PHYSICAL REVIEW A **84**, 032118 (2011)

Equivalence of the measures of non-Markovianity for open two-level systems

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(Received 21 March 2011; published 26 September 2011)

Different measures have been presented to depict the deviation of quantum time evolution in open systems from Markovian processes. We demonstrate that the measure proposed by Breuer, Laine, and Piilo [Phys. Rev. Lett. **103**, 210401 (2009)] and the two measures proposed by Rivas, Huelga, and Plenio [Phys. Rev. Lett. **105**, 050403 (2010)] have exactly the same non-Markovian time-evolution intervals and thus are really equivalent to each other when they are applied to open two-level systems coupled to environments via the Jaynes-Cummings or dephasing models. This equivalence implies that the three measures, in different ways, capture the intrinsic character of the non-Markovianity of quantum evolutional processes. We also show that the maximization in the definition of the first measure can be actually removed for the considered models without influencing the sensibility of the measure to detect non-Markovianity.

DOI: 10.1103/PhysRevA.84.032118 PACS number(s): 03.65.Yz, 03.65.Ta, 42.50.Lc

I. INTRODUCTION

The evolution of open quantum systems can be divided into two basic types: Markovian and non-Markovian processes [1]. For Markovian processes, the correlation time between the system and environment is considered to be infinitesimally small so that the dynamical map does not carry any memory effects, leading to a monotonic flow of the information from the system to the environment. In contrast, non-Markovian processes with memory have different dynamical traits which give rise to the backflow of information from the environment to the system [2,3]. Recently, people found that non-Markovian processes can lead to distinctly different effects on decoherence and disentanglement [4,5] of open systems compared to Markovian processes. Many relevant physical systems, such as the quantum optical system [1], quantum dot [6], and color-core spin in semiconductor [7], could not be described simply by Markovian dynamics. Some problems in quantum chemistry [8] and the excitation transfer of a biological system [9] also need to be treated as non-Markovian processes. Because of these distinct properties and extensive applications, more and more attention and interest have been devoted to the study of non-Markovian processes of open systems, including the measures of non-Markovianity [2,3,10–13], the positivity [14–16], and some other dynamical properties [17–20] and approaches [21,22]. Experimentally, the simulation [23,24] of non-Markovian environment has been realized.

The measure of non-Markovianity of quantum evolution is a fundamental problem which aims to detect whether a quantum process is non-Markovian and how much it deviates from a Markovian one. So far, almost all measures can only be applied to the former, i.e., as the sufficient (not necessary) condition for the emergence of non-Markovianity. It still remains elusive and, in some sense, controversial for measuring the non-Markovianity of quantum processes. Based on the distinguishability of quantum states, Breuer, Laine, and Piilo (BLP) [2] proposed a measure to detect the non-Markovianity of quantum processes for the flow of information between the system and environment. Alternatively, Rivas, Huelga, and Plenio (RHP) [10] also presented two measures of non-Markovianity by exploiting the dynamical behavior of quantum entanglement under the local trace-preserving completely positive (CP) maps. Other measures of non-Markovianity include the one proposed by Wolf *et al.* [11] based on the breakdown of the semigroup property, the one proposed by Lu *et al.* [13] using quantum Fisher information flow, and the one proposed by Usha Devi *et al.* [12] using relative entropy difference and fidelity difference. It is worthwhile to stress that these measures of non-Markovianity are not generally equivalent. Thus studying the relations among all of these measures under some specific models becomes very important. Very recently, Haikka *et al.* [25] studied the links between the BLP measure and one of the RHP measures for a laser-driven qubit system embedded in a structured Lorentzian environment. They showed, both analytically and numerically, that the two measures agree in the nonsecular regime. But for other cases, no definite result has been presented.

