Distance between quantum states in the presence of initial qubit-environment correlations: A comparative study

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The time evolution of the *trace distance* between two states of an open quantum system may increase due to initial system-environment correlations, thus exhibiting a breakdown of distance contractivity of the reduced dynamics. We analyze how the time evolution of the distance depends on the chosen distance measure. Here we elucidate the behavior of the trace distance, the Hilbert-Schmidt distance, the Bures distance, the Hellinger distance, and the quantum Jensen-Shannon divergence for two system-environment setups, namely a qubit bilinearly coupled to an infinite and a finite-size environment with the latter composed of harmonic oscillators.

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I. INTRODUCTION

Quantum states represented by density matrices ρ can be determined by quantum state tomography and compared using various quantifiers. Distances and other similarity measures provide a quantitative method to evaluate how close two states are together or how precisely a quantum channel can transmit information. Unfortunately, there is no single, ideal measure of distinguishability of different states. There are no criteria for the distance measure to be "better than another." Even the natural requirement that the distance between states should have properties of a metric (i.e., identity of indiscernibles, symmetry, and the triangle inequality) is relaxed in a case of *fidelity* which is a celebrated statistical similarity measure. Loosely speaking, two states are close to each other if the distance is small. We also expect that two different distances are equivalent if any two states that become closer to one another in the sense of one distance measure also become closer in the sense of the second, and vice versa.

There are diverse ways of introducing a notion of distance between two quantum states [1]. Examples of such distance measures comprise the trace distance, Hilbert-Schmidt distance, Bures distance, Hellinger distance, and Jensen-Shannon divergence, to mention a few, see also Refs. [1–5]. These metrics possess distinct properties like being Riemannian, monotone (contractive), with bounds and relations among them [6].

Let us recall that any positive and trace-preserving map $\mathcal{E}$ defined on the whole space of operators ρ on the Hilbert space is contractive with respect to a given distance $D[\rho_1,\rho_2]$ if

$$D[\mathcal{E}(\rho_1),\mathcal{E}(\rho_2)] \leqslant D[\rho_1,\rho_2]. \quad (1)$$

In particular, when $\mathcal{E} = \mathcal{E}_t$ is a completely positive quantum dynamical semigroup such that $\rho(t) = \mathcal{E}_t\rho(0)$, then contractivity means that

$$D[\rho_1(t),\rho_2(t)] \leqslant D[\rho_1(s),\rho_2(s)], \quad \text{for} \quad t > s. \quad (2)$$

As a consequence, the distance cannot increase in time and the distinguishability of any states cannot increase above an initial value. In particular, if a quantum open system and its environment are initially prepared in an uncorrelated state, the reduced dynamics is completely positive and hence contractive with respect to some metrics. In consequence, the distance $D[\rho_1,\rho_2]$ between two states can tend to zero when the system approaches a unique steady state (i.e., the dynamics is relaxing).

We emphasize that contractivity is not a universal feature but depends on the metric: Quantum evolution may be contractive with respect to a given metric and may not be contractive with respect to other metric measures. Moreover, contractivity of quantum evolution can break down provided that the system is initially correlated with its environment. Effects induced by such correlations have been studied in various context [7–10]. First experiments on initial system-environment correlations were reported in Ref. [11]. Examples of an exact reduced dynamics which fail contractivity with respect to the *trace distance* are presented in Refs. [12,13]: The trace distance of different states grows above its initial value and the distinguishability growth occurs not only at the short time scales but is shown to be a feature of the long-time limit as well. The trace metric is likely the most important measure of the size for distance in quantum information processing, and according to Ref. [13], an increase of the distance can be interpreted in terms of the exchange of information between the system and its environment. If the distance increases over its initial value, information which is locally inaccessible at the initial time is transferred to the open system. This transfer of information enlarges the distinguishability of the open-system states which suggests various ways for the experimental detection of initial correlations. With this study we demonstrate that the correlation-induced distinguishability growth is not generic with respect to distance measures, but distinctly depends on the assumed form of the metric measure.

The outline of the paper is as follows. In Sec. II we list several forms of the distance measure. In Sec. III, we define a dephasing model of the qubit plus environment [14] and the environment is assumed to be infinite. We also present the reduced dynamics of the qubit for a particular initial qubit-environment state which is correlated (entangled). Properties of time evolution of the distance between two states of the qubit are demonstrated for selected metrics. In Sec. IV we consider the similar model, but now with a finite-size environment consisting of just one boson. We study distances between two states and analyze its properties. Finally, Sec. V provides our summary and some conclusions.

II. A SELECTION OF DIFFERENT DISTANCE MEASURES

The question of similarity between quantum states can have very different meanings depending on the context in which the question is posed. One can distinguish at least two main classes of problems. The first is related to the geometric structure of a set of states, and the second is related to the statistical content of quantum states. These two classes are not disjoint due to the richness of links joining different quantifiers [6]. Here we limit our consideration to measures which are, or are expected (as the Jensen-Shannon divergence discussed below) to be a *metric*. We will consider the following types of the distance between any two states ρ_1 and ρ_2:

(1) The use of the trace distance

$$D_T[\rho_1,\rho_2] = \tfrac{1}{2}\mathrm{Tr}\sqrt{(\rho_1-\rho_2)^2}, \tag{3}$$

presents a contraction in the sense discussed in the Introduction and is limited to the unit interval

$$0 \leqslant D_T[\rho_1,\rho_2] \leqslant 1.$$

The trace distance, being Euclidean, has apart from its geometric characteristics, also a profound statistical meaning as a quantifier for "statistical distinguishability" of quantum states [5]. Due to its universal character the trace distance has been considered in the context of a contractivity breakdown caused by the system-environment correlations [12,13]. In this paper it will serve as a natural reference for other measures to be compared with.

