PHYSICAL REVIEW A **84**, 023830 (2011)

Geometrical interpretation of optical absorption

J. J. Monzón,[1] A. G. Barriuso,[1] L. L. Sánchez-Soto,[1] and J. M. Montesinos-Amilibia[2]
[1]*Departamento de Óptica, Facultad de Física, Universidad Complutense, E-28040 Madrid, Spain*
[2]*Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad Complutense, E-28040 Madrid, Spain*
(Received 15 March 2011; published 17 August 2011)

We reinterpret the transfer matrix for an absorbing system in very simple geometrical terms. In appropriate variables, the system appears as performing a Lorentz transformation in a (1 + 3)-dimensional space. Using homogeneous coordinates, we map that action on the unit sphere, which is at the realm of the Klein model of hyperbolic geometry. The effects of absorption appear then as a loxodromic transformation, that is, a rhumb line crossing all the meridians at the same angle.

DOI: 10.1103/PhysRevA.84.023830 PACS number(s): 42.25.Bs, 03.30.+p, 78.20.Ci, 78.67.Pt

I. INTRODUCTION

One-dimensional continuous models are very useful to provide a detailed account of the behavior of a variety of systems [1,2]. Frequently, linear approximations are enough to capture the essential features with sufficient accuracy. The nature of the actual particles, or states, or elementary excitations, as they may be variously called, is irrelevant for many purposes: there are always two input and two output channels related by a 2×2 transfer matrix [3–6]. Such a matrix is just a compact way of setting out the integration of the differential equations involved in the model with the pertinent boundary conditions; this is what makes the method so effective.

However, a quick look at the literature immediately reveals the very different backgrounds and habits in which the transfer matrix is used and the very little "cross talk" between them, which sometimes leads to confusion. To fill this gap, in recent years a number of geometrical concepts have been introduced to display the topic in a unifying mathematical scenario that can be clarifying for the different applications [7–10].

In this paper we continue this programm and extend the theory to absorbing systems. In that case, the transfer matrix is an element of the group SL(2,$\mathbb{C}$) of unimodular complex 2×2 matrices. The algebraic structure of these matrices is very appealing; in particular, SL(2,$\mathbb{C}$) is locally isomorphic to the Lorentz group SO(1,3) in (1 + 3) dimensions [11]. Apart from a relativistic presentation of the topic, which has interest in its own, this gives rise to a bilinear transformation that nicely depicts the physics in the complex plane. Hyperbolic geometry allows then for an exhaustive classification of these actions. For the problem at hand, it turns out that the system induces a loxodromic transformation when viewed in the proper variables, so it is exactly a rhumb line in standard navigation (a line on the surface of a sphere that always makes an equal angle with every meridian). The consequences of this remarkable fact are fully explored in what follows.

II. TRANSFER MATRIX: RELATIVISTIC VARIABLES

We restrict our analysis to the optical transfer matrix, although the discussion is independent of the example employed and could be easily translated to electronic states, plasmons, long wave acoustic or piezoelectric modes, or whatever.

To simplify the details as much as possible, we look at the simplest system we can imagine: an isolated absorbing (homogeneous and isotropic) film embedded between two identical ambient a and substrate s media, which we take to be air. The generalization to more involved structures is rather obvious. A monochromatic, linearly polarized plane wave falls from the ambient with amplitude $E_{+,a}$, as well as another plane wave of the same frequency and polarization, and amplitude $E_{-,s}$ from the substrate (see Fig. 1 for a scheme of all the fields). Without loss of generality, we assume normal incidence, so we do not need to make a distinction between p and s polarizations.