In this paper, we study the equivalence of the three measures of non-Markovianity proposed recently by BLP [2] and by RHP [10] for a two-level system coupled to its environment via a damped Jaynes-Cummings or dephasing model. Two important results are found: First, the three measures are exactly equivalent in the aspect of detecting non-Markovianity for the involved models; second, for our considered models, we find that the maximization in the definition of the BLP measure may actually be removed. This is the problem extensively explored [26,27]. Our work has the following features: the interaction models considered here are most fundamental in the theoretical studies of dynamics of open quantum systems, and we do not need to assume any specific spectral density for the structured environment. Thus our results possess good adaptability. Moreover, our deductions are completely analytical and hence the results are more convincing.

The article is organized as follows. In Sec. II we briefly review the three measures proposed by BLP and RHP. The definition of equivalence between different measures is also presented. In Sec. III and Sec. IV we study the equivalence of

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the three measures for a two-level system interacting with the environment via the damped Jaynes-Cummings model and the dephasing model. The conclusion is presented in Sec. V.

II. REVIEW OF THE THREE MEASURES

In Ref. [2], BLP presented a measure of non-Markovianity for quantum processes of open systems based on the idea that Markovian processes tend to continuously reduce the trace distance between any two states of a quantum system. Thus an increase of the trace distance during any time interval implies the emergence of non-Markovianity. The authors further linked the changes of trace distance to the flow of information between the system and its environment, and concluded that the backflow of information from environment to the system is the key feature of a non-Markovian dynamics. Considering that the measure should reveal the total feature of a whole quantum process, they thus suggested the quantity

$$\mathcal{N} = \max_{\rho_{1,2}(0)} \int_{\sigma>0} dt \sigma[t,\rho_{1,2}(0)] \tag{1}$$

as the measure of non-Markovianity of a quantum process, where $\sigma[t,\rho_{1,2}(0)]$ denotes the time derivative of the trace distance for a pair of dynamical states with initial values $\rho_{1,2}(0)$ of the considered system. The time integration is extended over all intervals in which σ is positive, and the maximum is taken over all pairs of initial states. For any Markovian process, we have $\mathcal{N} = 0$; if $\mathcal{N} > 0$, the process must be non-Markovian.

Alternatively, RHP [10] presented another method to measure the non-Markovianity of a quantum process which is based on the monotonic drop of quantum entanglement between bipartite systems under the influence of local Markovian environments. Suppose a system of interest is initially prepared in a maximally entangled state with an ancillary particle, where only the system is influenced by a noise environment and the ancillary particle is noise-free. Then the quantity for measuring the non-Markovianity of the quantum process is defined as

$$\mathcal{I}^{(E)} = \int_{t_0}^{t_{\max}} \left| \frac{dE[\rho_{SA}(t)]}{dt} \right| dt - \triangle E, \tag{2}$$

where the time derivative in the integrand is for the dynamical entanglement between the system and ancillary particle. $\triangle E = E[\rho_{SA}(t_0)] - E[\rho_{SA}(t_{\max})]$ denotes the difference of entanglement at the initial time t_0 and the final time $t_{\max}$ of the interest quantum process.

The above two measures, $\mathcal{N}$ and $\mathcal{I}^{(E)}$, are introduced, respectively, through the monotonicity of trace distance or quantum entanglement under CP (or local CP) maps. Their mathematical forms are actually similar. It is not difficult to find that Eq. (2) may be equivalently rewritten as

$$\mathcal{I}^{(E)} = 2\int_{\dot{E}(t)>0} \dot{E}(t) dt, \tag{3}$$

with $\dot{E}(t)$ the time derivative of the dynamical entanglement $E(t)$. Obviously, the measure $\mathcal{I}^{(E)}$ is formally very similar to $\mathcal{N}$, with the time derivative $\dot{E}(t)$ of the dynamical entanglement replaced by the time derivative $\sigma = \dot{D}$ of the dynamical trace distance $D(\rho_1(t),\rho_2(t))$. The visible difference is as follows: the measure $\mathcal{N}$ involves a maximum over all pairs of initial states, while $\mathcal{I}^{(E)}$ is defined via a given maximally entangled initial state and thus escapes from the optimization problem. Actually, the escape of the optimization is just one of the original intentions for the measure $\mathcal{I}^{(E)}$ to be proposed. In the following sections, we will demonstrate that for the models under consideration, the maximization process can actually be removed.