(2) The space of density matrices describing states of a quantum system can be equipped with a very natural scalar product [5] leading to the Hilbert-Schmidt distance

$$D_{\mathrm{HS}}[\rho_1,\rho_2] = \sqrt{\mathrm{Tr}(\rho_1-\rho_2)^2}. \tag{4}$$

This distance is restricted by the inequality relation

$$0 \leqslant D_{\mathrm{HS}}[\rho_1,\rho_2] \leqslant 2D_T[\rho_1,\rho_2].$$

The Hilbert-Schmidt distance is of Riemann type. Unfortunately, it generally does not possess the "contractivity property" discussed in the Introduction. Fortunately enough, however, archetype quantum systems such as qubits constitute useful exceptions, as will be discussed in further detail below.

(3) There is a very elegant and deep geometric structure useful for studying general quantum systems, namely the Hilbert-Schmidt fiber bundle [6]. Its base manifold is equipped with a natural metric [6] (i.e., the Bures distance)

$$D_B^2[\rho_1,\rho_2] = 2[1-\sqrt{F(\rho_1,\rho_2)}]. \tag{5}$$

The Bures distance is contractive and can be expressed by the fidelity

$$F(\rho_1,\rho_2) = [\mathrm{Tr}\sqrt{\sqrt{\rho_1}\ \rho_2\ \sqrt{\rho_1}}]^2 \tag{6}$$

and hence, additionally to its geometric character, the Bures distance inherits a clear statistical interpretation. In this case

$$0 \leqslant D_B[\rho_1,\rho_2] \leqslant \sqrt{2}.$$

(4) Among the variety of distances between states there are measures whose definition originate from the statistical interpretation of quantum states [6]. One of them is the so-called Hellinger distance

$$\begin{aligned} D_H^2[\rho_1,\rho_2] &= \mathrm{Tr}(\sqrt{\rho_1}-\sqrt{\rho_2})^2 \\ &= 2[1-A(\rho_1,\rho_2)], \end{aligned} \tag{7}$$

where the quantum affinity reads

$$A(\rho_1,\rho_2) = \mathrm{Tr}(\sqrt{\rho_1}\ \sqrt{\rho_2}). \tag{8}$$

The Hellinger distance assumes values from the interval

$$0 \leqslant D_H[\rho_1,\rho_2] \leqslant \sqrt{2}.$$

(5) The notion of (information) entropy occurs in almost all branches of physics as a tool of quantifying information or relative information contained in states, either classical or quantum. There are certain technical difficulties in using certain types of information entropies [15]. These measures are, in general, not metrics. The Jensen-Shannon divergence is a tool which allows one to overcome this sort of problem. It is defined in terms of a symmetrized relative entropy between states; here, however, we use instead the following expression [3]:

$$D_{\mathrm{JS}}^2[\rho_1,\rho_2] = H_N\left(\frac{\rho_1+\rho_2}{2}\right) - \frac{1}{2}H_N(\rho_1) - \frac{1}{2}H_N(\rho_2), \tag{9}$$

where

$$H_N(\rho) = -\mathrm{Tr}[\rho\ln\rho]$$

is the von Neumann entropy. This quantity takes values from the unit interval

$$0 \leqslant D_{\mathrm{JS}}[\rho_1,\rho_2] \leqslant 1. \tag{10}$$

Whether the Jensen-Shannon divergence is a metric for all mixed states remains an unsolved problem [15,16].

Below, we will consider one-qubit system (with an $N=2$-dimensional Hilbert space) for which one can represent the density matrices in the form

$$\rho_i = \tfrac{1}{2}[1+\vec{r}_i\vec{\sigma}], \quad i=1,2, \tag{11}$$

where $\vec{r}_i = [x_i,y_i,z_i]$ is the Bloch vector and $\vec{\sigma} = [\sigma_x,\sigma_y,\sigma_z]$ are the Pauli matrices. In this case, the trace and Hilbert-Schmidt distances are equivalent, namely [17]

$$D_{\mathrm{HS}}[\rho_1,\rho_2] = \sqrt{2}D_T[\rho_1,\rho_2]. \tag{12}$$

This distance is equal to the ordinary Euclidean distance between the two states on the Bloch sphere (i.e., $D_{\mathrm{HS}}(\rho_1,\rho_2) = |\vec{r}_1-\vec{r}_2|$). Moreover, the expression for the Bures distance simplifies because the fidelity assumes the form [6]

$$F(\rho_1,\rho_2) = \mathrm{Tr}(\rho_1\rho_2) + 2\sqrt{\det\rho_1\det\rho_2}. \tag{13}$$

The Helinger distance can explicitly be calculated using the relation for the affinity (8). Then the affinity is expressed by the relation [2]

$$A(\rho_1,\rho_2) = \frac{\left(1+\sqrt{1-r_1^2}\right)\left(1+\sqrt{1-r_2^2}\right)+\vec{r}_1\vec{r}_2}{(\sqrt{1+r_1}+\sqrt{1-r_1})(\sqrt{1+r_2}+\sqrt{1-r_2})}, \tag{14}$$

where $r_i^2 = x_i^2 + y_i^2 + z_i^2$. The Jensen-Shannon divergence (9) is expressed by the von Neumann entropy which is given by

$$H_N(\rho_i) = \ln 2 - \frac{1}{2}\ln\left(1-r_i^2\right) - \frac{r_i}{2}\ln\frac{1+r_i}{1-r_i}. \tag{15}$$

It has been proved that for qubits the Jensen-Shannon divergence is a metric [16].