As a result of multiple reflections in the interfaces of the film, we have a backward-traveling plane wave in the ambient, denoted $E_{-,a}$, and a forward-traveling plane wave in the substrate, denoted $E_{+,s}$. If the field amplitudes are treated as a column vector

$$\mathbf{E} = \begin{pmatrix} E_- \\ E_+ \end{pmatrix}, \tag{2.1}$$

the amplitudes at both the ambient and the substrate sides are related by

$$\mathbf{E}_a = \mathbf{M}\,\mathbf{E}_s, \tag{2.2}$$

where $\mathbf{M}$ is the transfer matrix. For the case in point it can be shown that [12]

$$\mathbf{M} = \begin{pmatrix} (T^2 - R^2)/T & R/T \\ -R/T & 1/T \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.3}$$

the complex numbers R and T being, respectively, the reflection and transmission coefficients

$$R = \frac{r[1 - \exp(-i2\delta)]}{1 - r^2 \exp(-i2\delta)}, \quad T = \frac{(1 - r^2)\exp(-i\delta)}{1 - r^2 \exp(-i2\delta)}. \tag{2.4}$$

Here, r is the Fresnel reflection coefficient for the interface of air and film, and $\delta = 2\pi N d/\lambda$ is the normal-incidence film phase thickness, where N is the complex refractive index, d the film thickness, and λ the wavelength in vacuo. Note that in this symmetric configuration the coefficients R and T do not depend on which side (ambient or substrate) the light is impinging on, and, consequently, we have the constraint $b = -c$ in $\mathbf{M}$. Additionally, since the ambient and substrate are identical, we have that det $\mathbf{M} = +1$, so the transfer matrices belong to the group SL(2,$\mathbb{C}$).

 ©2011 American Physical Society

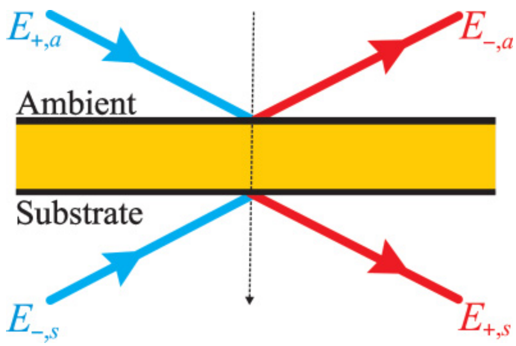

FIG. 1. (Color online) Scheme of the input (blue arrows) and output (red arrows) fields in an absorbing film. The subscripts + and − refer to the positive and negative directions of the Z axis, while a and s refer to the media (ambient and substrate) in which the field propagates. We assume normal incidence.

To proceed further, let us construct the matrices

$$\mathscr{E} = \begin{pmatrix} |E_-|^2 & E_-^* E_+ \\ E_- E_+^* & |E_+|^2 \end{pmatrix} \tag{2.5}$$

for both a and s variables. They are quite reminiscent of the coherence matrix in optics or the density matrix in quantum mechanics [13]. One can readily verify that

$$\mathscr{E}_a = \mathbf{M}\, \mathscr{E}_s\, \mathbf{M}^\dagger \,, \tag{2.6}$$

so they transform under the action of the transfer matrix $\mathbf{M}$ by conjugation.

Let now σ^μ (the Greek indices run from 0 to 3) be the set of four Hermitian matrices $\sigma^0 = \mathbb{1}$ (the identity) and $\boldsymbol{\sigma}$ (the standard Pauli matrices). They constitute a natural basis of the vector space of 2×2 complex matrices, and we can define the coordinates e^μ with respect to that basis as

$$e^\mu = \tfrac{1}{2}\,\mathrm{Tr}(\mathscr{E}\sigma^\mu)\,, \tag{2.7}$$

obtaining

$$\begin{aligned} e^0 &= \tfrac{1}{2}(|E_-|^2 + |E_+|^2)\,, & e^1 &= \mathrm{Re}(E_-^* E_+)\,, \\ e^2 &= \mathrm{Im}(E_-^* E_+)\,, & e^3 &= \tfrac{1}{2}(|E_-|^2 - |E_+|^2)\,. \end{aligned} \tag{2.8}$$

As a result, the conjugation (2.6) induces a transformation on the variables e^μ of the form

$$e_a^\mu = \Lambda^\mu_\nu(\mathbf{M})\, e_s^\nu \,, \tag{2.9}$$

where Λ^μ_ν can be found to be

$$\Lambda^\mu_\nu(\mathbf{M}) = \tfrac{1}{2}\,\mathrm{Tr}(\sigma^\mu \mathbf{M} \sigma_\nu \mathbf{M}^\dagger)\,, \tag{2.10}$$

and it turns out to be a Lorentz transformation [14]. Clearly, the matrices $\mathbf{M}$ and $-\mathbf{M}$ generate the same Λ, so this homomorphism is two-to-one.

