then we can write

$$|j=1,m=1\rangle = |m_1=1/2,m_2=1/2\rangle, \tag{28}$$

$$|j=1,m=-1\rangle = |m_1=-1/2,m_2=-1/2\rangle, \tag{29}$$

and

$$|j=1,m=0\rangle = \frac{1}{\sqrt{2}}[|m_1=1/2,m_2=-1/2\rangle + |m_1=-1/2,m_2=1/2\rangle] \tag{30}$$

[18]. Thus, $|\psi\rangle$ in Eq. (26) can be written as

$$|\psi\rangle = C_1|1/2,1/2\rangle + \frac{1}{\sqrt{2}}C_2[|1/2,-1/2\rangle + |-1/2,1/2\rangle] + C_3|-1/2,-1/2\rangle. \quad (31)$$

This state vector is symmetric under the exchange of two atoms.

Since $|\psi\rangle$ is arbitrary, the quantities $\langle\psi|\hat{J}_x|\psi\rangle$, $\langle\psi|\hat{J}_y|\psi\rangle$, and $\langle\psi|\hat{J}_z|\psi\rangle$ have arbitrary values and, hence, $\langle\hat{\mathbf{J}}\rangle = \langle\psi|\hat{\mathbf{J}}|\psi\rangle$ has arbitrary direction. We now perform a rotation of the coordinate system from $\{x,y,z\}$ to $\{x',y',z'\}$ such that the vector $\langle\hat{\mathbf{J}}\rangle$ points along the z' axis. In doing so, we assume that the vector $\langle\hat{\mathbf{J}}\rangle$ was in the first octant of the coordinate system $\{x,y,z\}$. After the rotation, the components $\{\hat{J}_{x'},\hat{J}_{y'},\hat{J}_{z'}\}$ in the rotated frame $\{x',y',z'\}$ are related to $\{\hat{J}_x,\hat{J}_y,\hat{J}_z\}$ in the unrotated frame $\{x,y,z\}$ as

$$\hat{J}_{x'} = \hat{J}_x\cos\theta\cos\phi + \hat{J}_y\cos\theta\sin\phi - \hat{J}_z\sin\theta, \quad (32)$$

$$\hat{J}_{y'} = -\hat{J}_x\sin\phi + \hat{J}_y\cos\phi, \quad (33)$$

$$\hat{J}_{z'} = \hat{J}_x\sin\theta\cos\phi + \hat{J}_y\sin\theta\sin\phi + \hat{J}_z\cos\theta, \quad (34)$$

where

$$\cos\theta = \frac{\langle\hat{J}_z\rangle}{|\langle\hat{\mathbf{J}}\rangle|}, \quad (35)$$

$$\cos\phi = \frac{\langle\hat{J}_x\rangle}{\sqrt{\langle\hat{J}_x\rangle^2 + \langle\hat{J}_y\rangle^2}}. \quad (36)$$

We can check using Eqs. (32), (35), and (36), that for the arbitrary state $|\psi\rangle$ we have

$$\langle\hat{J}_{x'}\rangle = \langle\hat{J}_x\rangle\cos\theta\cos\phi + \langle\hat{J}_y\rangle\cos\theta\sin\phi - \langle\hat{J}_z\rangle\sin\theta \quad (37)$$

$$= \frac{1}{|\langle\hat{\mathbf{J}}\rangle|\sqrt{\langle\hat{J}_x\rangle^2 + \langle\hat{J}_y\rangle^2}}[\langle\hat{J}_x\rangle^2\langle\hat{J}_z\rangle + \langle\hat{J}_y\rangle^2\langle\hat{J}_z\rangle - \langle\hat{J}_z\rangle(\langle\hat{J}_x\rangle^2 + \langle\hat{J}_y\rangle^2)] = 0. \quad (38)$$

Similarly, using Eqs. (33), (34), (35), and (36), we have,

$$\langle\hat{J}_{y'}\rangle = 0, \quad (39)$$

$$\langle\hat{J}_{z'}\rangle = |\langle\hat{\mathbf{J}}\rangle|. \quad (40)$$

Thus, the mean pseudo-spin vector is now along the z' axis.

We now calculate the quantum fluctuations in the components of $\hat{\mathbf{J}}$ along two mutually orthogonal directions, in a plane perpendicular to $\langle\hat{\mathbf{J}}\rangle$. For simplicity, we take the above-mentioned two orthogonal directions along the x' and y' axes, respectively. Therefore, we calculate the quantum fluctuations $\Delta J^2_{x'}$ and $\Delta J^2_{y'}$.

Now, we have already shown in Eqs. (38) and (39) that we have here $\langle\hat{J}_{x'}\rangle = \langle\hat{J}_{y'}\rangle = 0$. Therefore, according to Eqs. (19), (32), and (33), we obtain for the quantum state $|\psi\rangle$

$$\Delta J^2_{x'} = \langle\hat{J}^2_{x'}\rangle = \langle\hat{J}^2_x\rangle\cos^2\theta\cos^2\phi + \langle\hat{J}^2_y\rangle\cos^2\theta\sin^2\phi + \langle\hat{J}^2_z\rangle\sin^2\theta + \tfrac{1}{2}\langle\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x\rangle\cos^2\theta\sin 2\phi - \tfrac{1}{2}\langle\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x\rangle\sin 2\theta\cos\phi - \tfrac{1}{2}\langle\hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y\rangle\sin 2\theta\sin\phi \quad (41)$$

and

$$\Delta J^2_{y'} = \langle\hat{J}^2_{y'}\rangle = \langle\hat{J}^2_x\rangle\sin^2\phi + \langle\hat{J}^2_y\rangle\cos^2\phi - \tfrac{1}{2}\langle\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x\rangle\sin 2\phi, \quad (42)$$

respectively.

It is to be mentioned here that we are not using the forms of $\cos\theta$, $\cos\phi$, etc., as given in Eqs. (35) and (36), respectively, as to keep the mathematical expressions neat. At the end of the calculation, we use the above-mentioned equations to ensure that we are calculating the fluctuations in a plane perpendicular to $\langle\hat{\mathbf{J}}\rangle$.