In prior works [12,13], examples showing that the trace distance of different states can grow above its initial value have been presented. Our objective here is to investigate whether the growth of distance measure is preserved as well for the other metric measures introduced above.

III. MODEL A: QUBIT COUPLED TO INFINITE ENVIRONMENT OF OSCILLATORS

In this section, we consider the same model as in Ref. [12]. For the readers convenience and to keep the paper self-contained, we provide all necessary definitions and notation. The model consists of a qubit Q (two-level system) coupled to its environment B and we limit our considerations to the case when the process of energy dissipation is negligible and only pure dephasing is acting as the mechanism responsible for decoherence of the qubit dynamics [14]. Such a system can be described by the Hamiltonian (with $\hbar = 1$)

$$H = H_Q \otimes \mathbb{I}_B + \mathbb{I}_Q \otimes H_B + S^z \otimes H_I, \tag{16}$$

$$H_Q = \varepsilon S^z, \quad H_B = \int_0^\infty d\omega\, h(\omega) a^\dagger(\omega) a(\omega), \tag{17}$$

$$H_I = \int_0^\infty d\omega [g^*(\omega) a(\omega) + g(\omega) a^\dagger(\omega)], \tag{18}$$

where S^z is the z component of the spin operator and is represented by the diagonal matrix $S^z = \mathrm{diag}[1,-1]$ of elements 1 and -1. The parameter ε is the qubit energy splitting, $\mathbb{I}_Q$ and $\mathbb{I}_B$ are identity operators (matrices) in corresponding Hilbert spaces of the qubit Q and the environment B, respectively. The operators $a^\dagger(\omega)$ and $a(\omega)$ are the bosonic creation and annihilation operators, respectively. The real-valued spectrum function $h(\omega)$ characterizes the environment. The coupling is described by the function $g(\omega)$ and the function $g^*(\omega)$ is the complex conjugate to $g(\omega)$. The Hamiltonian (16) can be rewritten in the block-diagonal structure [18]

$$H = \mathrm{diag}[H_+, H_-], \quad H_\pm = H_B \pm H_I \pm \varepsilon \mathbb{I}_B. \tag{19}$$

As an example, we assume a correlated initial state of the total system in the form similar to that in Ref. [12], namely

$$|\Psi(0)\rangle = b_+|1\rangle \otimes |\Omega_0\rangle + b_-|-1\rangle \otimes |\Omega_\lambda\rangle. \tag{20}$$

The states $|1\rangle$ and $|-1\rangle$ denote the excited and ground states of the qubit, respectively. The nonzero complex numbers b_+ and b_- are chosen such that $|b_+|^2 + |b_-|^2 = 1$. The state $|\Omega_0\rangle$ is the ground (vacuum) state of the environment and

$$|\Omega_\lambda\rangle = C_\lambda^{-1}[(1-\lambda)|\Omega_0\rangle + \lambda|\Omega_f\rangle], \tag{21}$$

where $|\Omega_f\rangle = D(f)|\Omega_0\rangle$ is the coherent state. The displacement (Weyl) operator $D(f)$ reads [19]

$$D(f) = \exp\left\{\int_0^\infty d\omega [f(\omega) a^\dagger(\omega) - f^*(\omega) a(\omega)]\right\} \tag{22}$$

for an arbitrary square-integrable function f. The constant C_λ normalizes the state (20) and is given by the expression

$$C_\lambda^2 = (1-\lambda)^2 + \lambda^2 + 2\lambda(1-\lambda)\mathrm{Re}\langle\Omega_0|\Omega_f\rangle, \tag{23}$$

where Re is a real part of the scalar product $\langle\Omega_0|\Omega_f\rangle$ of two states in the environment Hilbert space. The parameter $\lambda \in [0,1]$ controls the initial entanglement of the qubit with the environment. For $\lambda = 0$ the qubit and the environment are initially uncorrelated while for $\lambda = 1$ the entanglement is most prominent for a given class of initial states.

The initial state (20) of the total system evolves according to the formula

$$|\Psi(t)\rangle = b_+|1\rangle \otimes |\psi_+(t)\rangle + b_-|-1\rangle \otimes |\psi_-(t)\rangle, \tag{24}$$

where

$$\begin{aligned} |\psi_+(t)\rangle &= \exp(-iH_+t)|\Omega_0\rangle, \\ |\psi_-(t)\rangle &= \exp(-iH_-t)|\Omega_\lambda\rangle. \end{aligned} \tag{25}$$

