Initial cooperative decay rate and cooperative Lamb shift of resonant atoms in an infinite cylindrical geometry

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We obtain in both the scalar and vector photon models the analytical expressions for the initial cooperative decay rate and the cooperative Lamb shift for an ensemble of resonant atoms distributed uniformly in an infinite cylindrical geometry for the case that the initial state of the system is prepared in a phased state modulated in the direction of the cylindrical axis. We find that qualitatively the scalar and vector theories give different results.

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I. INTRODUCTION

The cooperative decay rate (CDR) of a collection of identical resonant atoms was first introduced by Dicke [1] in his seminal work on superradiance from small samples. The cooperative Lamb shift (CLS) [2] is an analog of the Lamb shift, where the virtual photon emitted by one atom is reabsorbed by another. These quantities have been discussed [1–32] for various geometries in the literature, primarily for the "slab" [2,3,9,10,27] and the sphere [2,7,8,11–16,18–20,23,25,26,30]. More recently, similar discussions have appeared [5,24,28,29] for a cylindrical geometry. The cylindrical geometry is the natural one for certain types of experiment [22]. Because of the long-range nature of photon exchange, some features of the phenomena are highly geometry dependent, and we believe it useful to explore the cylindrical case in detail.

In [2] and many more recent papers, the quantities calculated refer only to the initial state of the sample, prepared in some simple way. In [1] and many papers since, it was supposed that this state would remain spatially unchanged in the course of decay; it is now evident, however, that significant change takes place [5,10,12,14–18,20,25,26,28–30]. This change can be explored by various methods, notably by finding the eigenmodes of the system [5,8–11,13,15,16,19,20,24–26,28,29].

Nevertheless, it is useful to develop formulas for the initial CDR and CLS, as both the calculations and the results are considerably simpler for this case, and the most feasible experiments detect mainly the initial burst. Although "initial" calculations have been developed quite far for the sphere [2,4,7,15,16,19,20,23,25,26,30] and ellipsoid [21], we are not aware of such a calculation for the cylinder.

We also note that much previous work, both on initial values and on time development, has been restricted to the "scalar photon model" [5–8,12–15,18–21,24,29,30] in which the dipole-dipole interaction kernel $(i/k_0^3)\exp(ik_0\Re)[(\vec{\vec{I}} - 3\hat{\Re}\hat{\Re})/\Re^3] - (\vec{\vec{I}} - \hat{\Re}\hat{\Re})(k_0^2/\Re)$ of vector electrodynamics is replaced by a long-range simplification $\exp(ik_0\Re)/ik_0\Re$. The scalar model has yielded much valid understanding; however, its use in the cylindrical geometry is equivalent to restricting the analysis to "TM" polarization in which the electric field points along the cylinder axis.

In this paper, we obtain the initial values of both CDR and CLS for an ensemble of resonant atoms uniformly distributed in an infinite cylindrical geometry in both the scalar photon and the vector photon (quantum electrodynamics) theories.

In the scalar photon theory, both initial CDR and initial CLS are obtained analytically for the case that the system is prepared in a cylindrically symmetric state in which the amplitude of the excitation is constant and the phase is modulated along the direction of the cylinder axis with wave number k.

For the system uniformly excited ($k = 0$), our two main results in the scalar photon theory are (i) that for discrete values of cylinder radius at the zeros of $J_1(k_0R)$, where R is the cylinder radius, the CDR initially vanishes, making the system metastable (compare [25]); and (ii) that the CLS changes sign at $k_0R = 1.15685$.

For $0 < k^2 < k_0^2$, both the CDR and the CLS can be obtained directly from the case ($k = 0$) by a simple scaling transformation. For $k^2 > k_0^2$, however, the initial CDR is identically zero for all values of the cylinder radius R, and the CLS takes a new form.

In the vector photon theory (realistic electrodynamics) we distinguish between two polarization states, parallel and perpendicular to the cylinder axis (TM and TE, respectively). We find analytic expressions for both cases. For TM polarization the CDR and CLS have essentially the same form as in the scalar model. For TE polarization, however, both quantities exhibit a qualitatively different dependence both on R and on k.

In Sec. II, we shall review the general expressions for the CDR and CLS: they originate from real and imaginary parts of a single analytic expression. In Sec. III, we shall compute both quantities in the scalar photon model and in Sec. IV in the vector photon model (quantum electrodynamics). Conclusions will follow in Sec. V.