The variables e^μ are coordinates in a Minkowskian (1 + 3)-dimensional space and the action of the system can be seen as a Lorentz transformation. To give a physical feeling of this point, let us recall that any matrix $\mathbf{M} \in \mathrm{SL}(2,\mathbb{C})$ can be decomposed in one and only one way in the form $\mathbf{M} = \mathbf{HU}$, where $\mathbf{H}$ is a positive-definite Hermitian ("modulus") and $\mathbf{U}$ is unitary ("phase"). Under the homomorphism (2.10), $\mathbf{H}$ generates a boost of velocity $\boldsymbol{\beta} = \mathbf{v}/c$, while $\mathbf{U}$ induces a pure spatial rotation. The parameters of the boost and the rotation can be easily related to those of $\mathbf{M}$; the explicit expression can be found, e.g., in Ref. [15].

To gain physical insights, let us focus for a moment on a lossless film, for which such a decomposition appears particularly transparent. The transfer matrix (2.3) reduces then to

$$\mathbf{M}_{\mathrm{lossless}} = \begin{pmatrix} 1/T^* & R/T \\ R^*/T^* & 1/T \end{pmatrix}, \tag{2.11}$$

so it belongs to the group SU(1,1). By simple inspection it can be checked that $\mathbf{M}_{\mathrm{lossless}}$ can be decomposed as

$$\mathbf{M}_{\mathrm{lossless}} = \mathbf{HU} = \begin{pmatrix} 1/|T| & R/|T| \\ R^*/|T| & 1/|T| \end{pmatrix} \begin{pmatrix} e^{i\tau} & 0 \\ 0 & e^{-i\tau} \end{pmatrix}, \tag{2.12}$$

where we have written

$$R = |R| \exp(i\rho), \quad T = |T| \exp(i\tau)\,, \tag{2.13}$$

so ρ and τ are the phase changes in reflection and transmission, respectively. The unitary component $\mathbf{U}$ generates, under the homomorphism (2.10), the matrix

$$\mathbf{R}(\mathbf{U}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\tau) & \sin(2\tau) & 0 \\ 0 & -\sin(2\tau) & \cos(2\tau) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.14}$$

that is, a rotation in the plane e^1-e^2 of angle twice the phase of the transmission coefficient.

The Hermitian component $\mathbf{H}$ generates the boost

$$\mathbf{L}(\mathbf{H}) = \begin{pmatrix} \gamma & -\gamma\beta\cos\rho & -\gamma\beta\sin\rho & 0 \\ -\gamma\beta\cos\rho & 1 + (\gamma - 1)\cos^2\rho & (\gamma - 1)\cos\rho\sin\rho & 0 \\ -\gamma\beta\sin\rho & (\gamma - 1)\cos\rho\sin\rho & 1 + (\gamma - 1)\sin^2\rho & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.15}$$

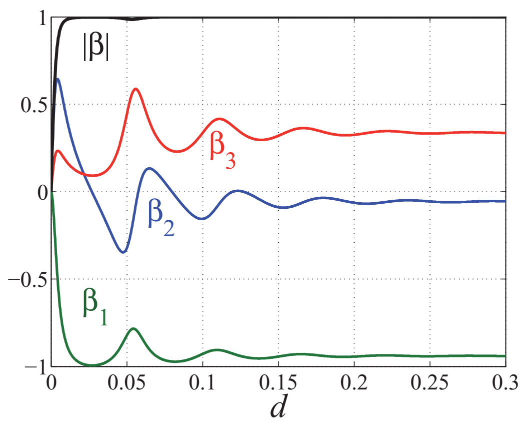

FIG. 2. (Color online) Components of the velocity $\boldsymbol{\beta} = (\beta_1,\beta_2,\beta_3)$ and its modulus $|\boldsymbol{\beta}|$ of the boost generated by a film of germanium when its thickness varies from 0 to 0.3 μm. We work at $\lambda = 0.6199$ μm, and the complex refractive index is $N = 5.588 - 0.933\,i$.

where the velocity $\beta = |\boldsymbol{\beta}|$ and the relativistic factor $\gamma = 1/\sqrt{1-\beta^2}$ are given by

$$\beta = \frac{2|R|}{1+|R|^2}, \quad \gamma = \frac{1+|R|^2}{1-|R|^2}. \tag{2.16}$$

The matrix $\mathbf{L}(\mathbf{H})$ is then a boost to a reference frame moving with a constant velocity β in the plane e^1-e^2, in a direction forming a counterclockwise angle ρ with the axis e^1.