The density matrix of the total (isolated) system is $\varrho(t) = |\Psi(t)\rangle\langle\Psi(t)|$. In turn, the partial trace Tr_B over the environment B yields the density matrix $\rho_\lambda(t) = \mathrm{Tr}_B \varrho(t)$ of the qubit. It can be expressed in the matrix form as

$$\rho_\lambda(t) = \begin{pmatrix} |b_+|^2 & b_+ b_-^* A_\lambda(t) \\ b_+^* b_- A_\lambda^*(t) & |b_-|^2 \end{pmatrix}, \tag{26}$$

where the dephasing function $A_\lambda(t)$ reads

$$A_\lambda(t) = C_\lambda^{-1} e^{-2i\varepsilon t - r(t)}[1 - \lambda + \lambda e^{-2i\Phi(t) + s(t)}], \tag{27}$$

and [18]

$$\begin{aligned} r(t) &= 4\int_0^\infty d\omega g_h^2(\omega)[1-\cos(\omega t)], \\ s(t) &= 2\int_0^\infty d\omega g_h(\omega) f(\omega)[1-\cos(\omega t)] \\ &\quad - \frac{1}{2}\int_0^\infty d\omega f^2(\omega), \end{aligned} \tag{28}$$

where $g_h(\omega) = g(\omega)/h(\omega)$ and

$$\Phi(t) = \int_0^\infty d\omega g_h(\omega) f(\omega) \sin(\omega t). \tag{29}$$

Without loss of generality we have assumed here that the functions $g(\omega)$ and $f(\omega)$ are real valued.

A. Analysis of different distance measures

For the analysis of distance properties of the model considered, we still have to specify two quantities: the spectral density $g_h(\omega) = g(\omega)/h(\omega)$ and the coherent state determined by the function $f(\omega)$. The spectral density function $g_h(\omega)$ completely defines the coupling and modes of the environment. Typically the spectral function is taken as some continuous function of frequency to indicate that the environment can be treated as infinite compared to the system. With this study we restrict ourselves to the case in which this function assumes the explicit form

$$g_h^2(\omega) = \alpha\, \omega^{\mu-1} \exp(-\omega/\omega_c), \tag{30}$$

where $\alpha > 0$ is the qubit-environment coupling constant, ω_c is a cutoff frequency, and $\mu > -1$ is the "Ohmicity" parameter: the case $-1 < \mu < 0$ corresponds to the sub-Ohmic, $\mu = 0$ to the Ohmic, and $\mu > 0$ to the super-Ohmic environments, respectively. Comparing this equation with the expression for the standard spectral function $J(\omega)$ (see, e.g., Refs. [20,21]), one can find the relation [18]

$$J(\omega) = \omega^2 g_h^2(\omega). \tag{31}$$

As follows from our previous study, only in the case of super-Ohmic environment, the trace distance can increase. Therefore below we analyze only this regime.

To determine the coherent state $|\Omega_f\rangle$, we can propose any integrable function $f(\omega)$ but for convenience let

$$f^2(\omega) = \gamma\, \omega^{\nu-1} \exp(-\omega/\omega_c). \tag{32}$$

The only reason for such a choice is the possibility to calculate explicit formulas for the functions in Eqs. (28) and (29). As a result one gets

$$\begin{aligned} r(t) &= 4\mathcal{L}(\alpha,\mu,t),\\ s(t) &= 2\mathcal{L}(\sqrt{\alpha\gamma},(\mu+\nu)/2,t) - \frac{1}{2}\gamma\Gamma(\nu)\omega_c^{\nu},\\ \Phi(t) &= \sqrt{\alpha\gamma}\;\Gamma\left(\frac{\mu+\nu}{2}\right)\omega_c^{\frac{\mu+\nu}{2}}\;\frac{\sin\left[\frac{\mu+\nu}{2}\arctan(\omega_c t)\right]}{\left(1+\omega_c^2 t^2\right)^{\kappa/2}},\\ \mathcal{L}(\alpha,\mu,t) &= \alpha\Gamma(\mu)\omega_c^{\mu}\left\{1 - \frac{\cos[\mu\arctan(\omega_c t)]}{\left(1+\omega_c^2 t^2\right)^{\mu/2}}\right\}, \end{aligned} \tag{33}$$

and $\Gamma(z)$ is the Euler gamma function.

We next examine the time evolution of the distance for all four distance measures: namely the trace distance D_T, the Bures distance D_B, the Hellinger distance D_H, and the quantum Jensen-Shannon measure $D_{\rm JS}$. We recall that the trace and Hilbert-Schmidt distances are equivalent. As shown in Ref. [12], the only chance to observe an increase of the distance between two states is to vary the parameters of the environment encoded in $|\Omega_\lambda\rangle$ in Eq. (21). The simplest theoretical possibility is to manipulate the *correlation parameter* λ. When two different states are determined by two different sets of numbers $b_\pm^{(k)}$ $(k = 1,2)$ in Eq. (20) for the same state $|\Omega_\lambda\rangle$ then $A_{\lambda_1}(t) = A_{\lambda_2}(t)$ and an increasing growth of the distance becomes not possible.