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II. INITIAL COOPERATIVE DECAY RATE AND INITIAL COOPERATIVE LAMB SHIFT

As in [23], we take the initial CDR and CLS to be $2\,\mathrm{Re}(\Sigma)$ and $-\mathrm{Im}(\Sigma)$, respectively, where

$$\Sigma = -\frac{\sum_a \vec{\mathbf{P}}^*(a)\cdot\vec{\mathbf{P}}(a)}{\sum_a |\vec{\mathbf{P}}(a)|^2}, \tag{2.1}$$

in which a denotes the ath atom. In the continuum approximation, $\vec{P}(a)$ is replaced by $\vec{P}(\vec{r}_a)$ and its time development will be given by

$$\dot{P}_j(\vec{r}) = -\frac{n\wp^2 k_0^3}{\hbar}\sum_{i=1}^{3}\int d^3\vec{r}'\, G_{i,j}(\vec{r}-\vec{r}')P_i(\vec{r}'), \tag{2.2}$$

where $\wp$ and k_0 are the dipole strength and the wave number of the atomic transition, n is the number density of the atoms in the cloud, and $\wp^2k_0^3/\hbar$ is proportional to the spontaneous decay rate of excitation probability in the isolated atom, $\gamma_1 = \frac{4}{3}(\wp^2k_0^3/\hbar)$ by Fermi's golden rule. The subscripts i,j refer to the three directions of space, and the Green function $\vec{G}$ is given by

$$\vec{\vec{G}}(\vec{\Re}) = \frac{i}{k_0^3}\exp(ik_0\Re)\left[\frac{\vec{\vec{I}}-3\hat{\Re}\hat{\Re}}{\Re^3}(1-ik_0\Re) - (\vec{\vec{I}}-\hat{\Re}\hat{\Re})\frac{k_0^2}{\Re}\right], \tag{2.3}$$

where $\vec{\Re} = \vec{r}-\vec{r}'$.

In the scalar model, P_j is replaced in Eq. (2.2) by a scalar quantity b and $G_{i,j}$ by G:

$$\dot{b}(\vec{r},t) = -\frac{n\gamma_1}{2}\int d^3\vec{r}'\, G(\vec{r}-\vec{r}')b(\vec{r}',t), \tag{2.4}$$

where

$$G(\vec{\Re}) = \frac{\exp(ik_0\Re)}{ik_0\Re}. \tag{2.5}$$

(The appearance in Eq. (2.2) of $\wp^2k_0^3/\hbar = (3/4)\gamma_1 = (3/2)(\gamma_1/2)$ rather than $\gamma_1/2$ is related to the factor $2/3$ in the radiation rate from an oscillating dipole.)

For the scalar model, we shall consider only cylindrically symmetric plane-wave initial states, $b(z,t=0) = \exp(ikz)$, where the z axis is the axis of the cylinder. For the vector model, we shall consider both TM (transverse magnetic) and TE (transverse electric) polarization. In the TM case, initial $\vec{P}$ will be cylindrically symmetric, $\vec{P}(z,t=0) = \hat{e}_z\exp(ikz)$; and in the TE case $\vec{P}(z,t=0) = \hat{e}_x\exp(ikz)$.

In all cases we shall allow k to be arbitrary, but we shall single out for special attention the assumption $k=0$, which in all cases makes initial b or $\vec{P}$ constant throughout the sample. [We note that in [5,24,28] the phase factor is transverse, $\exp(ikx)$, rather than $\exp(ikz)$. Hence the only overlap between those and the present paper is in the case of $k=0$.]

The atoms are supposed to be uniformly distributed throughout a cylinder of radius R and infinite length in the z direction. Thus Eq. (2.1) can be written

$$\Sigma_S = \frac{n\gamma_1}{2\int d^3\vec{r}\ldots b^*(\vec{r})\,b(\vec{r})}\times\int d^3\vec{r}'\int d^3\vec{r}\,b(\vec{r}')G(\vec{r}-\vec{r}')b^*(\vec{r}) \tag{2.6}$$

for the scalar model, and

$$\Sigma_V = \frac{n\wp^2k_0^3}{\hbar\sum_i^3\int d^3\vec{r}\,P_i(\vec{r})P_i^*(\vec{r})}\times\int d^3\vec{r}'\int d^3\vec{r}\sum_{i,j}^{3}P_i(\vec{r}')G_{i,j}(\vec{r}-\vec{r}')P_j^*(\vec{r}) \tag{2.7}$$

for the vector model.

Strictly, the sums in the numerator and denominator of Eq. (2.1) are both infinite because of the infinite length of the cylinder, but this makes no trouble. For fixed $\vec{r}$ the integral over $\vec{r}'$ in Eq. (2.2) or Eq. (2.4) is finite, and one is left with both the numerator and denominator of Eq. (2.6) or Eq. (2.7) independent of z, so that the integrals over z can be omitted.