We now express these fluctuations as an algebraic sum of the quantum fluctuations of the individual constituent atoms and their correlation terms. From Eq. (5), we have for a system of two two-level atoms,

$$\hat{J}_x = \hat{J}_{1_x} + \hat{J}_{2_x}, \quad (43)$$

$$\hat{J}_y = \hat{J}_{1_y} + \hat{J}_{2_y}, \quad (44)$$

$$\hat{J}_z = \hat{J}_{1_z} + \hat{J}_{2_z}. \quad (45)$$

Therefore, we have

$$\langle\hat{J}^2_{x,y,z}\rangle = \langle\hat{J}^2_{1_{x,y,z}}\rangle + \langle\hat{J}^2_{2_{x,y,z}}\rangle + \langle\hat{J}_{1_{x,y,z}}\hat{J}_{2_{x,y,z}} + \hat{J}_{2_{x,y,z}}\hat{J}_{1_{x,y,z}}\rangle. \quad (46)$$

Now, using Eqs. (43) and (44), we have

$$\langle\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x\rangle = \langle\hat{J}_{1_x}\hat{J}_{1_y} + \hat{J}_{1_y}\hat{J}_{1_x}\rangle + \langle\hat{J}_{2_x}\hat{J}_{2_y} + \hat{J}_{2_y}\hat{J}_{2_x}\rangle + \langle\hat{J}_{1_x}\hat{J}_{2_y} + \hat{J}_{2_y}\hat{J}_{1_x}\rangle + \langle\hat{J}_{1_y}\hat{J}_{2_x} + \hat{J}_{2_x}\hat{J}_{1_y}\rangle. \quad (47)$$

Similarly, using Eqs. (43), (44), and (45), we have

$$\langle\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x\rangle = \langle\hat{J}_{1_x}\hat{J}_{1_z} + \hat{J}_{1_z}\hat{J}_{1_x}\rangle + \langle\hat{J}_{2_x}\hat{J}_{2_z} + \hat{J}_{2_z}\hat{J}_{2_x}\rangle + \langle\hat{J}_{1_x}\hat{J}_{2_z} + \hat{J}_{2_z}\hat{J}_{1_x}\rangle + \langle\hat{J}_{1_z}\hat{J}_{2_x} + \hat{J}_{2_x}\hat{J}_{1_z}\rangle \quad (48)$$

and

$$\langle\hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y\rangle = \langle\hat{J}_{1_y}\hat{J}_{1_z} + \hat{J}_{1_z}\hat{J}_{1_y}\rangle + \langle\hat{J}_{2_y}\hat{J}_{2_z} + \hat{J}_{2_z}\hat{J}_{2_y}\rangle + \langle\hat{J}_{1_y}\hat{J}_{2_z} + \hat{J}_{2_z}\hat{J}_{1_y}\rangle + \langle\hat{J}_{1_z}\hat{J}_{2_y} + \hat{J}_{2_y}\hat{J}_{1_z}\rangle. \quad (49)$$

It is to be noted here that though the operators of atom 1 commute with those of atom 2, we are not taking advantage of that as to keep the expressions symmetric with respect to the indices 1 and 2.

Using Eqs. (46) to (49) in Eq. (41) and (42), we get

$$\Delta J^2_{x'} = \sum_{i=1}^{2}[\langle\hat{J}^2_{i_x}\rangle\cos^2\theta\cos^2\phi + \langle\hat{J}^2_{i_y}\rangle\cos^2\theta\sin^2\phi + \langle\hat{J}^2_{i_z}\rangle\sin^2\theta] + \sum_{i=1}^{2}\sum_{\substack{l=1\\ l\neq i}}^{2}[\langle\hat{J}_{i_x}\hat{J}_{l_x}\rangle\cos^2\theta\cos^2\phi + \langle\hat{J}_{i_y}\hat{J}_{l_y}\rangle\cos^2\theta\sin^2\phi + \langle\hat{J}_{i_z}\hat{J}_{l_z}\rangle\sin^2\theta] + \frac{1}{2}\sum_{i=1}^{2}\sum_{l=1}^{2}[\langle\hat{J}_{i_x}\hat{J}_{l_y} + \hat{J}_{l_y}\hat{J}_{i_x}\rangle\cos^2\theta\sin 2\phi$$

$$-\langle \hat{J}_{i_x}\hat{J}_{l_z} + \hat{J}_{l_z}\hat{J}_{i_x}\rangle \sin 2\theta \cos\phi$$
$$-\langle \hat{J}_{i_y}\hat{J}_{l_z} + \hat{J}_{l_z}\hat{J}_{i_y}\rangle \sin 2\theta \sin\phi] \quad (50)$$

and

$$\Delta J_{y'}^2 = \sum_{i=1}^{2}[\langle \hat{J}_{i_x}^2\rangle \sin^2\phi + \langle \hat{J}_{i_y}^2\rangle \cos^2\phi] + \sum_{i=1}^{2}\sum_{\substack{l=1\\ l\neq i}}^{2}[\langle \hat{J}_{i_x}\hat{J}_{l_x}\rangle \sin^2\phi + \langle \hat{J}_{i_y}\hat{J}_{l_y}\rangle \cos^2\phi] - \frac{1}{2}\sum_{i=1}^{2}\sum_{l=1}^{2}\langle \hat{J}_{i_x}\hat{J}_{l_y} + \hat{J}_{l_y}\hat{J}_{i_x}\rangle \sin 2\phi. \quad (51)$$

Now, since $|\psi\rangle$ is symmetric under the exchange of two atoms and both the atoms have been treated on equal footing in the state $|\psi\rangle$, we have

$$\langle \hat{J}_{1_x}\rangle = \langle \hat{J}_{2_x}\rangle, \quad (52)$$
$$\langle \hat{J}_{1_y}\rangle = \langle \hat{J}_{2_y}\rangle, \quad (53)$$
$$\langle \hat{J}_{1_z}\rangle = \langle \hat{J}_{2_z}\rangle, \quad (54)$$
$$\langle \hat{J}_{1_x}\hat{J}_{2_y}\rangle = \langle \hat{J}_{1_y}\hat{J}_{2_x}\rangle, \quad (55)$$
$$\langle \hat{J}_{1_x}\hat{J}_{2_z}\rangle = \langle \hat{J}_{1_z}\hat{J}_{2_x}\rangle, \quad (56)$$
$$\langle \hat{J}_{1_y}\hat{J}_{2_z}\rangle = \langle \hat{J}_{1_z}\hat{J}_{2_y}\rangle, \quad (57)$$
$$\langle \hat{J}_{1_x}\hat{J}_{1_y} + \hat{J}_{1_y}\hat{J}_{1_x}\rangle = \langle \hat{J}_{2_x}\hat{J}_{2_y} + \hat{J}_{2_y}\hat{J}_{2_x}\rangle, \quad (58)$$
$$\langle \hat{J}_{1_x}\hat{J}_{1_z} + \hat{J}_{1_z}\hat{J}_{1_x}\rangle = \langle \hat{J}_{2_x}\hat{J}_{2_z} + \hat{J}_{2_z}\hat{J}_{2_x}\rangle, \quad (59)$$
$$\langle \hat{J}_{1_y}\hat{J}_{1_z} + \hat{J}_{1_z}\hat{J}_{1_y}\rangle = \langle \hat{J}_{2_y}\hat{J}_{2_z} + \hat{J}_{2_z}\hat{J}_{2_y}\rangle. \quad (60)$$