If, as it is usual, we introduce the rapidity ζ from the relations

$$\beta = \tanh\zeta, \quad \gamma = \cosh\zeta, \tag{2.17}$$

we have the following very appealing identification of the reflection and transmission coefficients with the parameters of the Lorentz transformation:

$$R = \tanh(\zeta/2)\exp(i\rho), \quad T = \operatorname{sech}(\zeta/2)\exp(i\tau). \tag{2.18}$$

Therefore, $|R| = \tanh(\zeta/2)$ behaves as a velocity (and they add accordingly), while $|T|$ behaves as $1/\gamma$.

As an illustrative example, in Fig. 2 we have plotted the components of the boosts generated by a film of germanium when its thickness d changes continuously from 0 to 0.3 μm. First of all, $|\boldsymbol{\beta}|$ is almost 1 (except for values close to $d=0$), which shows, perhaps unexpectedly, that the film operates at the ultrarelativistic limit. The components of $\boldsymbol{\beta}$ show damped oscillations, tending fast to constant values as the thickness increases. When no absorption is present, the coordinate e^3 (that is, the semidifference of the fluxes at each side of the film) remains invariant and $\beta_3 = 0$, while the other two components depend periodically on d. In consequence, the appearance of β_3 is a clear evidence of absorption.

III. GEOMETRY OF THE TRANSFER MATRIX

The value of the interval (for both a and s) is

$$(e^0)^2 - (e^1)^2 - (e^2)^2 - (e^3)^2 = 0, \tag{3.1}$$

so it is lightlike. Equation (3.1) defines the light cone in special relativity. Since it is impossible to plot it in a three-dimensional Euclidean world, we consider the intersection of the hyperplane $e^0 = 1$ and the future light cone (i.e., the one with $e^0 > 0$), which gives the unit sphere $\mathscr{S}_2$. This can be alternatively characterized by using homogeneous coordinates

$$\mathfrak{e}^1 = \frac{e^1}{e^0}, \quad \mathfrak{e}^2 = \frac{e^1}{e^0}, \quad \mathfrak{e}^3 = \frac{e^1}{e^0}, \tag{3.2}$$

in terms of which (3.1) can be recast as

$$(\mathfrak{e}^1)^2 + (\mathfrak{e}^2)^2 + (\mathfrak{e}^3)^2 = 1. \tag{3.3}$$

In this way we have established a correspondence between points (e^0,e^1,e^2,e^3) and points $(\mathfrak{e}^1,\mathfrak{e}^2,\mathfrak{e}^3)$ in $\mathscr{S}_2$. This also transfers the Minkowski metric $[ds^2 = (de^0)^2 - (de^1)^2 - (de^2)^2 - (de^3)^2]$ to $\mathscr{S}_2$, giving the projective or Klein model of the hyperbolic geometry [16].

Moreover, we can map the points of $\mathscr{S}_2$ onto the complex plane $\mathbb{C}$ (identified with the plane $\mathfrak{e}^3 = 0$) by the stereographic projection from the north pole. The point $(\mathfrak{e}^1,\mathfrak{e}^2,\mathfrak{e}^3)$ becomes then

$$z = \frac{\mathfrak{e}^1 + i\mathfrak{e}^2}{1+\mathfrak{e}^3} = \frac{E_-}{E_+}. \tag{3.4}$$

This confirms that what matters here are the transformation properties of the field quotients rather than the fields themselves. Therefore, the relativistic Lorentz transformation (2.9) can also be interpreted as the Möbius (or bilinear) transformation [17]

$$z_a = \frac{a z_s + b}{c z_s + d}, \tag{3.5}$$

induced by $\mathbf{M}$. When no light is incident from the substrate, $z_s = 0$ and then $z_a = R$. For the inverse matrix $\mathbf{M}^{-1}$ the transform of the origin is $-b/a = R/(R^2 - T^2)$.