The decay rate (rate of change of the spatial average of the amplitude square) at $t=0$ is then

$$\Gamma_{\mathrm{CDR}} = 2\,\mathrm{Re}(\Sigma), \tag{2.8}$$

while the CLS, at $t=0$, is given by

$$\Delta\Omega_{\mathrm{CLS}} = -\mathrm{Im}(\Sigma). \tag{2.9}$$

Given the simpler mathematical structure of the scalar photon theory, we shall compute these quantities for this model first.

III. SCALAR PHOTON MODEL

With $b=\exp(ikz)$, Eq. (2.7) becomes

$$\Sigma_S = \frac{n\gamma_1}{2}\frac{1}{\pi R^2\ell}\int d^3\vec{r}\int d^3\vec{r}'\,G(\vec{r}-\vec{r}')\exp[-ik(z-z')], \tag{3.1}$$

where ℓ denotes the infinite integral $\int_{-\infty}^{\infty}dz$.

Using cylindrical coordinates $\vec{r} = (\rho,\varphi,z)$ and $\vec{r}' = (\rho',\varphi',z')$, we introduce the standard form [33] for Eq. (2.5),

$$G(\vec{r}-\vec{r}') = \frac{1}{2k_0}\sum_{nm=-\infty}^{\infty}\exp[im(\phi-\phi')]\times\int_{-\infty}^{\infty}dk_z\exp[ik_z(z-z')]J_m(k_T\rho_<)H_m^{(1)}(k_T\rho_>), \tag{3.2}$$

where k_z is a dummy wave number, $k_T = \sqrt{k_0^2-k_z^2}$, and $\rho_<,\rho_>$ are the lesser and greater of ρ,ρ'.

When we substitute Eq. (3.2) into Eq. (3.1), we encounter several simplifications: (a) the integrals over ϕ,ϕ' vanish unless $m=0$; (b) the integral over z' yields a factor $2\pi\delta(k_z-k)$; (c) the integral over k_z then results in setting $k_z\to k$, and $k_T\to\kappa = \sqrt{k_0^2-k^2}$; and (d) the integral over z is then $\int_{-\infty}^{\infty}dz$ canceling ℓ in the denominator.

The result is

$$\Sigma_S = \frac{n\gamma_1}{2}\frac{1}{\pi R^2}\frac{1}{2k_0}2\pi\int_0^{2\pi}d\phi\int_0^{2\pi}d\phi'\int_0^R\rho\,d\rho \times\int_0^R\rho'\,d\rho'\,J_0(\kappa\rho_<)H_0^{(1)}(\kappa\rho_>). \tag{3.3}$$

Since the integrand is symmetric under $\rho\leftrightarrow\rho'$, we may restrict the integral in ρ' to $\rho'\leqslant\rho$ and compensate by multiplying by 2. This gives

$$\Sigma_S = \frac{n\gamma_1}{2}\frac{1}{\pi R^2}\frac{1}{2k_0}(2\pi)^3 2\int_0^R\rho\,d\rho\,H_0^{(1)}(\kappa\rho) \times\int_0^\rho\rho'\,d\rho'\,J_0(\kappa\rho') = \frac{n\gamma_1}{2}\frac{1}{\pi R^2}\frac{1}{2k_0}(2\pi)^3 2 \times\int_0^R\rho\,d\rho\,H_0^{(1)}(\kappa\rho)\frac{1}{\kappa}\rho J_1(\kappa\rho). \tag{3.4}$$

Now by differentiating

$$\frac{d}{dw}\left[wH_1^{(1)}(w)wJ_1(w)\right] = w^2\left[H_0^{(1)}(w)J_1(w)+H_1^{(1)}(w)J_0(w)\right] = 2w^2H_0^{(1)}(w)J_1(w)+w^2\left[H_1^{(1)}(w)J_0(w)-H_0^{(1)}(w)J_1(w)\right] \tag{3.5}$$

and applying the Wronskian identity

$$H_0^{(1)}(w)J_1(w)-H_1^{(1)}(w)J_0(w)=\frac{2i}{\pi w}, \tag{3.6}$$

we find that

$$\int_0^W dw\,2w^2H_0^{(1)}(w)J_1(w) = W^2H_1^{(1)}(W)J_1(W)+i\frac{W^2}{\pi}. \tag{3.7}$$

Hence

$$\Sigma_S = \frac{n\gamma_1}{2}\frac{1}{\pi R^2}\frac{1}{2k_0}(2\pi)^3\frac{1}{\kappa}\frac{1}{\kappa^3}(\kappa R)^2\left[H_1^{(1)}(\kappa R)J_1(\kappa R)+\frac{i}{\pi}\right] = \frac{n\gamma_1}{2}\frac{4\pi^2}{k_0^3}\tilde{\Sigma}_S, \tag{3.8}$$

where

$$\tilde{\Sigma}_S = \frac{1}{\alpha}\left[H_1^{(1)}(k_0R\sqrt{\alpha})J_1(k_0R\sqrt{\alpha})+\frac{i}{\pi}\right], \tag{3.9}$$

with $\alpha = 1-(k^2/k_0^2)$.