In Fig. 1, we depict the time evolution of the distances $D(t) = D[\rho_0(t),\rho_\lambda(t)]$ between the initially noncorrelated and correlated states for four metrics. We observe that only for the trace metric, the distance $D[\rho_0(t),\rho_\lambda(t)]$ can increase above its

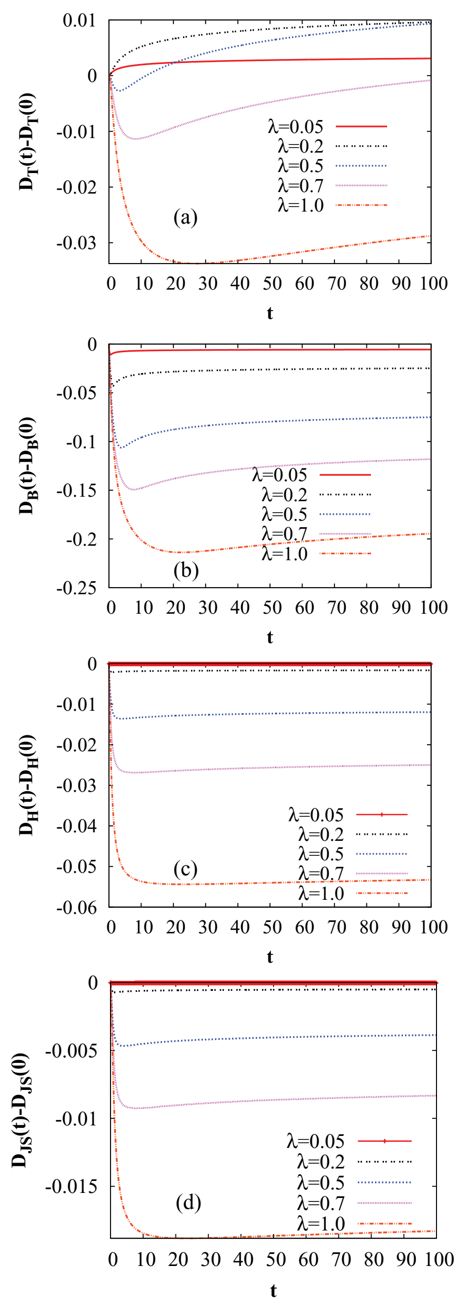

FIG. 1. (Color online) Time evolution of distances between two qubit states in the case of infinite environment. (a) The trace $D_T = D_{\rm HS}/\sqrt{2}$, (b) Bures D_B, (c) Hellinger D_H, and (d) quantum Jensen-Shannon $D_{\rm JS}$ distances, respectively. The distances $D(t) = D[\rho_0(t),\rho_\lambda(t)]$ are between the initially noncorrelated and correlated states for selected values of the correlation parameter λ. Time is in units of ω_c, the dimensionless coupling $\alpha\omega_c^{\mu} = 0.01$ and $\gamma\omega_c^{\nu} = 0.05$. The remaining parameters are $\varepsilon = 1, \mu = 0.01$, $\nu = 0.2$, and $|b_+^{(1)}|^2 = |b_+^{(2)}|^2 = 1/2$.

initial value and there is some optimal value of the correlation parameter $0 < \lambda < 1$ for which the distinguishability of final states is the best. Because this case was studied in Ref. [12], we do not present the details here for the trace distance properties. In the remaining three cases, the distance between states at arbitrary time $t > 0$ is always smaller than the distance at time $t = 0$ and the distinguishability of final states is weaker than for the initial states. An interesting feature is the appearance of the absolute minimal distance at some time $t_m > 0$ during the time evolution of the qubit. At an early stage of time evolution, the distance decreases, reaching a minimum before it increases again and eventually saturates at asymptotic long times. The conclusion from our analysis depicted in Fig. 1 hence is as follows: An increase of the distance above its initial value between two qubit states presents not a universal property of the correlated initial state, but instead is rather sensitive to the chosen metric measure. Among our chosen five different metric measures, only the trace and the Hilbert-Schmidt metrics exhibit this typical property for the considered decoherence model.

IV. MODEL B: QUBIT COUPLED TO A FINITE ENVIRONMENT

The preparation of an initial state as determined by Eq. (20) requires highly sophisticated quantum engineering tools which presently seem not feasible or at best difficult to realize. Fortunately, interesting features of distances between states resulting from initial system-environment correlations can be studied with a simplified setup. Following such reasoning we next study a qubit that is coupled to a *finite*-size environment. In this case the notion of decoherence is absent in a strict sense of the term. Nevertheless, the considered qubit constitutes an open system. Our choice of a finite bosonic environment is motivated by recent progress in quantum engineering of nonclassical electromagnetic fields which can be prepared in various states, both in the optical [22] and in the microwave [23] energetic regimes. As an example, we consider a single boson mode. The total Hamiltonian (19) then reduces to the form

$$H = \mathrm{diag}[H_+, H_-],$$
$$H_\pm = \omega a^\dagger a \pm g_0(a + a^\dagger) \pm \varepsilon \mathbb{I}_B, \tag{34}$$

where g_0 is a coupling constant. The initial state of the total system is, in general, correlated, namely

$$|\Psi(0)\rangle = b_+|1\rangle \otimes |0\rangle + b_-|-1\rangle \otimes |\Omega_\lambda\rangle, \tag{35}$$

where

$$|\Omega_\lambda\rangle = C_\lambda^{-1}[(1-\lambda)|0\rangle + \lambda|F\rangle]. \tag{36}$$

The state $|0\rangle$ is a vacuum state (a ground state) of the boson and the choice for the state $|F\rangle$ is limited to two classes studied in quantum optics, being known to be distinct with respect to their nonclassical character. First we use $|F\rangle = |z\rangle$ to be a coherent state. Next we analyze the case when $|F\rangle = |N\rangle$ is a number eigenstate. The density matrix of the qubit assumes the same structure as in Eq. (26), but now with the modified function $A_\lambda(t)$.