The third quantity for measuring the non-Markovianity of a quantum process is also based on the entanglement dynamics between the system and an ancillary particle. Given the maximally entangled state $|\Phi\rangle$ of the system plus the ancillary particle, a locally complete positive map ε will keep the positivity of the density operator $\rho = |\Phi\rangle\langle\Phi|$ invariable, i.e., $\varepsilon(|\Phi\rangle\langle\Phi|) \geqslant 0$. Starting from this point of view, a measure of non-Markovianity of quantum processes is then born [10],

$$\mathcal{I} = \int_0^\infty g(t)dt, \tag{4}$$

with

$$g(t) = \lim_{\epsilon\to 0^+} \frac{\|[I + (\mathcal{L}_t \otimes I)\epsilon]|\Phi\rangle\langle\Phi|\| - 1}{\epsilon}, \tag{5}$$

where $\mathcal{L}_t$ is the super operator in the non-Markovian dynamical master equation $\frac{d\rho}{dt} = \mathcal{L}_t(\rho)$ for the open system.

The above three measures $\mathcal{N}$, $\mathcal{I}^{(E)}$, and $\mathcal{I}$, for the non-Markovianity of a quantum process are introduced in different ways. The physical meanings are different. A question naturally arises: Are they equivalent to each other? In this paper, we demonstrate that the three measures are equivalent to each other when they are applied to open two-level systems with damped Jaynes-Cummings or dephasing models. This equivalence implies that the measures, from different sides, well capture the intrinsical characters of non-Markovianity of quantum evolutional processes.

Before we conclude this section, let us elaborate on the implication for the different but equivalent measures. When we mention that two measures M_1 and M_2 are equivalent, what we mean is that for a given quantum process, if the non-Markovianity emerges tested by the measure M_1, then this also works by the measure M_2. Conversely, if the non-Markovianity does not emerge using M_1, nothing would appear using the measure M_2. The exact logistics are as follows: $M_1 > 0 \Longleftrightarrow M_2 > 0$ and $M_1 = 0 \Longleftrightarrow M_2 = 0$.

III. THE CASE OF THE JAYNES-CUMMINGS MODEL

In this section, we will demonstrate the equivalence of the three measures of non-Markovianity by the Jaynes-Cummings model describing a two-level system coupled to its environment. To this end, we should first derive the expressions of the three measures given the specific interaction model. For the calculation of $\mathcal{N}$, the environment is assumed to be initially in a vacuum state, but its spectral density is arbitrary. This case can be solved exactly. For any pair of initial atomic states, i.e., $\rho_1(0)$ and $\rho_2(0)$, we obtain the time derivative of the distance of the corresponding dynamical states as [3]

$$\sigma(t,\rho_{1,2}(0)) = \frac{2|G(t)|^2a^2 + |b|^2}{\sqrt{|G(t)|^2a^2 + |b|^2}} \frac{d}{dt}|G(t)|, \tag{6}$$

where $a = \langle 1|\rho_1(0)|1\rangle - \langle 1|\rho_2(0)|1\rangle$ and $b = \langle 1|\rho_1(0)|0\rangle - \langle 1|\rho_2(0)|0\rangle$ are the differences of populations and of coherence, respectively, between the two given initial states. The function $G(t)$ is defined as the solution of the integrodifferential equation,

$$\frac{d}{dt}G(t) = -\int_0^t dt_1 f(t-t_1)G(t_1), \tag{7}$$

with the initial condition $G(0)=1$. The kernel $f(t-t_1)$ denotes the two-point reservoir correlation function which is the Fourier transformation of the spectral density. Introducing the time-dependent decay rate,