We briefly recall that the fixed points are of great significance for the characterization of Möbius transformations. These points are defined as the field configurations such that $z_a = z_s \equiv z_f$ in Eq. (3.5), whose solutions are

$$z_{f\pm} = \frac{(a-d) \pm \sqrt{[\mathrm{Tr}(\mathbf{M})]^2 - 4}}{2c}. \tag{3.6}$$

So they are determined by the trace of $\mathbf{M}$. When the trace is a real number, the induced Möbius transformations are called elliptic, hyperbolic, or parabolic, when $[\mathrm{Tr}(\mathbf{M})]^2$ is lesser than, greater than, or equal to 4, respectively. The canonical representatives of those matrices are [18]

$$\underbrace{\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}}_{\text{elliptic}}, \quad \underbrace{\begin{pmatrix} e^{\xi/2} & 0 \\ 0 & e^{-\xi/2} \end{pmatrix}}_{\text{hyperbolic}}, \quad \underbrace{\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}}_{\text{parabolic}}, \tag{3.7}$$

while the induced geometrical actions are

$$z_a = z_s e^{i\theta}, \quad z_a = z_s e^{\xi}, \quad z_a = z_s + \alpha, \tag{3.8}$$

that is, a rotation of angle $\boldsymbol{\theta}$, a squeezing of parameter ξ, and a parallel displacement of magnitude α, respectively. The relativistic versions of these transformations can be easily worked out using Eq. (2.10).

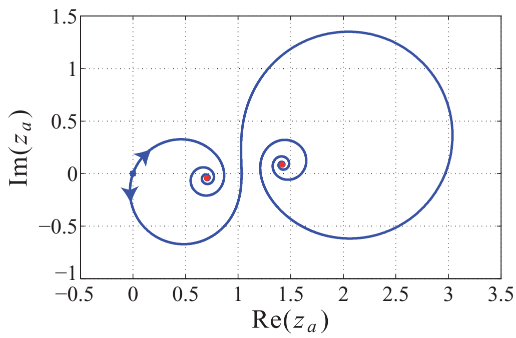

FIG. 3. (Color online) Transform of the origin $z_s = 0$ (blue point) by the matrix **M** of an absorbing film of germanium when its thickness varies continuously, as in Fig. 2. The two fixed points (marked in red) are r (left) and $1/r$ (right).

For an absorbing system the transfer matrix has, in general, a complex trace. The canonical representative is then a combination of a hyperbolic and an elliptic transformation:

$$\begin{pmatrix} e^{\xi/2} & 0 \\ 0 & e^{-\xi/2} \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} . \tag{3.9}$$

This global action gives a logarithmic spiral. Because of its unique mathematical properties, it was named by Jacob Bernouilli as *spira mirabilis* (marvelous spiral) and appears in many instances in nature [19].

In Fig. 3, we have plotted the Möbius transform of the origin (i.e., the reflection coefficient R) for the same film of germanium as before when its thickness changes continuously. When d is thick enough, the film tends to be a bulk medium, the interferential effects gradually disappear, and $\lim_{d\to\infty} R = r$, which is precisely a fixed point of the transformation. In fact, a simple exercise shows that **M** has two fixed points: r and $1/r$. The first acts as an attractor for the action of **M**, while the second is the attractor of $\mathbf{M}^{-1}$. For these reasons, the total curve in Fig. 3 involves necessarily both actions (indicated by the corresponding arrows), despite the fact that $\mathbf{M}^{-1}$ does not correspond to any physically feasible system. For a lossless film, this spiral reduces to a circle with its center at the fixed point.

To better illustrate this behavior, in Fig. 4 we represent the same transformation, but viewed on the sphere $\mathscr{S}_2$, defined in Eq. (3.3). We have also drawn the meridians through the two fixed points. As we can nicely appreciate, the trajectory traced by the system is indeed a rhumb line crossing all these meridians at the same angle, which is known as a loxodrome [20]. The stereographic projection from the north pole of the sphere gives precisely the curve in Fig. 3 in the complex plane, with the same comportment with respect to the projected meridians.