Therefore, using Eqs. (52), (53), and (54), we can reduce $\cos\theta$ and $\cos\phi$ given in Eqs. (35) and (36), respectively, as

$$\cos\theta = \frac{\langle \hat{J}_{1_z}\rangle}{|\langle \hat{\mathbf{J}}_1\rangle|}, \quad (61)$$

$$\cos\phi = \frac{\langle \hat{J}_{1_x}\rangle}{\sqrt{\langle \hat{J}_{1_x}\rangle^2 + \langle \hat{J}_{1_y}\rangle^2}}, \quad (62)$$

where

$$|\langle \hat{\mathbf{J}}_1\rangle| = \sqrt{\langle \hat{J}_{1_x}\rangle^2 + \langle \hat{J}_{1_y}\rangle^2 + \langle \hat{J}_{1_z}\rangle^2}. \quad (63)$$

Using Eqs. (32), (43), (61), and (62), it can be shown that

$$\langle \hat{J}_{i_{x'}}\rangle = \langle \hat{J}_{i_x}\rangle \cos\theta\cos\phi + \langle \hat{J}_{i_y}\rangle \cos\theta\sin\phi - \langle \hat{J}_{i_z}\rangle \sin\theta = 0. \quad (64)$$

Therefore, from Eqs. (19) and (32), we have

$$\Delta J_{i_{x'}}^2 = \langle \hat{J}_{i_{x'}}^2\rangle = \langle \hat{J}_{i_x}^2\rangle \cos^2\theta\cos^2\phi + \langle \hat{J}_{i_y}^2\rangle \cos^2\theta\sin^2\phi + \langle \hat{J}_{i_z}^2\rangle \sin^2\theta + \tfrac{1}{2}\langle \hat{J}_{i_x}\hat{J}_{i_y} + \hat{J}_{i_y}\hat{J}_{i_x}\rangle \cos^2\theta\sin 2\phi - \tfrac{1}{2}\langle \hat{J}_{i_x}\hat{J}_{i_z} + \hat{J}_{i_z}\hat{J}_{i_x}\rangle \sin 2\theta\cos\phi - \tfrac{1}{2}\langle \hat{J}_{i_y}\hat{J}_{i_z} + \hat{J}_{i_z}\hat{J}_{i_y}\rangle \sin 2\theta\sin\phi, \quad (65)$$

where, $i = 1,2$. Using the above equation we can write Eq. (50) as

$$\Delta J_{x'}^2 = \sum_{i=1}^{2}\Delta J_{i_{x'}}^2 + \sum_{i=1}^{2}\sum_{\substack{l=1\\ l\neq i}}^{2}[\langle \hat{J}_{i_x}\hat{J}_{l_x}\rangle \cos^2\theta\cos^2\phi + \langle \hat{J}_{i_y}\hat{J}_{l_y}\rangle \cos^2\theta\sin^2\phi + \langle \hat{J}_{i_z}\hat{J}_{l_z}\rangle \sin^2\theta + \langle \hat{J}_{i_x}\hat{J}_{l_y}\rangle \cos^2\theta\sin 2\phi - \langle \hat{J}_{i_x}\hat{J}_{l_z}\rangle \sin 2\theta\cos\phi - \langle \hat{J}_{i_y}\hat{J}_{l_z}\rangle \sin 2\theta\sin\phi]. \quad (66)$$

Similarly, it can be shown that

$$\Delta J_{i_{y'}}^2 = \langle \hat{J}_{i_{y'}}^2\rangle = \langle \hat{J}_{i_x}^2\rangle \sin^2\phi + \langle \hat{J}_{i_y}^2\rangle \cos^2\phi - \tfrac{1}{2}\langle \hat{J}_{i_x}\hat{J}_{i_y} + \hat{J}_{i_y}\hat{J}_{i_x}\rangle \sin 2\phi \quad (67)$$

and, hence, we have

$$\Delta J_{y'}^2 = \sum_{i=1}^{2}\Delta J_{i_{y'}}^2 + \sum_{i=1}^{2}\sum_{\substack{l=1\\ l\neq i}}^{2}[\langle \hat{J}_{i_x}\hat{J}_{l_x}\rangle \sin^2\phi + \langle \hat{J}_{i_y}\hat{J}_{l_y}\rangle \cos^2\phi - \langle \hat{J}_{i_x}\hat{J}_{l_y}\rangle \sin 2\phi]. \quad (68)$$

We now take advantage of the fact that the operators of atom 1 commute with those of atom 2 and obtain, using Eqs. (58) to (60) in Eq. (66),

$$\Delta J_{x'}^2 = \Delta J_{1_{x'}}^2 + \Delta J_{2_{x'}}^2 + 2\langle \hat{J}_{1_x}\hat{J}_{2_x}\rangle \cos^2\theta\cos^2\phi + 2\langle \hat{J}_{1_y}\hat{J}_{2_y}\rangle \cos^2\theta\sin^2\phi + 2\langle \hat{J}_{1_z}\hat{J}_{2_z}\rangle \sin^2\theta + 2\langle \hat{J}_{1_x}\hat{J}_{2_y}\rangle \cos^2\theta\sin 2\phi - 2\langle \hat{J}_{1_x}\hat{J}_{2_z}\rangle \sin 2\theta\cos\phi - 2\langle \hat{J}_{1_y}\hat{J}_{2_z}\rangle \sin 2\theta\sin\phi. \quad (69)$$

Thus, the quantum fluctuation $\Delta J_{x'}^2$ of a composite system of two two-level atoms is equal to the sum of the fluctuations $\Delta J_{1_{x'}}^2$ and $\Delta J_{2_{x'}}^2$ of the individual constituent atoms and the correlation terms $\langle \hat{J}_{1_{x,y,z}}\hat{J}_{2_{x,y,z}}\rangle$, $\langle \hat{J}_{1_x}\hat{J}_{2_y}\rangle$, $\langle \hat{J}_{1_x}\hat{J}_{2_z}\rangle$, and $\langle \hat{J}_{1_y}\hat{J}_{2_z}\rangle$, which depend upon the correlation among the two atoms.

In a similar fashion, we have

$$\Delta J_{y'}^2 = \Delta J_{1_{y'}}^2 + \Delta J_{2_{y'}}^2 + 2\langle \hat{J}_{1_y}\hat{J}_{2_y}\rangle \cos^2\phi + 2\langle \hat{J}_{1_x}\hat{J}_{2_x}\rangle \sin^2\phi - 2\langle \hat{J}_{1_x}\hat{J}_{2_y}\rangle \sin 2\phi, \quad (70)$$

where the last three terms represent the correlation among the two atoms.

Thus, we can see from Eqs. (69) and (70) that, by expressing the quantum fluctuations $\Delta J_{x'}^2$ and $\Delta J_{y'}^2$ of the composite system of two two-level atoms in the above way, we can separate out the correlation terms among the two atoms from their intrinsic quantum fluctuations. This helps to visualize and study solely the quantum correlations existing among the two atoms.