$$\gamma(t) = -2\mathrm{Re}\left(\frac{\dot{G}(t)}{G(t)}\right) = -\frac{2}{|G(t)|}\frac{d}{dt}|G(t)|, \tag{8}$$

then the derivative of the trace distance may be rewritten as

$$\sigma(t,\rho_{1,2}(0)) = -\gamma(t)F(t). \tag{9}$$

Here the completely positive real function $F(t)$ is defined as

$$F(t) = \frac{a^2 e^{-\frac{3}{2}\Gamma(t)} + |b|^2 e^{-\frac{1}{2}\Gamma(t)}}{\sqrt{a^2 e^{-\Gamma(t)} + |b|^2}}, \tag{10}$$

with $\Gamma(t)=\int_0^t dt'\gamma(t')$. According to Eq. (1), the non-Markovian measure $\mathcal{N}$ may be expressed as

$$\mathcal{N} = -\int_{\gamma(t)<0} \gamma(t)F(t)dt, \tag{11}$$

where we have assumed that the pair of initial states $\rho_{1,2}(0)$ just maximizes $\mathcal{N}$ so that the maximization symbol is removed. Actually, due to the complete positivity of the function $F(t)$, the time intervals in which the trace distance increases monotonously, or equivalently to the intervals in which non-Markovianity emerges (we call them non-Markovian intervals below), are uniquely determined by the condition $\gamma(t)<0$. The change of initial state $\rho_{1,2}(0)$ would not alter the positions and lengths of these non-Markovianity intervals, i.e., it would not alter their distribution. No matter how the initial states change, $\mathcal{N}$ would not shift from positive to zero, or vice versa. In this sense, the maximization to Eq. (11) may be removed without influencing the sensibility of $\mathcal{N}$ to detect non-Markovianity. This is an important result which solves the problem being explored by many researchers [26,27].

We turn to the calculation of the second measure $\mathcal{I}^{(E)}$. Consider a maximally entangled state of two-level atoms,

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|10\rangle_{\mathrm{SA}} + |01\rangle_{\mathrm{SA}}), \tag{12}$$

where atom 1 is our considered system which couples to an environment via Jaynes-Cummings interaction and atom 2 is an ancillary particle which remains isolated from the environment. The whole Hamiltonian reads

$$\begin{aligned} H &= H_0 + H_I, \\ H_0 &= \sum_{i=1}^{2} \omega_0 \sigma_+^{(i)}\sigma_-^{(i)} + \sum_k \omega_k {a_k}^\dagger a_k, \\ H_I &= \sum_k g_k \sigma_+^{(1)} a_k + \mathrm{H.c.}, \end{aligned} \tag{13}$$

where $\sigma_+^{(i)}$ and $\sigma_-^{(i)}$ denote ladder operators for the ith atom, ω_k and a_k are the frequency and annihilation operator regarding the kth harmonic oscillator of the environment, and g_k is the coupling constant. We assume that the two atoms have the same transition frequency ω_0. In the interaction picture with respect to H_0, one has

$$\tilde{H}_I = \sigma_+^1 \sum_k g_k a_k \exp[(\omega_0 - \omega_k)t] + \mathrm{H.c.} \tag{14}$$

Assume that the two atoms are initially in the entangled state $|\Phi\rangle$ and environment is initially in the vacuum state $|0\rangle$. The time evolution of the wave function for the compound system including both the atoms and environment may be expressed as

$$\begin{aligned} |\psi(t)\rangle = {} & c_1(t)|10\rangle_{\mathrm{SA}}|0\rangle_E + c_2(t)|01\rangle_{\mathrm{SA}}|0\rangle_E \\ & + \sum_k c_k(t)|00\rangle_{\mathrm{SA}}|1_k\rangle_E. \end{aligned} \tag{15}$$