In the "symmetric" (uniform) initial state with $k=0$, we have $\kappa=k_0$ and Eq. (3.9) becomes

$$\tilde{\Sigma}_S(k=0) = \left[J_1(k_0R)H_1^{(1)}(k_0R)+\frac{i}{\pi}\right] = \left\{J_1^2(k_0R)+i\left[J_1(k_0R)N_1(k_0R)+\frac{1}{\pi}\right]\right\}. \tag{3.10}$$

In Figs. 1 and 2, we plot the real and imaginary parts of $\tilde{\Sigma}_s(k=0)$ as a function of $u=k_0R$.

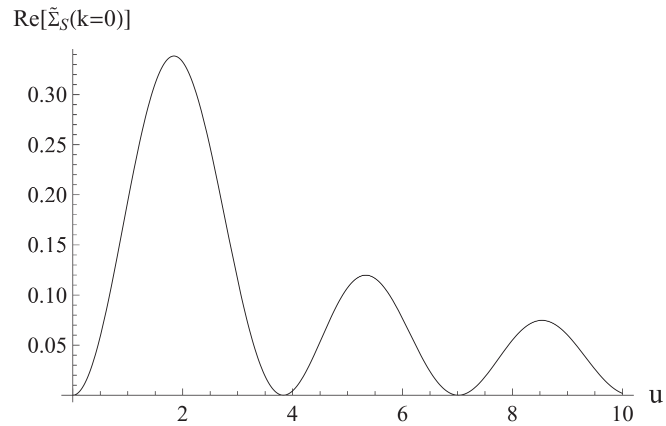

FIG. 1. $\mathrm{Re}[\tilde{\Sigma}_S(k=0)]$ is plotted as a function of u.

We note that

(1) The value of the CDR vanishes at the zeros of $J_1(u)$. For cylinders with these values of the radius, the system is metastable, i.e., the decay rate is initially zero and builds up slowly with time.

(2) The CLS changes sign at $k_0R = 1.15685$.

(3) For $k_0R\ll 1$,

$$\mathrm{Re}[\Sigma_S(k=0)] \approx \frac{n\gamma_1}{2}\frac{\pi^2}{k_0^3}(k_0R)^4, \tag{3.11}$$

$$\mathrm{Im}[\Sigma_S(k=0)] \approx \frac{n\gamma_1}{2}\frac{\pi}{2k_0^3}[-1+4\gamma-4\ln(2) +4\ln(k_0R)](k_0R)^2, \tag{3.12}$$

where γ is the Euler gamma constant.

(4) For $0<\alpha<1$, $0<k^2<k_0^2$, $\alpha\tilde{\Sigma}_S$ is a universal function of $u\sqrt{\alpha}$. Therefore Figs. 1 and 2 remain valid with simultaneous rescaling of ordinate and abscissa.

For $\alpha<0$, κ is imaginary, and Bessel functions of imaginary arguments appear in Eq. (3.9). Recalling the relations

$$I_n(x) = i^{-n}J_n(ix), \tag{3.13}$$

$$K_n(x) = \frac{\pi i}{2}\exp\left(in\frac{\pi}{2}\right)H_n^{(1)}(ix), \tag{3.14}$$

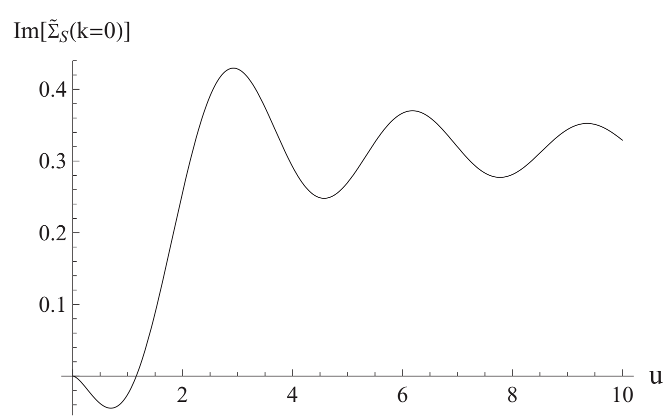

FIG. 2. $\mathrm{Im}[\tilde{\Sigma}_S(k=0)]$ is plotted as a function of u.