We now calculate $\Delta J_{1_{x',y'}}^2$ and $\Delta J_{2_{x',y'}}^2$ for the state $|\psi\rangle$. Using the expression of $|\psi\rangle$ given in Eq. (31), we get

$$\langle \hat{J}_{i_x}^2\rangle = \langle \hat{J}_{i_y}^2\rangle = \langle \hat{J}_{i_z}^2\rangle = \tfrac{1}{4} \quad (71)$$

and

$$\langle \hat{J}_{i_x}\hat{J}_{i_y} + \hat{J}_{i_y}\hat{J}_{i_x}\rangle = \langle \hat{J}_{i_x}\hat{J}_{i_z} + \hat{J}_{i_z}\hat{J}_{i_x}\rangle = \langle \hat{J}_{i_y}\hat{J}_{i_z} + \hat{J}_{i_z}\hat{J}_{i_y}\rangle = 0, \quad (72)$$

where $i = 1,2$. Therefore, using these equations and also Eqs. (61) and (62) in Eq. (65), we get

$$\Delta J^2_{i_{x'}} = \Delta J^2_{i_{y'}} = \tfrac{1}{4}. \tag{73}$$

Now, using the expressions of $\cos\theta$ and $\cos\phi$ given in Eqs. (61) and (62), respectively, and also using Eq. (73) in Eq. (69), we obtain

$$\begin{aligned}\Delta J^2_{x'} &= \frac{1}{4} + \frac{1}{4} + \frac{2\langle \hat{J}_{1_z}\rangle^2}{|\langle \hat{\mathbf{J}}_1\rangle|^2\left(\langle \hat{J}_{1_x}\rangle^2 + \langle \hat{J}_{1_y}\rangle^2\right)} \\ &\quad \times \left[\langle \hat{J}_{1_x}\hat{J}_{2_x}\rangle\langle \hat{J}_{1_x}\rangle^2 + 2\langle \hat{J}_{1_x}\hat{J}_{2_y}\rangle\langle \hat{J}_{1_x}\rangle\langle \hat{J}_{1_y}\rangle \right. \\ &\quad \left. + \langle \hat{J}_{1_y}\hat{J}_{2_y}\rangle\langle \hat{J}_{1_y}\rangle^2\right] + \frac{2\langle \hat{J}_{1_z}\hat{J}_{2_z}\rangle}{|\langle \hat{\mathbf{J}}_1\rangle|^2}\left[\langle \hat{J}_{1_x}\rangle^2 + \langle \hat{J}_{1_y}\rangle^2\right] \\ &\quad - \frac{4\langle \hat{J}_{1_z}\rangle}{|\langle \hat{\mathbf{J}}_1\rangle|^2}\left[\langle \hat{J}_{1_x}\hat{J}_{2_z}\rangle\langle \hat{J}_{1_x}\rangle + \langle \hat{J}_{1_y}\hat{J}_{2_z}\rangle\langle \hat{J}_{1_y}\rangle\right] \qquad (74) \\ &= \frac{1}{2} + \mathcal{C}x, \qquad (75)\end{aligned}$$

where $\mathcal{C}x$ is the sum of last seven terms in Eq. (74). It represents the quantum correlation existing among the two atoms.

Similarly, using Eqs. (61), (62), and (73) in Eq. (70), we get

$$\begin{aligned}\Delta J^2_{y'} &= \frac{1}{4} + \frac{1}{4} + \frac{2}{\left(\langle \hat{J}_{1_x}\rangle^2 + \langle \hat{J}_{1_y}\rangle^2\right)}\left[\langle \hat{J}_{1_x}\hat{J}_{2_x}\rangle\langle \hat{J}_{1_y}\rangle^2 \right. \\ &\quad \left. + \langle \hat{J}_{1_y}\hat{J}_{2_y}\rangle\langle \hat{J}_{1_x}\rangle^2 - 2\langle \hat{J}_{1_x}\hat{J}_{2_y}\rangle\langle \hat{J}_{1_x}\rangle\langle \hat{J}_{1_y}\rangle\right] \qquad (76) \\ &= \frac{1}{2} + \mathcal{C}y, \qquad (77)\end{aligned}$$

where $\mathcal{C}y$ represents the correlation among the two atoms.

We now see what happens to the correlation terms $\mathcal{C}x$ and $\mathcal{C}y$ for an unentangled state. As mentioned in the earlier section, an unentangled state $|\psi\rangle$ of the composite system of two atoms can be expressed as a direct product of the individual atomic state vectors of the two constituent atoms as

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle, \tag{78}$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are the atomic state vectors corresponding to the two constituent atoms. Now, it is easy to see that for these kinds of states the following conditions are satisfied.

$$\begin{aligned}&\langle \hat{J}_{1_x}\hat{J}_{2_x}\rangle = \langle \hat{J}_{1_x}\rangle\langle \hat{J}_{2_x}\rangle, \quad \langle \hat{J}_{1_y}\hat{J}_{2_y}\rangle = \langle \hat{J}_{1_y}\rangle\langle \hat{J}_{2_y}\rangle, \\ &\langle \hat{J}_{1_z}\hat{J}_{2_z}\rangle = \langle \hat{J}_{1_z}\rangle\langle \hat{J}_{2_z}\rangle, \quad \langle \hat{J}_{1_x}\hat{J}_{2_y}\rangle = \langle \hat{J}_{1_x}\rangle\langle \hat{J}_{2_y}\rangle, \qquad (79) \\ &\langle \hat{J}_{1_x}\hat{J}_{2_z}\rangle = \langle \hat{J}_{1_x}\rangle\langle \hat{J}_{2_z}\rangle, \quad \langle \hat{J}_{1_y}\hat{J}_{2_z}\rangle = \langle \hat{J}_{1_y}\rangle\langle \hat{J}_{2_z}\rangle.\end{aligned}$$

Hence, using the above equations and also Eqs. (52), (53), and (54) in the expressions of $\mathcal{C}x$ and $\mathcal{C}y$ as given in Eqs. (74) and (76), respectively, we get

$$\mathcal{C}x = \mathcal{C}y = 0. \tag{80}$$

Thus, for an unentangled state $\mathcal{C}x$ and $\mathcal{C}y$ are zero, and we have

$$\Delta J^2_{x',y'}\big|_{\text{un-ent}} = \Delta J^2_{1_{x',y'}} + \Delta J^2_{2_{x',y'}} = \tfrac{1}{2}. \tag{81}$$

That is, the quantum fluctuations of the composite state is just the algebraic sum of the corresponding fluctuations of the individual constituent atoms.