Tracing out the environment, we obtain

$$\begin{aligned} \rho_{\mathrm{SA}}(t) = {} & [1-|c_1(t)|^2-|c_2(t)|^2]|00\rangle\langle 00| + |c_1(t)|^2|10\rangle\langle 10| \\ & + |c_2(t)|^2|01\rangle\langle 01| + c_1(t)c_2^*(t)|10\rangle\langle 01| \\ & + c_1^*(t)c_2(t)|01\rangle\langle 10|, \end{aligned} \tag{16}$$

where the time-dependent coefficients are governed by the Schrödinger equation:

$$-i\frac{\partial}{\partial t}|\psi(t)\rangle = \tilde{H}_I(t)|\psi(t)\rangle. \tag{17}$$

After a straightforward deduction, we find $c_2(t)=c_2(0)=1/\sqrt{2}$ and $c_1(t)=G(t)/\sqrt{2}$ with $G(t)$ determined by Eq. (7). If we use concurrence to describe the entanglement between the system and the ancillary particle, we have from Eq. (16),

$$C[\rho_{\mathrm{SA}}(t)] = 2|c_1^*(t)c_2(t)| = |G(t)|, \tag{18}$$

and thus

$$\frac{d}{dt}C[\rho_{\mathrm{SA}}(t)] = -\frac{1}{2}\gamma(t)e^{-\Gamma(t)/2}, \tag{19}$$

where the definition of $\gamma(t)$ in Eq. (8) has been used. Then Eq. (3) immediately produces

$$\mathcal{I}^{(E)} = -\int_{\gamma(t)<0} \gamma(t)e^{-\Gamma(t)/2}dt. \tag{20}$$

It shows again that the non-Markovian intervals for measure $\mathcal{I}^{(E)}$ are uniquely determined by the condition $\gamma(t)<0$.

For calculating the third measure $\mathcal{I}$, we solve the master equation of the open two-level system [1],

$$\frac{d\rho}{dt} = -\frac{i}{2}S(t)[\sigma_+\sigma_-,\rho] + \gamma(t)\left[\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-,\rho\}\right], \tag{21}$$

where $S(t)=-2\mathrm{Im}(\frac{\dot{G}(t)}{G(t)})$ with $G(t)$ determined by Eq. (7) and $\gamma(t)$ is defined by Eq. (8). Straightforward calculation leads to

$$g(t) = \begin{cases} 0 & \text{for} \quad \gamma(t) \geqslant 0 \\ -\gamma(t) & \text{for} \quad \gamma(t) < 0 \end{cases} \tag{22}$$

and thus

$$\mathcal{I} = -\int_{\gamma(t)<0} \gamma(t)dt. \tag{23}$$

Once again, the non-Markovian intervals for this measure are determined by $\gamma(t) < 0$.

Through the arguments above, we conclude that for an open two-level system with a damped Jaynes-Cummings model, the distributions of non-Markovian intervals for the three measures $\mathcal{N}$, $\mathcal{I}^{(E)}$, and $\mathcal{I}$ are exactly the same, which are determined uniquely by the condition $\gamma(t) < 0$. According to the viewpoint of equivalence for different measures mentioned in Sec. II, we thus conclude that the three measures are equivalent for detecting the existence of non-Markovianity of a two-level system interacting with the environment via the Jaynes-Cummings model. In the argument above, we did not assume any specific structure of spectral density. Thus the result has good adaptability. The results are valid for an arbitrary structured environment (Lorentzian, Ohmic, waveguide spectrum, etc.), arbitrary detunings (including resonance) between the system and its environment, and arbitrary coupling strengths. For a Lorentzian structured environment, the exact expression [28] of $\gamma(t)$ was evaluated, which is related to the spectral width, the atomic free-decay rate, and the detuning between the center frequency of the spectrum and atomic transition. The existence of non-Markovian intervals for some choices of the microscopic degrees of freedom of the bath has been demonstrated [2,28].