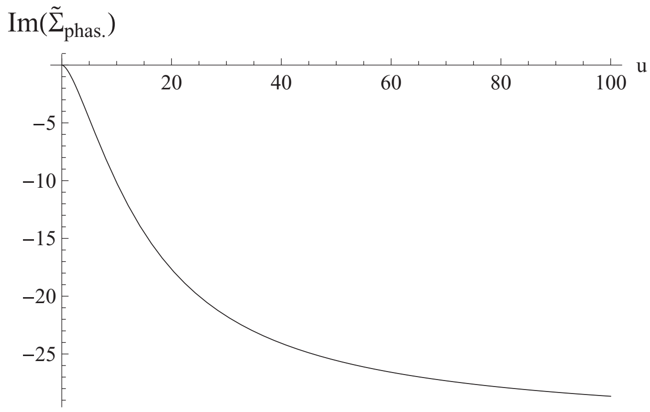

FIG. 3. $\text{Im}(\tilde{\Sigma}_{phas}) = \text{Im}(\tilde{\Sigma}_S)$ is plotted as function of u, for $\alpha = -0.01$.

we find that $\tilde{\Sigma}_S$, in this case, is purely imaginary and is given by

$$\tilde{\Sigma}_S = -\frac{i}{|\alpha|\pi}[1 - 2K_1(k_0R\sqrt{|\alpha|})I_1(k_0R\sqrt{|\alpha|})]. \tag{3.15}$$

We plot in Fig. 3, for $\alpha < 0$, $\text{Im}(\tilde{\Sigma}_S)$ as a function of u, for a fixed value of α, and in Fig. 4, $\text{Im}(\tilde{\Sigma}_S)$ as a function of α, for a fixed value of u.

In concluding this section, we note that the vanishing of CDR for $k = 0$ at zeros of $J_1(k_0R)$ is similar to its behavior for the sphere, where the vanishing is at zeros of $j_1(k_0R)$. This latter behavior was noted in [25], where the instantaneous decay rate was also followed through time by the eigenfunction method. It was shown that for radii near these critical values, the decay rate begins small and rises to a peak only after a lapse of time comparable to the half-life at other radii.

The vanishing of CDR at all radii in the regime $k > k_0$ is equivalent to the well-known phenomenon of internal reflection in a dielectric waveguide. Radiation cannot escape because the wave equation in vacuo requires an imaginary wave number in the radial direction.

The quantity $\text{Im}(\Sigma)$ cannot be observed in this regime as a frequency shift in emitted radiation, since none is emitted. Nevertheless, the cooperative Lamb shift is really present in the evolution of the atomic phase, and may be amenable to more sophisticated measurements.

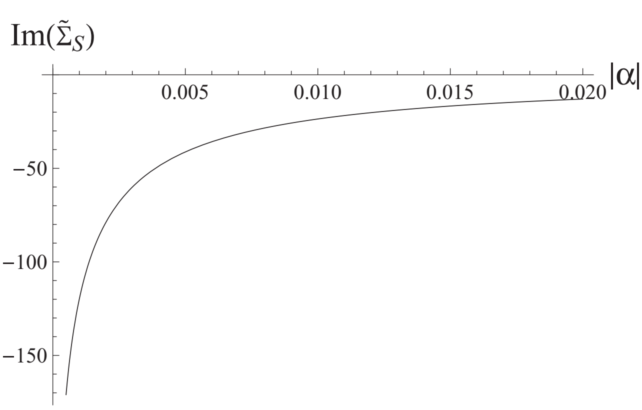

FIG. 4. $\text{Im}(\tilde{\Sigma}_S)$ is plotted as function of $|\alpha|$, $\alpha < 0$, for $u = 12\pi$.

IV. VECTOR PHOTON MODEL

The Green's function in the vector photon model, given in Eq. (2.3), can also be written as

$$G_{i,j}(\vec{\Re}) = G(\vec{\Re})\delta_{i,j} + \frac{1}{k_0^2}\frac{\partial}{\partial \Re_i}\frac{\partial}{\partial \Re_j}G(\vec{\Re}), \tag{4.1}$$

where $G(\vec{\Re})$ is the scalar Green's function, Eq. (2.2).

We shall use this form of the electrodynamic Green's function in the computations in this section. We shall consider two cases (i) TM (initial polarization parallel to the axis of the cylinder) and (ii) TE (initial polarization transverse to the axis of the cylinder).