The terms $\mathcal{C}x$ and $\mathcal{C}y$ are nonzero when the atomic state vector is entangled. We can give a physical interpretation of $\mathcal{C}x$ and $\mathcal{C}y$ in the following way. If $\Delta J^2_{x'}$ and $\Delta J^2_{y'}$ are the fluctuations of an entangled state, then using Eqs. (75), (77), and (81), we can write

$$\begin{aligned}\mathcal{C}x &= \Delta J^2_{x'} - \tfrac{1}{2} \qquad (82) \\ &= \Delta J^2_{x'} - \Delta J^2_{x'}\big|_{\text{un-ent}}, \qquad (83) \\ \mathcal{C}y &= \Delta J^2_{y'} - \tfrac{1}{2} \qquad (84) \\ &= \Delta J^2_{y'} - \Delta J^2_{y'}\big|_{\text{un-ent}}. \qquad (85)\end{aligned}$$

Thus, $\mathcal{C}x$ and $\mathcal{C}y$ are the measures of the deviations of the quantum fluctuations of an entangled state from those of an unentangled one. Whenever $\mathcal{C}x$ and $\mathcal{C}y$ for a quantum state of a composite system are nonzero, we can conclude that the corresponding quantum state is an entangled state. We can construct a parameter out of $\mathcal{C}x$ and $\mathcal{C}y$ for the detection and quantification of quantum entanglement. Since $\mathcal{C}x$ and $\mathcal{C}y$ may have opposite signs, and also to treat both of them on equal footing, we construct a parameter S as,

$$S = \tfrac{1}{2}(\mathcal{C}x^2 + \mathcal{C}y^2), \tag{86}$$

such that the nonzero value of S implies the presence of quantum entanglement in the corresponding system. We call S the quantum entanglement parameter. S is the mean squared deviation of the quantum fluctuations in the two quadratures (x' and y') of a quantum mechanically entangled state from those of an unentangled one.

Thus, whenever we have

$$S = 0, \tag{87}$$

the corresponding quantum state is unentangled, and whenever we have

$$S > 0, \tag{88}$$

the corresponding quantum state is entangled. The condition $S > 0$ is the necessary and sufficient condition for the presence of quantum entanglement. We can prove it in this way. We know that whenever a quantum state for a composite system of two two-level atoms is entangled, the corresponding quantum state vector cannot be written as a direct product of the individual atomic state vectors. In that case, the conditions in Eq. (79) are not satisfied and hence $\mathcal{C}x$ and $\mathcal{C}y$ are nonzero, implying that $S > 0$. This shows that the condition $S > 0$ forms the necessary condition for the presence of quantum entanglement. We now prove that the condition is sufficient also for the presence of entanglement in this way. Whenever $S > 0$, either $\mathcal{C}x$ or $\mathcal{C}y$ or both of them are nonzero. This means that all the conditions in Eq. (79) are not satisfied, implying that the corresponding quantum state vector cannot be expressed as a direct product of the individual atomic state vectors, and, hence, the quantum state is entangled. Thus, we have proved that the condition $S > 0$ forms the necessary and sufficient condition for the presence of quantum entanglement.

We can see from Eqs. (75) that, if $\mathcal{C}x < 0$, then

$$\Delta J^2_{x'} < \tfrac{1}{2}, \tag{89}$$

and, hence, the quantum state $|\psi\rangle$ is a spin squeezed state having squeezing in the x' quadrature. We see from Eq. (77) that, at the same time we should have $\mathcal{C}y > 0$ (no squeezing in the y' quadrature), so that

$$\Delta J_{y'}^2 > \tfrac{1}{2}, \tag{90}$$

and Heisenberg's uncertainty principle [Eq. (24)] is restored. Similarly, when $\mathcal{C}y < 0$, there is squeezing in the y' quadrature and no squeezing in the x' one.

Now, we see from the above discussion that when we have spin squeezing in a quantum state, either $\mathcal{C}x$ or $\mathcal{C}y$ is less than zero and hence $S > 0$, implying the presence of quantum entanglement. Thus, whenever there is spin squeezing there is quantum entanglement. But the reverse is not true. It may happen that some quantum state does not show spin squeezing at all, that is, $\mathcal{C}x$ and $\mathcal{C}y$ are never less than zero and instead they are always greater than zero. In that case, we have $S > 0$, implying the presence of quantum entanglement. This shows that a quantum state that is not spin squeezed may show quantum entanglement.

For the purpose of quantification of entanglement, we notice that since S is the mean squared deviation of the quantum fluctuations in the two quadratures (x' and y') of an entangled state from the corresponding fluctuations of an unentangled one, we can take S itself to be proportional to the amount of quantum entanglement in a system. As S for a system increases, the entanglement of the system also increases. As S decreases, the entanglement also decreases. Thus, the value of S itself can be a measure of entanglement in a system. The question now arises about how to measure S. In the next section, we establish a connection between S and experimentally measurable quantities.

III. RELATIONSHIP BETWEEN THE QUANTUM ENTANGLEMENT PARAMETER *S* AND THE EXPERIMENTALLY MEASURABLE QUANTITIES

In this section, we show how we can measure the qunatum entanglement parameter S. We rewrite Eqs. (75) and (77) as

$$\mathcal{C}x = \Delta J_{x'}^2 - \tfrac{1}{2}, \tag{91}$$

$$\mathcal{C}y = \Delta J_{y'}^2 - \tfrac{1}{2}. \tag{92}$$

Therefore,

$$\mathcal{C}x^2 = \Delta J_{x'}^4 - \Delta J_{x'}^2 + \tfrac{1}{4}, \tag{93}$$

$$\mathcal{C}y^2 = \Delta J_{y'}^4 - \Delta J_{y'}^2 + \tfrac{1}{4}. \tag{94}$$

Hence,

$$S = \tfrac{1}{2}\big[\Delta J_{x'}^2\big(\Delta J_{x'}^2 - 1\big) + \Delta J_{y'}^2\big(\Delta J_{y'}^2 - 1\big) + \tfrac{1}{2}\big]. \tag{95}$$

Multiplying and dividing $\Delta J_{x'}^2$ and $\Delta J_{y'}^2$ in the above expression by $2/j$ ($j = \frac{N}{2} = 1$), we get

$$S = \frac{1}{2}\bigg[\frac{2\Delta J_{x'}^2 j}{2j}\bigg(\frac{2\Delta J_{x'}^2 j}{2j} - 1\bigg) + \frac{2\Delta J_{y'}^2 j}{2j}\bigg(\frac{2\Delta J_{y'}^2 j}{2j} - 1\bigg) + \frac{1}{2}\bigg] \tag{96}$$

$$= \frac{1}{2}\bigg[\frac{Q_x{}^2 j}{2}\bigg(\frac{Q_x{}^2 j}{2} - 1\bigg) + \frac{Q_y{}^2 j}{2}\bigg(\frac{Q_y{}^2 j}{2} - 1\bigg) + \frac{1}{2}\bigg], \tag{97}$$

where Q_x and Q_y are the spin squeezing parameters introduced in Sec. I. Since Q_x and Q_y are experimentally measurable quantities, the parameter S gets connected directly with the experiment. We can obtain numerical values of S by measuring Q_x and Q_y by experiment and using the above formula. Thus, we can measure S for a system experimentally.