IV. THE CASE OF THE DEPHASING MODEL

As the second example to demonstrate the equivalence of the non-Markovianity measures, we consider a two-level atom which is coupled to a reservoir of harmonic oscillators via the dephasing model. The Hamiltonian in the Schrödinger picture is taken to be

$$H = \frac{\omega_0}{2}\sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k \sigma_z(\lambda_k b_k^\dagger + \lambda_k^* b_k), \tag{24}$$

where ω_0 is the transition frequency and $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$ is the Pauli operator of the atom. ω_k and b_k are, respectively, the frequency and annihilation operators for the kth harmonic oscillator of the reservoir. The coupling strength λ_k is assumed to be complex in general. This dephasing model, which is extensively used to simulate the decoherence of a qubit coupled to its environment in quantum information science, can be solved exactly. For the initial state of the total system

$$\rho_{\mathrm{SB}}(0) = \rho(0) \otimes \rho_B, \tag{25}$$

with $\rho(0)$ the initial state of the atom and ρ_B the initial thermal equilibrium state of the reservoir, the evolution of the elements of the reduced density matrix of the atom may be written as [1]

$$\begin{gathered}\rho_{11}(t) = \rho_{11}(0), \rho_{00}(t) = \rho_{00}(0),\\ \rho_{10}(t) = \rho_{01}^*(t) = \rho_{10}(0)e^{\Gamma_p(t)}.\end{gathered} \tag{26}$$

Here the negative dephasing function $\Gamma_p(t)$ is defined as

$$\Gamma_p(t) = -\int_0^\infty d\omega J(\omega)\coth(\omega/2k_B T)\frac{1-\cos\omega t}{\omega^2}, \tag{27}$$

with $J(\omega)$ the spectral density of the reservoir, k_B the Boltzmann constant, and T the reservoir temperature in thermal equilibrium. For any pair of initial states $\rho_1(0)$ and $\rho_2(0)$ of the atom, one can easily obtain $\rho_1(t)$ and $\rho_2(t)$, and then get the dynamical trace distance as

$$D(t,\rho_{1,2}(0)) = \sqrt{a^2 + |b|^2 e^{2\Gamma_p(t)}}, \tag{28}$$

where $a = \langle 1|\rho_1(0)|1\rangle - \langle 1|\rho_2(0)|1\rangle$ and $b = \langle 1|\rho_1(0)|0\rangle - \langle 1|\rho_2(0)|0\rangle$ are the differences of the population and of coherence for the two given initial states, respectively. Inserting this trace distance into Eq. (1), we immediately get to the non-Markovian measure for this dephasing model as

$$\mathcal{N}_p = -2\int_{\gamma_{p(t)}<0} dt\,\gamma_p(t)\frac{|b|^2 e^{2\Gamma_p(t)}}{\sqrt{a^2 + |b|^2 e^{2\Gamma_p(t)}}}, \tag{29}$$

where the dephasing rate is defined as $\gamma_p(t) = -\frac{1}{2}\dot{\Gamma}_p(t)$, and based on the same reason as before, we neglect the maximization to $\mathcal{N}_p$. It shows that the non-Markovian intervals are uniquely determined by $\gamma_p(t) < 0$.

In order to calculate the expression of the second measure $\mathcal{I}^{(E)}$, we introduce an isolated ancillary atom which is initially prepared in a maximally entangled state $|\Phi\rangle$ in Eq. (12) with the system atom. According to the evolution of Eq. (26) and employing the trick from Ref. [29], one can easily obtain the time evolution of the density matrix for the system and ancillary atoms,

$$\begin{aligned}\rho_{\mathrm{SA}}(t) = \tfrac{1}{2}[&|10\rangle\langle 10| + e^{\Gamma_p(t)}|01\rangle\langle 10|\\ &+ e^{\Gamma_p(t)}|10\rangle\langle 01| + |01\rangle\langle 01|],\end{aligned} \tag{30}$$

where the first state refers to the system and the second one to the ancillary atom. It is easy to find that the concurrence for this state is $C(t) = \exp[\Gamma_p(t)]$, and thus Eq. (3) produces

$$\mathcal{I}_p^{(E)} = -4\int_{\gamma_p(t)<0} \gamma_p(t)e^{\Gamma_p(t)}dt. \tag{31}$$

Again, the non-Markovian intervals for measure $\mathcal{I}_p^{(E)}$ are also uniquely determined by $\gamma_p(t) < 0$.