From a physical perspective, the spiral can be understood as the combined effect of interference and absorption. In Fig. 5, the reflectance and absorptance of the same germanium film are plotted as a function of the thickness d, showing oscillations (approximately out of phase between them). When both magnitudes become almost thickness independent, we are in the bulk limit and the transmittance vanishes.

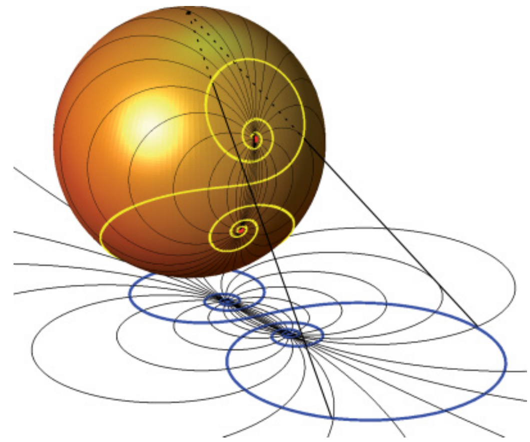

FIG. 4. (Color online) Transformation induced on the sphere $\mathscr{S}_2$ given in Eq. (3.3) by the same germanium film as before. The meridians are also shown to stress that the trajectory is a rhumb line crossing all of them at the same angle. The stereographic projection from the north pole reproduces Fig. 3. For an easier visualization, we have projected on the tangent plane to the south pole ($\mathfrak{e}^3 = -1$), instead of the equatorial one ($\mathfrak{e}^3 = 0$).

To round up our exposition, we observe that the matrix **M** can always be diagonalized:

$$\mathbf{C\,M\,C^{-1}} = \begin{pmatrix} m_+ & 0 \\ 0 & m_- \end{pmatrix}, \tag{3.10}$$

with $\mathbf{C} \in \mathrm{SL}(2,\mathbb{C})$. As this conjugation preserves the determinant, the eigenvalues are inverse one of the other, $m_+ = 1/m_-$. The trace is also preserved, so that $\mathrm{Tr}(\mathbf{M}) = m_+ + m_-$, whose solutions are [21]

$$m_\pm = \tfrac{1}{2}\{\mathrm{Tr}(\mathbf{M}) \pm \sqrt{[\mathrm{Tr}(\mathbf{M})]^2 - 4}\}, \tag{3.11}$$

wherefrom we can conclude that $m_\pm = \exp(\pm i\delta)$, where δ is the film phase thickness. In addition, the matrix in (3.10) must be of the canonical form (3.9), which fixes the parameters of the squeezing and the rotation of the loxodrome

$$\theta = 4\pi\frac{d}{\lambda}\,\mathrm{Re}(N), \quad \xi = 4\pi\frac{d}{\lambda}\,\mathrm{Im}(N). \tag{3.12}$$

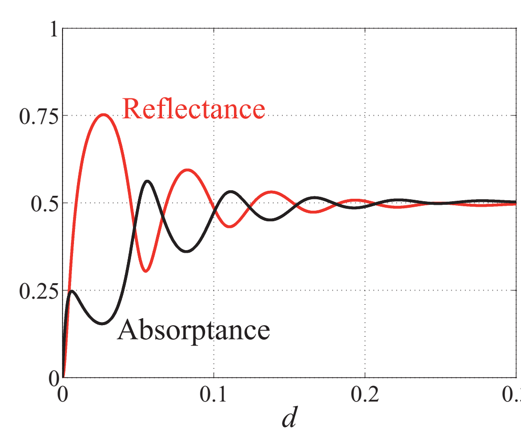

FIG. 5. (Color online) Reflectance and absorptance of the same germanium film as before in terms of its thickness (in microns).

This remarkable formula shows a clear relation between the physical parameters of the absorbing film and those of the geometrical transformation induced by it.

IV. CONCLUDING REMARKS

Modern geometry provides a useful and, at the same time, simple language in which numerous physical ideas and concepts may be clearly formulated and effectively treated.

This paper is yet another example of the advantages of these methods: we have devised a geometrical tool to analyze optical absorption in a concise way that, in addition, is closely related to other fields of physics.