If we now multiply and divide Q_x^2 and Q_y^2 in Eq. (97) by $j/|\langle\hat{\mathbf{J}}\rangle|^2$, we get

$$\begin{aligned} S &= \frac{1}{2}\bigg[\frac{Q_x{}^2 j^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j|\langle\hat{\mathbf{J}}\rangle|^2}\bigg(\frac{Q_x{}^2 j^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j|\langle\hat{\mathbf{J}}\rangle|^2} - 1\bigg) + \frac{Q_y{}^2 j^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j|\langle\hat{\mathbf{J}}\rangle|^2}\bigg(\frac{Q_y{}^2 j^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j|\langle\hat{\mathbf{J}}\rangle|^2} - 1\bigg) + \frac{1}{2}\bigg] \\ &= \frac{1}{2}\bigg[\frac{\xi_{R_x}^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j}\bigg(\frac{\xi_{R_x}^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j} - 1\bigg) + \frac{\xi_{R_y}^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j}\bigg(\frac{\xi_{R_y}^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j} - 1\bigg) + \frac{1}{2}\bigg], \end{aligned} \tag{98}$$

where

$$\xi_{R_x} = \frac{j}{|\langle\hat{\mathbf{J}}\rangle|}Q_x, \tag{99}$$

$$\xi_{R_y} = \frac{j}{|\langle\hat{\mathbf{J}}\rangle|}Q_y \tag{100}$$

are called the spectroscopic squeezing parameters used in the context of Ramsey spectroscopy [15]. Thus, the quantum entanglement parameter S gets connected with the spectroscopic squeezing parameters, which are experimentally measurable.

In the next section, we extend these ideas to systems containing N number of two-level atoms.

IV. QUANTUM ENTANGLEMENT IN A SYSTEM OF *N* TWO-LEVEL ATOMS

An arbitrary symmetric pure state for a system of N two-level atoms in the $\{m_1,m_2,m_3,\ldots,m_N\}$ representation is given as

$$\begin{aligned} |\Psi\rangle = {} & G_1\bigg|\frac{1}{2},\frac{1}{2},\ldots.\frac{1}{2}\bigg\rangle + \frac{G_2}{\sqrt{{}^N C_1}}\bigg[\bigg|-\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots.\frac{1}{2}\bigg\rangle + \bigg|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\ldots.\frac{1}{2}\bigg\rangle + \ldots. + \bigg|\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots.-\frac{1}{2}\bigg\rangle\bigg] \\ & + \frac{G_3}{\sqrt{{}^N C_2}}\bigg[\bigg|-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\ldots.\frac{1}{2}\bigg\rangle + \bigg|-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\ldots.\frac{1}{2}\bigg\rangle + \ldots. + \bigg|\frac{1}{2},\frac{1}{2},\ldots.-\frac{1}{2},-\frac{1}{2}\bigg\rangle\bigg] \\ & + \ldots\ldots\ldots. + G_{N+1}\bigg|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\ldots.-\frac{1}{2}\bigg\rangle, \end{aligned} \tag{101}$$

where $G_1, G_2,..., G_{N+1}$ are constants and ${}^N C_r$ is given as

$$ {}^N C_r = \frac{N!}{r!(N-r)!}. \tag{102} $$

The quantum fluctuations $\Delta J_{x'}^2$ and $\Delta J_{y'}^2$ for this system can be written in analogy to Eqs. (66) and (68) as

$$ \begin{aligned} \Delta J_{x'}^2 = \sum_{i=1}^{N} \Delta J_{i_{x'}}^2 + \sum_{i=1}^{N}\sum_{\substack{l=1\\ l\neq i}}^{N} [\langle \hat{J}_{i_x}\hat{J}_{l_x}\rangle \cos^2\theta\cos^2\phi \\ + \langle \hat{J}_{i_y}\hat{J}_{l_y}\rangle \cos^2\theta\sin^2\phi + \langle \hat{J}_{i_z}\hat{J}_{l_z}\rangle \sin^2\theta \\ + \langle \hat{J}_{i_x}\hat{J}_{l_y}\rangle \cos^2\theta\sin 2\phi - \langle \hat{J}_{i_x}\hat{J}_{l_z}\rangle \sin 2\theta\cos\phi \\ - \langle \hat{J}_{i_y}\hat{J}_{l_z}\rangle \sin 2\theta\sin\phi] \end{aligned} \tag{103} $$

and

$$ \begin{aligned} \Delta J_{y'}^2 = \sum_{i=1}^{N} \Delta J_{i_{y'}}^2 + \sum_{i=1}^{N}\sum_{\substack{l=1\\ l\neq i}}^{N} [\langle \hat{J}_{i_x}\hat{J}_{l_x}\rangle \sin^2\phi \\ + \langle \hat{J}_{i_y}\hat{J}_{l_y}\rangle \cos^2\phi - \langle \hat{J}_{i_x}\hat{J}_{l_y}\rangle \sin 2\phi], \end{aligned} \tag{104} $$

where the upper index 2 in the summations in Eqs. (66) and (68) has been replaced by N.

Now, since the state $|\Psi\rangle$ is symmetric under the exchange of any two atoms and all the atoms have been treated on equal footing, we have for the state $|\Psi\rangle$,

$$ \langle \hat{J}_{1_x}\rangle = \langle \hat{J}_{2_x}\rangle = ... = \langle \hat{J}_{N_x}\rangle, \tag{105} $$
$$ \langle \hat{J}_{1_y}\rangle = \langle \hat{J}_{2_y}\rangle = ... = \langle \hat{J}_{N_y}\rangle, \tag{106} $$
$$ \langle \hat{J}_{1_z}\rangle = \langle \hat{J}_{2_z}\rangle = ... = \langle \hat{J}_{N_z}\rangle, \tag{107} $$
$$ \langle \hat{J}_{1_x}\hat{J}_{2_x}\rangle = \langle \hat{J}_{1_x}\hat{J}_{3_x}\rangle = ... = \langle \hat{J}_{N-1_x}\hat{J}_{N_x}\rangle, \tag{108} $$
$$ \langle \hat{J}_{1_y}\hat{J}_{2_y}\rangle = \langle \hat{J}_{1_y}\hat{J}_{3_y}\rangle = ... = \langle \hat{J}_{N-1_y}\hat{J}_{N_y}\rangle, \tag{109} $$
$$ \langle \hat{J}_{1_z}\hat{J}_{2_z}\rangle = \langle \hat{J}_{1_z}\hat{J}_{3_z}\rangle = ... = \langle \hat{J}_{N-1_z}\hat{J}_{N_z}\rangle, \tag{110} $$
$$ \langle \hat{J}_{1_x}\hat{J}_{2_z}\rangle = \langle \hat{J}_{1_x}\hat{J}_{3_z}\rangle = ... = \langle \hat{J}_{N-1_x}\hat{J}_{N_z}\rangle, \tag{111} $$
$$ \langle \hat{J}_{1_x}\hat{J}_{2_y}\rangle = \langle \hat{J}_{1_x}\hat{J}_{3_y}\rangle = ... = \langle \hat{J}_{N-1_x}\hat{J}_{N_y}\rangle, \tag{112} $$
$$ \langle \hat{J}_{1_y}\hat{J}_{2_z}\rangle = \langle \hat{J}_{1_y}\hat{J}_{3_z}\rangle = ... = \langle \hat{J}_{N-1_y}\hat{J}_{N_z}\rangle, \tag{113} $$

and, also,

$$ \Delta J_{1_{x'}}^2 = \Delta J_{2_{x'}}^2 = ... = \Delta J_{N_{x'}}^2 = \tfrac{1}{4}, \tag{114} $$
$$ \Delta J_{1_{y'}}^2 = \Delta J_{2_{y'}}^2 = ... = \Delta J_{N_{y'}}^2 = \tfrac{1}{4}. \tag{115} $$