(i) TM [$\vec{P}(t=0) = \hat{e}_z \exp(ikz)$]:

Equations (2.2) and (4.1) give at $t = 0$,

$$\dot{P}_z = -\frac{n\wp^2k_0^3}{\hbar}\int d^3\vec{r}'\left[G(\vec{r}-\vec{r}') + \frac{1}{k_0^2}\frac{\partial}{\partial z'}\frac{\partial}{\partial z'}G(\vec{r}-\vec{r}')\right]\exp(ikz'), \tag{4.2}$$

$$\dot{P}_x = -\frac{n\wp^2k_0^3}{\hbar}\int d^3\vec{r}'\frac{\partial}{k_0^2\partial x'}\frac{\partial}{\partial z'}[G(\vec{r}-\vec{r}')\exp(ikz')]. \tag{4.3}$$

Only $\dot{P}_z$ contributes to Σ, but $\dot{P}_x$may also be of experimental interest. We shall show presently that $\dot{P}_x$vanishes at $t = 0$.

The first term in Eq. (4.2) gives the same contribution to $\tilde{\Sigma}_{\text{TM}}$ as in the scalar photon theory, while the second term gives the same contribution multiplied by $-k^2/k_0^2$ by a double integration by parts since $G$and dG/dz' both vanish at $z' = \pm\infty$. Therefore this expression for Σ_{TM} is identical, apart from a prefactor, to $\alpha\Sigma_S$.

(ii) TE[$\vec{P}(t=0) = \hat{e}_x \exp(ikz)$]:

Equations (2.2) and (4.1) give

$$\dot{P}_x = -\frac{n\wp^2k_0^3}{\hbar}\int d^3\vec{r}'\left[G(\vec{r}-\vec{r}') + \frac{\partial}{k_0\,\partial x'}\frac{\partial}{k_0\,\partial x'}G(\vec{r}-\vec{r}')\right]\exp(ikz'), \tag{4.4}$$

$$\dot{P}_z = -\frac{n\wp^2k_0^3}{\hbar}\int d^3\vec{r}'\frac{\partial}{k_0^2\,\partial x'}\frac{\partial}{\partial z'}[G(\vec{r}-\vec{r}')\exp(ikz')]. \tag{4.5}$$

Here only $\dot{P}_x$contributes to Σ. This time the initial value of $\dot{P}_z$ vanishes (see below), because Eq. (4.5) is the same as Eq. (4.3).

We now calculate Eqs. (4.3) and (4.5), and the second term in Eq. (4.4). Expressing $\partial/\partial x'$ in cylindrical coordinates,

$$\frac{\partial}{\partial x'} = \hat{e}_{x'}\cdot\vec{\nabla}' = \cos(\phi')\frac{\partial}{\partial\rho'} - \frac{\sin(\phi')}{\rho'}\frac{\partial}{\partial\phi'} \tag{4.6}$$

and

$$\frac{\partial^2}{\partial x'^2} = \cos^2(\phi')\frac{\partial^2}{\partial\rho'^2} + \frac{\sin^2(\phi')}{\rho'}\frac{\partial}{\partial\rho'} + \frac{2\sin(\phi')\cos(\phi')}{\rho'^2}\frac{\partial}{\partial\phi'} - \frac{2\sin(\phi')\cos(\phi')}{\rho'}\frac{\partial^2}{\partial\rho'\,\partial\phi'} + \frac{\sin^2(\phi')}{\rho'^2}\frac{\partial^2}{\partial\phi'^2}. \tag{4.7}$$

In applying Eqs. (4.6) and (4.7) to G as given by Eq. (3.2), we may first replace $\partial/\partial\phi'$ by a factor $-im$, and then we see that all terms with $m \neq 0$ vanish on integration over ϕ'. There remain the terms with $m = 0$, and these vanish if they originally contained $\partial/\partial\phi'$. Consequently, only the first term of Eq. (4.6) and the first two terms of Eq. (4.7) survive with $\cos(\phi')$ replaced by 0 and both $\cos^2(\phi')$ and $\sin^2(\phi')$ replaced by1/2, giving finally

$$\frac{\partial}{\partial x'} \to 0, \tag{4.8a}$$

$$\frac{\partial^2}{\partial x'^2} \to \frac{1}{2}\left(\frac{\partial^2}{\partial \rho'^2} + \frac{1}{\rho'}\frac{\partial}{\partial \rho'}\right) = \frac{1}{2\rho'}\frac{\partial}{\partial \rho'}\left(\rho'\frac{\partial}{\partial \rho'}\right) \tag{4.8b}$$

after the integration over ϕ'. From Eq. (4.8a) we see that Eqs. (4.3) and (4.5) vanish, as stated above.