Finally, let us calculate the measure $\mathcal{I}$ of Eq. (4). It is not difficult from Eq. (26) to write the master equation for the dephasing of a two-level atom as

$$\frac{d\rho}{dt} = \gamma_p(t)[\sigma_z\rho\sigma_z - \rho], \tag{32}$$

where we again have used the relation $\dot{\Gamma}_p(t) = -2\gamma_p(t)$. Following the same deduction for Eqs. (21)–(23), we obtain

$$\mathcal{I}_p = -2\int_{\gamma_p(t)<0} \gamma_p(t)dt. \tag{33}$$

Once again, the non-Markovian intervals for this measure are uniquely determined by $\gamma_p(t) < 0$.

In a word, for a two-level system with a dephasing model, we again draw the conclusion that the distributions of non-Markovian intervals for the three measures are exactly the same and uniquely determined by the condition $\gamma_p(t) < 0$. Thus the three measures are equivalent. Again, the result is valid for any spectral density of environment. As an example, we consider the Ohmic spectrum $J(\omega) = \omega e^{-\omega/\Omega}$. It has a linear increase for small frequencies and an exponential

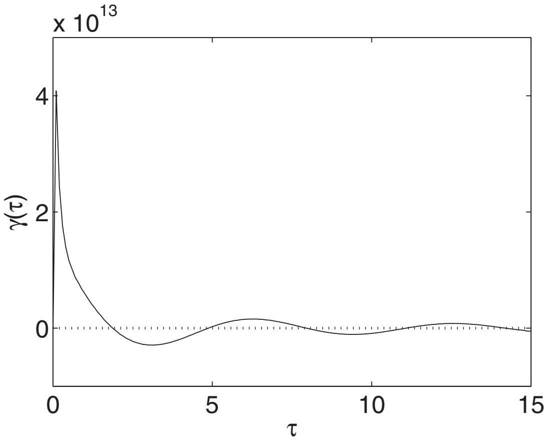

FIG. 1. Dephasing rate $\gamma_p(\tau)$ of Eq. (34) as a function of the dimensionless time τ for parameters $\tau_B = 10^{-13}$ s and $\widetilde{\Omega} = 20$. The time intervals in which $\gamma_p(\tau) < 0$ are the so-called non-Markovian intervals.

frequency cutoff at Ω. Such a form for the spectral density is typically obtained in a quantum optical regime [1]. For this special spectrum, the dephasing rate can be written as

$$\gamma_p(\tau) = \frac{1}{2\tau_B}\int_0^\infty d\widetilde{\omega} e^{-\widetilde{\omega}/\widetilde{\Omega}} \coth\left(\frac{\widetilde{\omega}}{2}\right)\sin(\widetilde{\omega}\tau), \quad (34)$$

where the thermal correlation time is defined as $\tau_B = 1/k_B T$ and $\widetilde{\omega} = \omega\tau_B$, $\widetilde{\Omega} = \Omega\tau_B$, $\tau = t/\tau_B$ are, respectively, the dimensionless frequencies and dimensionless time. In Fig. 1, we plot this dephasing rate as a function of dimensionless time for the parameters $\tau_B = 10^{-13}$ s and $\widetilde{\Omega} = 20$, where the non-Markovian intervals can be observed visibly.