In fact, this could be of great benefit in elucidating models that although apparently complex, display extra symmetries. This is the case of, e.g., parity-time (PT)–invariant potentials [22], whose physical interpretation is a touchy business [23]. Quite recently, optics has provided a fertile ground where PT-related notions can be implemented and experimentally investigated [24–26], so our formalism can pave the way toward understanding the intriguing and unexpected properties that rely on nonreciprocal light propagation.

Moreover, this picture permits us to transplant space-time phenomena to the more familiar arena of the optical world. However, note that this gateway works in both directions: here, it has allowed us to establish a relativistic presentation of absorption; but multilayer optics can be also used as a powerful instrument for visualizing special relativity [27]. This is more than an academic curiosity: in fact, some intricate relativistic effects, such as, e.g., the Wigner angle (or the Thomas precession), can be measured (and not merely inferred) by using very simple optical setups [28]. Our paper is one further step in this fruitful interplay between multilayer optics and relativity.

We stress that the behavior analyzed here is universal for any linear absorbing system. It is of application not only in optics, but also in all the fields in which the method of transfer matrix is suitable.

ACKNOWLEDGMENT

Financial support from the Spanish Research Agency (Grant No. FIS2008-04356) is gratefully acknowledged.

[1] E. H. Lieb and D. C. Matthis, *Mathematical Physics in One Dimension* (Academic, New York, 1966).
[2] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics* (American Mathematical Society, Providence, 2004).
[3] F. García-Moliner and V. R. Velasco, *Theory of Single and Multiple Interfaces* (World Scientific, Singapore, 1992).
[4] G. Monsivais, F. García-Moliner, and V. R. Velasco, J. Phys. Condens. Matter **7**, 5491 (1995).
[5] R. Pérez-Álvarez, C. Trallero-Herrero, and F. García-Moliner, Eur. J. Phys. **22**, 275 (2001).
[6] R. Pérez-Álvarez and F. García-Moliner, *Transfer Matrix, Green Function and Related Techniques* (Universitat Jaume I, Castelló, 2004).
[7] T. Yonte, J. J. Monzón, L. L. Sánchez-Soto, J. F. Cariñena, and C. López-Lacasta, J. Opt. Soc. Am. A **19**, 603 (2002).
[8] J. J. Monzón, T. Yonte, L. L. Sánchez-Soto, and J. F. Cariñena, J. Opt. Soc. Am. A **19**, 985 (2002).
[9] A. Barriuso, J. Monzón, and L. Sánchez-Soto, Opt. Lett. **28**, 1501 (2003).
[10] D. Sprung, G. Morozov, and J. Martorell, J. Phys. A **37**, 1861 (2004).
[11] A. O. Barut and R. Raczka, *Theory of Group Representations and Applications* (Polish Scientific Publishers PWN, Varsovia, 1977).
[12] J. Lekner, *Theory of Reflection* (Kluwer, Dordrecht, 1987).
[13] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995).
[14] G. L. Nater, *The Geometry of Minkowsky Spacetime* (Springer, New York, 1992).
[15] J. J. Monzón and L. L. Sánchez-Soto, J. Opt. Soc. Am. A **16**, 2013 (1999).
[16] W. Magnus, *Non-Euclidean Tessellations and Their Groups* (Academic Press, New York, 1974).
[17] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds* (Springer, New York, 1994).
[18] B. Iversen, *Hyperbolic Geometry* (Cambridge University Press, Cambridge, 1993).
[19] R. C. Archibald, Am. Math. Mon. **25**, 189 (1918).
[20] T. G. Feeman, Coll. Math. J. **32**, 334 (2001).
[21] E. Merzbacher, *Quantum Mechanics*, 3rd ed. (Wiley, New York, 1997).
[22] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[23] S. Weigert, Czech. J. Phys. **54**, 1139 (2004).
[24] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Phys. Rev. Lett. **103**, 093902 (2009).
[25] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nat. Phys. **6**, 192 (2010).
[26] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, Phys. Rev. Lett. **106**, 213901 (2011).
[27] J. J. Monzón and L. L. Sánchez-Soto, Opt. Commun. **162**, 1 (1999).
[28] J. J. Monzón and L. L. Sánchez-Soto, J. Mod. Opt. **48**, 21 (2001).