Now, applying Eq. (4.1) with Eq. (4.8) to Eq. (3.3), we see that the integral

$$\begin{aligned} I_S &= \int_0^R \rho\, d\rho \int_0^R \rho'\, d\rho'\, J_0(\kappa\rho_<) H_0^{(1)}(\kappa\rho_>) \\ &= \frac{R^2}{\kappa^2}\left[H_1^{(1)}(\kappa R) J_1(\kappa R) + \frac{i}{\pi}\right] \end{aligned} \tag{4.9}$$

[see Eqs. (3.4) and (3.8)] must be replaced in the TM case by

$$I_{\rm TM} = \left(1 - \frac{k^2}{k_0^2}\right) I_S = \alpha I_S. \tag{4.10a}$$

and in the TE case by

$$I_{\rm TE} = I_S + \Delta I, \tag{4.10b}$$

where

$$\begin{aligned} \Delta I = \frac{1}{k_0^2} &\int_0^R \rho\, d\rho \int_0^R \rho'\, d\rho' \frac{1}{2\rho'}\frac{\partial}{\partial \rho'} \\ &\times \left\{\rho' \frac{\partial}{\partial \rho'}\left[J_0(\kappa\rho_<) H_0^{(1)}(\kappa\rho_>)\right]\right\}. \end{aligned} \tag{4.11a}$$

Noting that the integration over ρ' is that of a perfect derivative, then

$$\Delta I = \frac{1}{2k_0^2}\int_0^R \rho\, d\rho \left\{\left(\rho'\frac{\partial}{\partial \rho'}\left[J_0(\kappa\rho_<) H_0^{(1)}(\kappa\rho_>)\right]\right)\right\}_{\rho'=0}^{\rho'=R}, \tag{4.11b}$$

which further simplifies to

$$\begin{aligned} \Delta I &= \frac{1}{2k_0^2}\int_0^R \rho\, d\rho \left\{R J_0(\kappa\rho)\frac{\partial}{\partial \rho'}\left[H_0^{(1)}(\kappa\rho')\right]\right\}_{\rho'=R} \\ &= -\frac{\kappa R}{2k_0^2} H_1^{(1)}(\kappa R)\int_0^R \rho\, d\rho\, J_0(\kappa\rho) \\ &= -\frac{R^2}{2k_0^2} H_1^{(1)}(\kappa R) J_1(\kappa R). \end{aligned} \tag{4.11c}$$

We now can write Σ as

$$\Sigma_{\rm TM} = \frac{4\pi^2 n \wp^2}{\hbar}\tilde{\Sigma}_{\rm TM}, \qquad \Sigma_{\rm TE} = \frac{4\pi^2 n \wp^2}{\hbar}\tilde{\Sigma}_{\rm TE}, \tag{4.12}$$

where [comparing Eqs. (4.10a) and (4.11c) with Eq. (4.9)]

$$\tilde{\Sigma}_{\rm TM} = \alpha\tilde{\Sigma}_S = \left[H_1^{(1)}(k_0 R\sqrt{\alpha}) J_1(k_0 R\sqrt{\alpha}) + \frac{i}{\pi}\right], \tag{4.13}$$

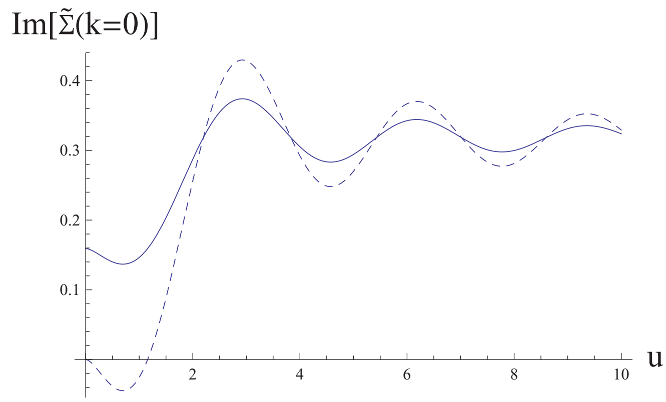

FIG. 5. (Color online) The CLS for initially TE (solid line) and for a system initially TM (dashed line) are plotted as function of u ($k = 0$, $\alpha = 1$).

$$\tilde{\Sigma}_{\rm TE} = \frac{1}{\alpha}\left[\left(1 - \frac{\alpha}{2}\right) H_1^{(1)}(k_0 R\sqrt{\alpha}) J_1(k_0 R\sqrt{\alpha}) + \frac{i}{\pi}\right]. \tag{4.14}$$