In the end of this section, we would like to point out that in our discussions, the master equations (21) and (32) both have Lindblad forms with single time-dependent decay rates $\gamma(t)$ or $\gamma_p(t)$. It is well known that the dynamical maps in these cases are completely positive. However, for generalized non-Lindblad types of master equations, the CP of the dynamical maps could not always be satisfied. As a simple example, consider the memory-type master equation of two-level systems,

$$\frac{d\rho(t)}{dt} = \int_0^t d\tau K(\tau)\rho(t-\tau), \quad (35)$$

with the memory kernel $K(\tau)\rho = k(\tau)[\sigma_-\rho\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_-]$. This equation is the straightforward generalization of the common Lindblad master equation to the memory non-Markovian form. For the exponential memory function [30] $k(\tau) = Ve^{-\gamma\tau}$ with $V > 0$, the equation can be solved exactly, which gives one of the density-matrix elements $\rho_{00}(t) = \rho_{00}(0) + [1 - \Lambda(t)]\rho_{11}(0)$ with $\Lambda(t) = e^{-\gamma t/2}[\cos(\frac{\Theta t}{2}) + \frac{\gamma}{\Theta}\sin(\frac{\Theta t}{2})]$ and $\Theta = \sqrt{4V - \gamma^2}$. Obviously, when $4V > \gamma^2$, the CP of the dynamical map can be violated. This example indicates that a memory-type master equation is not always be suitable for describing the evolution of a true open system. Our conclusion about equivalence of measures naturally could not be valid in this case, because the notion of quantum trace distance is no longer valid.

V. CONCLUSIONS AND DISCUSSIONS

In conclusion, we have compared the three measures of non-Markovianity proposed in Refs. [2] and [10], respectively, for two-level systems interacting with environments via a damped Jaynes-Cummings or dephasing model. The results show that the three measures have exactly the same distribution of non-Markovian intervals, and hence are actually equivalent for detecting non-Markovianity of quantum processes. This equivalence implies that these measures well capture the intrinsic character of non-Markovianity of quantum processes in different ways. We have also found that the maximization in the BLP measure can be removed for the considered models without influencing the sensibility of the measure to detect non-Markovianity. This result, which avoids the complicated mathematical calculations, is important. The dynamical models considered here, i.e., the damped Jaynes-Cummings and dephasing models, represent two kinds of fundamental coupling forms in studying problems of open systems. And we did not assume any specific spectral density for the structured environment. Thus our results have good adaptability. They apply to an arbitrary structured environment, arbitrary detunings between a system and its environment, and arbitrary coupling strengths.

In the discussion of the damped Jaynes-Cummings or dephasing model, the corresponding master equations (21) and (32) are in Lindblad form with a single time-dependent decay rate $\gamma(t)$ [or $\gamma_p(t)$]. In these cases, we find that $\gamma(t) < 0$ or $\gamma_p(t) < 0$ is the sufficient condition for the appearance of non-Markovianity. However, for Lindblad-like equations with multiple time-dependent decay rates, the situation would become more complicated. What about the conditions of non-Markovianity? Are the three measures still equivalent? These problems deserve further investigation.

In addition, the damped Jaynes-Cummings model is based on the rotating-wave approximation, which neglects in the microscopic system-reservoir interaction Hamiltonian, the counterrotating terms responsible for the virtual exchanges of energy between system and environment. It was shown for the damped harmonic oscillator system [31] that these virtual processes may strongly affect the short time behaviors of open quantum systems and are responsible for the non-Markovianity of the system dynamics. Thus it is meaningful to study the properties of non-Markovianity when the more realistic effect of non-rotating-wave terms is taken into account.

In a word, exposing the relations between different measures of non-Markovianity in different physical systems or different dynamical models is important, which helps to expose the nature of non-Markovian dynamics and find out effective methods to correctly identify and measure the non-Markovianity. Our work is only a start. We expect further research could be stimulated.

ACKNOWLEDGMENTS

We thank M. Feng for his help. This work is supported by the National Natural Science Foundation of China

(Grant No. 11075050), the National Fundamental Research Program of China (Grant No. 2007CB925204), the Program for Changjiang Scholars and Innovative Research Team in University under Grant No. IRT0964, the Construct Program of the National Key Discipline, and Hunan Provincial Natural Science Foundation under Grant No. 11JJ7001.

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