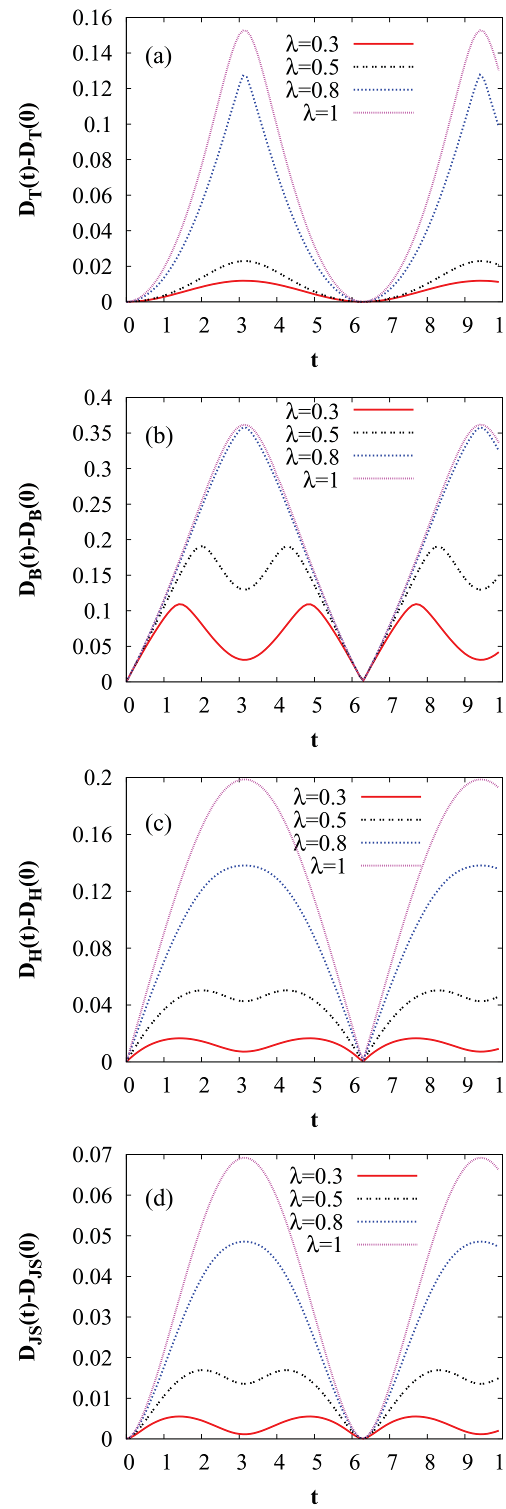

FIG. 2. (Color online) Time evolution of the (a) trace $D_T = D_{\mathrm{HS}}/\sqrt{2}$, (b) Bures D_B, (c) Hellinger D_H, and (d) Jensen-Shannon D_{JS} distances of qubit states for a finite environment. Initially the boson is in the mixture of the ground $|0\rangle$ and coherent states $|z\rangle$ with $z = |z|e^{i\phi}$. The impact of initial correlations quantified by the parameter λ is depicted. As in Fig. 1, the distances $D(t) = D[\rho_0(t), \rho_\lambda(t)]$ are between the initially noncorrelated and correlated states. Time is in units of ω. The chosen system parameters are $\varepsilon = 1, g = 0.1, |z| = 1, \phi = 0$, and $|b_+^{(1)}|^2 = |b_+^{(2)}|^2 = 1/2$.

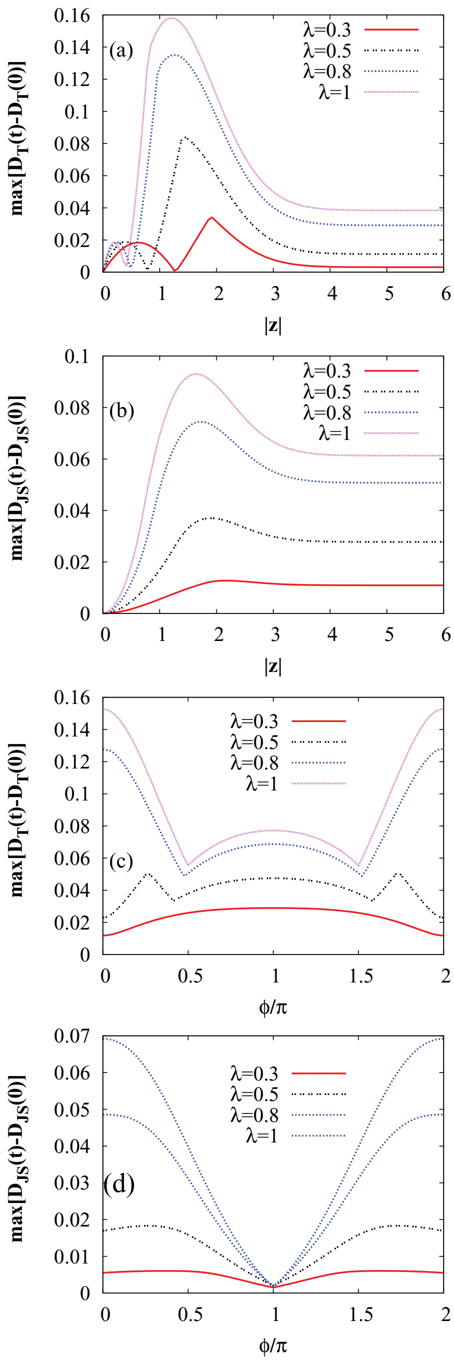

FIG. 3. (Color online) Illustration of the role of the amplitude $|z|$ [(a) and (b)] and phase ϕ [(c) and (d)] of the environment coherent state $|z\rangle$ ($z = |z|e^{i\phi}$) on trace and Jensen-Shannon distances of qubit states. Qualitatively, the Bures and Hellinger distances behave like the Jensen-Shannon distance. $\max[D(t) - D(0)]$ is the amplitude of distance time oscillations shown in Fig. 2. In (a) and (b) $\phi = 0$. In (c) and (d) $|z| = 1$. The remaining parameters are the same as in Fig. 2.