For $\alpha > 0$, the ratio of the real parts of $\tilde{\Sigma}$ in the two cases is

$$\frac{{\rm Re}(\tilde{\Sigma}_{\rm TE})}{{\rm Re}(\tilde{\Sigma}_{\rm TM})} = \frac{1}{\alpha}\left(1 - \frac{\alpha}{2}\right). \tag{4.15}$$

For $\alpha > 0$, we have

$${\rm Im}(\tilde{\Sigma}_{\rm TM}) = \left[N_1(k_0 R\sqrt{\alpha}) J_1(k_0 R\sqrt{\alpha}) + \frac{1}{\pi}\right], \tag{4.16}$$

but

$${\rm Im}(\tilde{\Sigma}_{\rm TE}) = \frac{1}{\alpha}\left[\left(1 - \frac{\alpha}{2}\right) N_1(k_0 R\sqrt{\alpha}) J_1(k_0 R\sqrt{\alpha}) + \frac{1}{\pi}\right]. \tag{4.17}$$

In Fig. 5 we plot ${\rm Im}(\tilde{\Sigma}_{\rm TM})$ and ${\rm Im}(\tilde{\Sigma}_{\rm TE})$ as a function of $u = k_0 R$, for $k = 0$. We note that at $u = 0$, ${\rm Im}[\tilde{\Sigma}_{\rm TE}(k = 0)] \neq 0$ because the leading cancellation between the two terms of Eq. (3.15) fails in Eq. (4.17). As $\alpha \to 0$, the solid curve will resemble the dotted curve, except for the scaling factor α. For any $\alpha > 0$, the minima and maxima of ${\rm Im}(\tilde{\Sigma}_{\rm TE})$ will fall at the same values of u as those of ${\rm Im}(\tilde{\Sigma}_{\rm TM})$; but in ${\rm Im}(\tilde{\Sigma}_{\rm TE})$ the first minimum is not negative for $\alpha < \alpha_{\rm crit} \approx 0.246$.

For $\alpha < 0$, we obtain

$${\rm Re}[\tilde{\Sigma}_{\rm TM}(\alpha < 0)] = {\rm Re}[\tilde{\Sigma}_{\rm TE}(\alpha < 0)] = 0, \tag{4.18}$$

$${\rm Im}[\tilde{\Sigma}_{\rm TM}(\alpha < 0)] = \frac{1}{\pi}[1 - 2K_1(k_0 R\sqrt{|\alpha|}) I_1(k_0 R\sqrt{|\alpha|})], \tag{4.19}$$

$$\begin{aligned} &{\rm Im}[\tilde{\Sigma}_{\rm TE}(\alpha < 0)] \\ &\quad = -\frac{1}{|\alpha|\pi}\left[1 - 2\left(1 + \frac{|\alpha|}{2}\right) K_1(k_0 R\sqrt{|\alpha|}) I_1(k_0 R\sqrt{|\alpha|})\right]. \end{aligned} \tag{4.20}$$

We note that $\mathrm{Im}(\tilde{\Sigma}_{\mathrm{TM}})$ is the same as$-|\alpha|\mathrm{Im}(\tilde{\Sigma}_S)$ [see Eq. (3.15)], but $\mathrm{Im}(\tilde{\Sigma}_{\mathrm{TE}})$ is not only scaled differently but also displaced, in the same way as for $\alpha > 0$.

V. CONCLUSION

As expected, the detailed form of CDR and CLS for the cylinder is quite different from that found for the slab or the sphere. Of interest is the vanishing of initial CDR at zeros of $J_1(k_0R)$, analogous to behavior in the sphere at zeros of j_1, and the vanishing of CDR at all radii for $k^2 > k_0^2$, which has no analog in the slab or sphere. We note that the CDR in TE is discontinuous across the resonance in k: when k approaches k_0 from below ($\alpha \to 0^+$), $\mathrm{Re}(\tilde{\Sigma}_{\mathrm{TE}}) \to \frac{1}{4}(k_0R)^2$ as seen from Eq. (3.9) or Eqs. (4.13) and (4.14). The k-dependent ratio of CDR between the two polarizations, seen in Eq. (4.15), may not be too difficult to observe.

It was seen in Eqs. (4.3) and (4.5) that in the vector model cross polarization (P_z from initial P_x, or P_x from initial P_z) does not occur in leading order in elapsed time t. At later times we expect this to remain true because of reflection symmetry in the y-z plane, even though individual eigenmodes no longer separate into TM and TE when $k^2 > 0$. Likewise, if the initial phase factor is not $\exp(ikz)$but $\exp(ikx)$ as in [5,24,28], cross polarization should remain zero at all times by reflection symmetry in the x-z plane. But cross polarization may be expected at later times when the initial phase has both an x and a z dependence.

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