Therefore, using the above equations we can reduce Eqs. (103) and (104) as

$$ \begin{aligned} \Delta J_{x'}^2 = N\Delta J_{1_{x'}}^2 + 2({}^N C_2)[\langle \hat{J}_{1_x}\hat{J}_{2_x}\rangle \cos^2\theta\cos^2\phi \\ + \langle \hat{J}_{1_y}\hat{J}_{2_y}\rangle \cos^2\theta\sin^2\phi + \langle \hat{J}_{1_z}\hat{J}_{2_z}\rangle \sin^2\theta \\ + \langle \hat{J}_{1_x}\hat{J}_{2_y}\rangle \cos^2\theta\sin 2\phi - \langle \hat{J}_{1_x}\hat{J}_{2_z}\rangle \sin 2\theta\cos\phi \\ - \langle \hat{J}_{1_y}\hat{J}_{2_z}\rangle \sin 2\theta\sin\phi] \end{aligned} \tag{116} $$

and

$$ \begin{aligned} \Delta J_{y'}^2 = N\Delta J_{1_{y'}}^2 + 2({}^N C_2)[\langle \hat{J}_{1_x}\hat{J}_{2_x}\rangle \sin^2\phi \\ + \langle \hat{J}_{1_y}\hat{J}_{2_y}\rangle \cos^2\phi - \langle \hat{J}_{1_x}\hat{J}_{2_y}\rangle \sin 2\phi], \end{aligned} \tag{117} $$

respectively. Now, according to Eqs. (5), (105), (106), and (107), we have

$$ \langle \hat{J}_x\rangle = N\langle \hat{J}_{1_x}\rangle, \tag{118} $$
$$ \langle \hat{J}_y\rangle = N\langle \hat{J}_{1_y}\rangle, \tag{119} $$
$$ \langle \hat{J}_z\rangle = N\langle \hat{J}_{1_z}\rangle. \tag{120} $$

Therefore, using the above three equations and Eqs. (35) and (36), we observe that the expressions of $\cos\theta$ and $\cos\phi$, in this case, have the same forms as given in Eqs. (61) and (62), respectively. Hence, using the expressions of $\cos\theta$, $\cos\phi$, $\sin\theta$, and $\sin\phi$ obtained from Eqs. (61) and (62), and also the expressions of $\Delta J_{1_{x',y'}}^2$ given in Eq. (114) and (115), in Eqs. (116) and (117), we get

$$ \begin{aligned} \Delta J_{x'}^2 = \frac{N}{4} + \frac{2({}^N C_2)\langle \hat{J}_{1_z}\rangle^2}{|\langle \hat{\mathbf{J}}_1\rangle|^2(\langle \hat{J}_{1_x}\rangle^2 + \langle \hat{J}_{1_y}\rangle^2)} \\ \times [\langle \hat{J}_{1_x}\hat{J}_{2_x}\rangle\langle \hat{J}_{1_x}\rangle^2 + 2\langle \hat{J}_{1_x}\hat{J}_{2_y}\rangle\langle \hat{J}_{1_x}\rangle\langle \hat{J}_{1_y}\rangle \\ + \langle \hat{J}_{1_y}\hat{J}_{2_y}\rangle\langle \hat{J}_{1_y}\rangle^2] + \frac{2({}^N C_2)\langle \hat{J}_{1_z}\hat{J}_{2_z}\rangle}{|\langle \hat{\mathbf{J}}_1\rangle|^2}[\langle \hat{J}_{1_x}\rangle^2 + \langle \hat{J}_{1_y}\rangle^2] \\ - \frac{4({}^N C_2)\langle \hat{J}_{1_z}\rangle}{|\langle \hat{\mathbf{J}}_1\rangle|^2}[\langle \hat{J}_{1_x}\hat{J}_{2_z}\rangle\langle \hat{J}_{1_x}\rangle + \langle \hat{J}_{1_y}\hat{J}_{2_z}\rangle\langle \hat{J}_{1_y}\rangle] \end{aligned} \tag{121} $$

$$ = \frac{N}{4} + \mathfrak{C}\mathfrak{x} \tag{122} $$

and

$$ \begin{aligned} \Delta J_{y'}^2 = \frac{N}{4} + \frac{2({}^N C_2)}{(\langle \hat{J}_{1_x}\rangle^2 + \langle \hat{J}_{1_y}\rangle^2)}[\langle \hat{J}_{1_x}\hat{J}_{2_x}\rangle\langle \hat{J}_{1_y}\rangle^2 \\ + \langle \hat{J}_{1_y}\hat{J}_{2_y}\rangle\langle \hat{J}_{1_x}\rangle^2 - 2\langle \hat{J}_{1_x}\hat{J}_{2_y}\rangle\langle \hat{J}_{1_x}\rangle\langle \hat{J}_{1_y}\rangle] \end{aligned} \tag{123} $$

$$ = \frac{N}{4} + \mathfrak{C}\mathfrak{y}, \tag{124} $$

respectively.

Here $\mathfrak{C}\mathfrak{x}$ is the sum of last seven terms on the right-hand side of Eq. (121) and $\mathfrak{C}\mathfrak{y}$ is the sum of last three terms on the right-hand side of Eq. (123), respectively. Thus, we observe from Eqs. (121) and (123) that the quantum fluctuations $\Delta J_{x'}^2$ and $\Delta J_{y'}^2$ of a system of N two-level atoms in an arbitrary symmetric pure state can be obtained by finding out the quatum fluctuations $\Delta J_{1_{x'}}^2$ and $\Delta J_{1_{y'}}^2$ of any single atom and the correlations among any two atoms only in the assembly. If the quantum state $|\Psi\rangle$ of the composite system is unentangled, then $|\Psi\rangle$ can be written as a direct product of the N individual atomic state vectors. In this case, the conditions like those expressed in Eq. (79) are satisfied and it can be shown that $\mathfrak{C}\mathfrak{x}$ and $\mathfrak{C}\mathfrak{y}$ are zero. Therefore, we have,

$$ \Delta J_{x'}^2|_{\text{un-ent}} = N\Delta J_{1_{x'}}^2 = \frac{N}{4}, \tag{125} $$
$$ \Delta J_{y'}^2|_{\text{un-ent}} = N\Delta J_{1_{y'}}^2 = \frac{N}{4}. \tag{126} $$