A. Case of initial coherent states

Let for any complex number $z = |z|e^{i\phi}$, the state $|F\rangle = |z\rangle$ be a coherent state of the boson. Then the function $A_\lambda(t)$ is given by

$$A_\lambda(t) = C_\lambda^{-1}\, e^{-2i\varepsilon t - R(t)}\,[1 - \lambda + \lambda e^{-2i\Lambda(t) + S(t)}], \qquad (37)$$

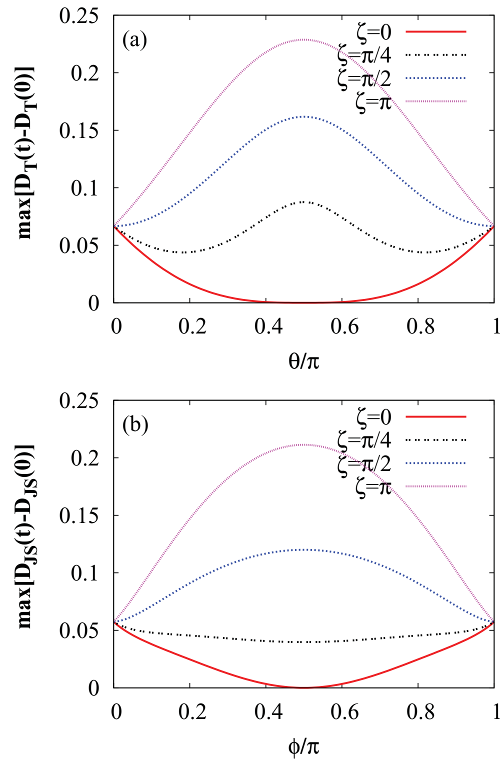

FIG. 4. (Color online) Illustration of the role of initial qubit states on trace and Jensen-Shannon distances. Qualitatively, the Bures and Hellinger distances behave like the Jensen-Shannon distance. $\max[D(t) - D(0)]$ is the amplitude of distance time oscillations shown in Fig. 2. The parameters characterizing two initial states (20) are $\lambda_1 = \lambda_2 = 1$, for the first state and $b_+ = b_- = 1/\sqrt{2}$, for the second state: $b_+ = \cos(\theta/2)$, $b_- = \exp(i\zeta)\sin(\theta/2)$ with angle parametrization θ and ζ on the Bloch sphere. The remaining parameters are $g = 0.1$, $z = 1$.

where

$$\begin{aligned} R(t) &= 4g^2[1 - \cos(\omega t)],\\ S(t) &= 2g|z|[\cos\phi - \cos(\omega t - \phi)] - \tfrac{1}{2}|z|^2, \\ \Lambda(t) &= g|z|[\sin(\omega t + \phi) + \sin\phi], \end{aligned} \qquad (38)$$

and $g = g_0/\omega$ is the rescaled coupling.

Because the total system is finite, time evolution of the qubit states is periodic. However, it is not unitary evolution. The distance between two states of the qubit is also a periodic function of time. Let us now inspect the time dependence of all four distances: trace $D_T = D_{\rm HS}/\sqrt{2}$, Bures D_B, Hellinger D_H, and Jensen-Shannon $D_{\rm JS}$ distances. In Fig. 2 we illustrate the role of the initial qubit-environment correlations in the case when two different initial states are determined by two different states $|\Omega_\lambda\rangle$ with different λ_1 and λ_2. The most peculiar feature is that for all measures, the distance at any time $t > 0$ is not smaller than at initial time. It is in clear contrast to the case of the infinite environment case when only the trace distance can increase about its initial value. Now, at the beginning, for $t > 0$, all distances increase above their initial value reaching the maximal value, which in turn grows when the correlation parameter $\lambda \to 1$. The maximal amplitude of distance oscillations is shown up for maximally entangled states (i.e., for $\lambda = 1$). It also depends on other system parameters, in particular on the state of environment which is determined by two quantities: the amplitude $|z|$ and phase ϕ of the coherent state $|z\rangle$. The inspection of the results revealed that there are regimes of optimal values of $|z|$ for which the distinguishability of two qubit states is most prominent. We present it in Fig. 3 for the trace and Jensen-Hellinger distances. The remaining two (Bures and Hellinger) distances exhibit similar behavior as the Jensen-Shannon distance. In the two bottom panels of Fig. 3 we demonstrate how the phase of the coherent state changes the distance. Again, as previously, we present only two cases. Two other cases are similar to the Jensen-Shannon one. Let us observe that in some regimes the trace distance possesses distinctive features which are different from other distance measures.