Thus, $\Delta J_{x'}^2$ and $\Delta J_{y'}^2$ are just the algebraic sum of the quantum fluctuations $\Delta J_{i_{x'}}^2$ and $\Delta J_{i_{y'}}^2$ $(i = 1,2,3,....,N)$, respectively, of all the N individual constituent atoms. If the quantum state of the composite system is entangled, the conditions like those given in Eq. (79) are not satisfied and, hence, $\mathfrak{C}\mathfrak{x}$

and $\mathfrak{C}\mathfrak{y}$ are nonzero. We see that here also $\mathfrak{C}\mathfrak{x}$ and $\mathfrak{C}\mathfrak{y}$ are the measures of the deviations of the quantum fluctuations $\Delta J_{x'}^2$ and $\Delta J_{y'}^2$ of an entangled state from those of an unentangled one. As mentioned in Sec. II, we also construct the quantum entanglement parameter S as the mean squared deviation of the quantum fluctuations in the two quadratures (x' and y') of an entangled state from the corresponding fluctuations of an unentangled one. According to Eqs. (86), (122), and (124), we have

$$\begin{aligned} S &= \frac{1}{2}[(\mathfrak{C}\mathfrak{x})^2 + (\mathfrak{C}\mathfrak{y})^2] \\ &= \frac{1}{2}\bigg[\Delta J_{x'}{}^4 - \frac{N}{2}\Delta J_{x'}{}^2 \\ &\quad + \Delta J_{y'}{}^4 - \frac{N}{2}\Delta J_{y'}{}^2 + \frac{N^2}{8}\bigg] \\ &= \frac{1}{2}\bigg[\Delta J_{x'}{}^2\left(\Delta J_{x'}{}^2 - \frac{N}{2}\right) \\ &\quad + \Delta J_{y'}{}^2\left(\Delta J_{y'}{}^2 - \frac{N}{2}\right) + \frac{N^2}{8}\bigg]. \end{aligned} \tag{127}$$

The necessary and sufficient condition for the presence of quantum entanglement in this system of N two-level atoms is $S > 0$. The proof is as follows. If the composite state vector $|\Psi\rangle$ is entangled, it cannot be expressed as a direct product of the N individual atomic state vectors. Then the conditions like those given in Eqs. (79) are not satisfied and, hence, either $\mathfrak{C}\mathfrak{x}$ or $\mathfrak{C}\mathfrak{y}$ or both of them are nonzero, implying that $S > 0$. Thus, $S > 0$ forms the necessary condition for the presence of entanglement. We now show that the condition is sufficient also. If $S > 0$, either $\mathfrak{C}\mathfrak{x}$ or $\mathfrak{C}\mathfrak{y}$ or both of them are nonzero, implying that the conditions like those expressed in Eq. (79) are not all satisfied and, hence, the corresponding quantum state is not expressible as a direct product of the N individual atomic state vectors, implying that the composite state is entangled. Thus, the condition $S > 0$ forms the sufficient condition for the presence of entanglement.

As in Sec. III, we now relate S with experimentally measurable quantities. Multiplying and dividing $\Delta J_{x'}^2$ and $\Delta J_{y'}^2$ in Eq. (127) by $2/j$, we get

$$\begin{aligned} S &= \frac{1}{2}\bigg[\frac{2\Delta J_{x'}^2 j}{2j}\left(\frac{2\Delta J_{x'}^2 j}{2j} - \frac{N}{2}\right) \\ &\quad + \frac{2\Delta J_{y'}^2 j}{2j}\left(\frac{2\Delta J_{y'}^2 j}{2j} - \frac{N}{2}\right) + \frac{N^2}{8}\bigg] \end{aligned} \tag{128}$$

$$\begin{aligned} &= \frac{1}{2}\bigg[\frac{Q_x{}^2 j}{2}\left(\frac{Q_x{}^2 j}{2} - \frac{N}{2}\right) \\ &\quad + \frac{Q_y{}^2 j}{2}\left(\frac{Q_y{}^2 j}{2} - \frac{N}{2}\right) + \frac{N^2}{8}\bigg], \end{aligned} \tag{129}$$

where Q_x and Q_y are the spin squeezing parameters introduced in Eqs. (20) and (21), respectively, in Sec. I. Thus, for a system of N two-level atoms, we can measure the quantum entanglement parameter by measuring the spin squeezing parameters Q_x and Q_y of the system.

If we now multiply and divide Q_x^2 and Q_y^2 in Eq. (129) by $j/|\langle\hat{\mathbf{J}}\rangle|^2$, we get

$$\begin{aligned} S &= \frac{1}{2}\bigg[\frac{Q_x{}^2 j^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j|\langle\hat{\mathbf{J}}\rangle|^2}\left(\frac{Q_x{}^2 j^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j|\langle\hat{\mathbf{J}}\rangle|^2} - \frac{N}{2}\right) \\ &\quad + \frac{Q_y{}^2 j^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j|\langle\hat{\mathbf{J}}\rangle|^2}\left(\frac{Q_y{}^2 j^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j|\langle\hat{\mathbf{J}}\rangle|^2} - \frac{N}{2}\right) + \frac{N^2}{8}\bigg] \\ &= \frac{1}{2}\bigg[\frac{\xi_{R_x}^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j}\left(\frac{\xi_{R_x}^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j} - \frac{N}{2}\right) \\ &\quad + \frac{\xi_{R_y}^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j}\left(\frac{\xi_{R_y}^2|\langle\hat{\mathbf{J}}\rangle|^2}{2j} - \frac{N}{2}\right) + \frac{N^2}{8}\bigg], \end{aligned} \tag{130}$$

where ξ_{R_x} and ξ_{R_y} are the spectroscopic squeezing parameters [15] introduced in Sec. III. Thus, we relate the quantum entanglement parameter S of a system of N two-level atoms with the experimentally measurable squeezing parameters used in the Ramsey spectroscopy.

V. SUMMARY AND CONCLUSION

We proposed the necessary and sufficient condition for the presence of quantum entanglement in arbitrary symmetric pure states of two-level atoms. We took the quantum fluctuations of the system in terms of the components of the pseudo-spin vector operator $\hat{\mathbf{J}}$ in two mutually orthogonal directions in a plane perpendicular to the mean pseudo-spin vector $\langle\hat{\mathbf{J}}\rangle$. We then expressed these fluctuations as an algebraic sum of the fluctuations of the individual constituent atoms and their correlation terms. We took these correlation terms in the two mutually orthogonal directions and in a plane perpendicular to $\langle\hat{\mathbf{J}}\rangle$ to construct a parameter S called the quantum entanglement parameter. We showed that this parameter can be used to detect and quantify quantum entanglement. The necessary and sufficient condition for the presence of quantum entanglement in such systems is $S > 0$. If a quantum state of the system is unentangled, we have $S = 0$. We also said that since S is the mean squared deviation of the quantum fluctuations of an entangled state from the corresponding fluctuations of an unentangled one, the numerical value of S can be taken as a measure of quantum entanglement in the system. We first made all these studies in case of two two-level atoms and then extended these ideas in case of systems containing N number of such atoms. We also established the relationship between the quantum entanglement parameter S and spin squeezing and spectroscopic squeezing parameters. This shows how we can measure S experimentally. We hope that our study may produce deeper insight into the subject.

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