Next, let us consider the case when two different states are determined by two different sets of numbers $b_\pm^{(k)}$ $(k = 1,2)$ in Eq. (20) but with the same state $|\Omega_\lambda\rangle$. One state is fixed by $b_+ = b_- = 1/\sqrt{2}$. The second state is conveniently parameterized by two angles θ and ζ on the Bloch sphere and is determined by the relations $b_+ = \cos(\theta/2)$ and $b_- = \exp(i\zeta)\sin(\theta/2)$. The result is depicted in Fig. 4 which shows that the amplitude of time-periodic oscillations of the distance can typically be increased by increasing the geometrical distance of initial states on the Bloch sphere. However, there are some exceptions such as, for example, for the case $\zeta = \pi/4$ in the case of the trace distance.

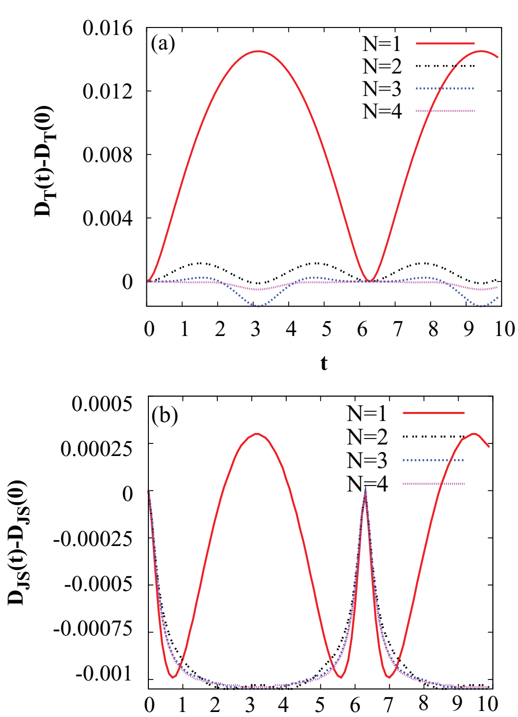

FIG. 5. (Color online) The trace and Jensen-Shannon distances of qubit states for the finite environment: boson in the mixture of the ground $|0\rangle$ and number $|N\rangle$ states. The Bures and Hellinger distances display similar time dependence as the Jensen-Shanon distance with the exception that they lie below zero. The parameters are $\lambda_1 = 0, \lambda_2 = 1, \epsilon = 1, g = 0.1$, and $|b_+^{(1)}|^2 = |b_+^{(2)}|^2 = 1/2$.

B. Case of initial number states

Let $|F\rangle = |N\rangle$ be a number eigenstate of the boson. Contrary to coherent states, such eigenstates are orthogonal and the state (35) becomes maximally entangled, that is, its partial trace, taken with respect to the bosonic degree of freedom, is an identity and it corresponds to the maximally mixed state of the qubit. In this case, the function $A_\lambda(t)$ assumes the form

$$A_\lambda(t) = C_\lambda^{-1}\, e^{-2i\varepsilon t - R(t)}\,[1 - \lambda + \lambda B_N(t)], \tag{39}$$

where

$$B_N(t) = \frac{(2g)^N}{\sqrt{N!}}(e^{-i\omega t} - 1)^N. \tag{40}$$

As in the first case, time evolution of the qubit states is time periodic and in consequence distance is also a periodic function of time. In Fig. 5, we present two forms of the distance, namely the trace and the Jensen-Shannon ones. Only these two distance measures can exhibit the increased distance over their initial value.

The "optimal" environment state is the first excited one (i.e., when $N = 1$). This state is highly nonclassical. The question of whether there is any relation between the nonclassical character of the environment and the distance between reduced qubit states remains open and will be postponed for further considerations. Further excited states diminish the positive value of difference $D(t) - D(0)$ or invert it into a negative value. Two remaining (Bures and Hellinger) distances behave in a similar way as the Jensen-Shannon one but they are removed down and never exceed their initial values.

V. SUMMARY

The objective to distinguish two quantum channels presents a most important challenge for quantum information processing tasks. The difficulty of the distinguishability issue leads naturally to a study of the problem on restricted classes of channels. With this work we presented two models and we have elucidated the properties of four distance measures for quantum states for the situation of a qubit which is coupled to an environment. At initial times, the system is in a correlated (entangled) state. Our chosen measures include the trace (and equivalent Hilbert-Schmidt), Bures, Hellinger, and Jensen-Shannon distances. We have considered two examples of the environment: namely an infinite one consisting of bosons and a finite one consisting of a single boson. We have demonstrated that in the case of the infinite environment, only the trace distance exhibits an increase above its initial value. All other remaining distances studied do not exhibit this property. In the case of a finite environment, however, some kind of universality is observed for the case when the boson consists in a mixture of ground and coherent states. In this second case, all distances behave more or less similarly: the distance measures oscillate with a common frequency between an initial value and some maximal positive value, which is different for differently chosen metrics. Nevertheless, their time dependence behaves qualitatively the same. This is not the case when the boson is in a mixture of the ground and excited states; only the trace and Jensen-Shannon distances are allowed to grow above the initial value.

Our main conclusion is as follows: The result of an increase of the distance measure above its initial value constitutes no universal property; its behavior upon evolving time strongly depends on the employed distance measure. in this respect, the trace distance receives a special status.

The authors are confident that this work may stimulate yet additional studies. Particularly, it would be interesting to investigate in some detail the objective of universally valid, initial-state-dependent and/or system-dependent properties of the various distance measures in use. The generalization of our results to (i) other classes of initial correlations between the system and environments of a different nature and (ii) for nonzero temperatures provide yet other appealing routes for future research.

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