Propagation and breakup of pulses in fiber amplifiers and dispersion-decreasing fibers with third-order dispersion

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(Received 22 February 2011; published 16 August 2011)

We develop a theory for pulses propagating in normal dispersion fiber amplifiers with constant and varying gain, and for dispersion-decreasing fibers, including the effect of third-order dispersion. These solutions of the generalized nonlinear Schrödinger equation are based on asymptotical methods, first-order perturbation theory, and a renormalization procedure. We have also found an explicit equation for the critical length corresponding to pulse breakup and a criterion which ensure the accuracy of the asymptotic solutions. This criterion is confirmed numerically, showing that the analytical description of the pulses and the critical length formulas developed here for fiber amplifiers and dispersion-decreasing fibers with third-order dispersion are very accurate.

DOI: 10.1103/PhysRevA.84.023823 PACS number(s): 42.81.Dp, 42.25.Fx, 42.65.Jx

I. INTRODUCTION

Self-similarity is a fundamental physical property that has been studied for many areas in physics and, in particular, in optics [1–5]. In addition, recent studies in nonlinear optics have revealed an important type of optical pulse (similaritons) with parabolic profiles and linear frequency chirps that propagate in nonlinear optical fibers with normal second-order group-velocity dispersion [6] and in optical fiber amplifiers with constant and distributed gain functions [7–9]. These propagating pulses in optical fiber amplifiers with normal dispersion are asymptotically self-similar and their asymptotic behavior depends only on the input energy. This remarkable property is connected with a global attractor [10] which attracts the trajectories of the pulses with different initial conditions to the same self-similar structurally stable asymptotic solution [10,11]. Moreover, this similariton solution is exact asymptotically when the propagation distance tends to infinity.

These results have been confirmed theoretically [7–9,12] and experimentally [7]. Self-similar parabolic pulses are of fundamental interest because they represent a new class of solution to the nonlinear Schrödinger equation (NLSE) with gain having wide-ranging practical significance, since their linear chirp leads to highly efficient pulse compression to the sub-100-fs domain [13]. The fiber amplifiers and lasers which use self-similar propagating pulses in the normal dispersion regime have been demonstrated experimentally to achieve high-energy pulses [7,14–16]. The amplifier similariton evolution also yields practical features such as parabolic output pulses with high energies, and the shortest pulses to date from a normal-dispersion laser [17]. In addition, the self-similar dynamics in fibers with longitudinally varying parameters is connected with pulse shaping and pulse compression [18–20].

In recent papers the effect of third-order fiber dispersion (TOD) on similariton pulse propagation has been studied in fiber amplifiers and mode-locked lasers [21–24]. In fact, the impact of TOD on parabolic pulse propagation is to generate optical shock-type instabilities [25,26]. A recent study of a fiber amplifier with TOD and constant gain [27] has used asymptotical methods, first-order perturbation theory, and a renormalization procedure. We note that the solution presented in Ref. [27] describes all of the features induced by TOD. A fiber amplifier with TOD also has been studied previously using perturbation theory and some approximations [28] and without perturbation theory but using different approximations [29]. The analytical solutions in these papers differ quantitatively from the analytical description of the quasi-similaritons [27] for fiber amplifiers where the TOD effects are sizable.

In this paper, we derive the analytical solution generalizing the results presented in Ref. [27] for the nonlinear Schrödinger equation (NLS) with constant and distributed gain and TOD for fiber amplifiers, which is in close agreement with numerical simulations. Our approach is based on an asymptotical method, perturbation theory by TOD, and a renormalization procedure. We use in the perturbation theory a constant dimensionless parameter ϵ which is proportional to β_3. In the formulation below it is also introduced a nontrivial small dimensionless parameter $\varepsilon(z)$ which is distance dependent and leads to distortion of the similariton and to non-self-similar propagation of the pulse. We may, however, describe such pulses as quasi-similaritons which retain their integrity, although the pulse shape deviates from parabolic and the chirp becomes nonlinear.

We also formulate a renormalization procedure for the solution which yields the exact energy as a function of propagating distance and takes into account higher orders of the small parameter in the perturbation theory for the effective width of the pulse. This procedure is more general than in our previous work [27] and leads to a highly accurate description of the propagating pulses.

We present in this paper a transformation which maps the solution of the NLS for an arbitrary distributed gain function and TOD onto the solution for fiber with decreasing second-order and third-order dispersions. This transformation, for the particular case of constant gain and $\beta_3 = 0$, maps the parabolic solution [7] into the solution for a fiber with decreasing second-order dispersion [30].

We have also found in the case when the gain is constant an analytical solution for the critical distance z_c describing the pulse breakup due to the shock-type instabilities. In the general case for fiber amplifiers with TOD and an arbitrary varying

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gain function we have proved, in the first-order perturbation theory, that the critical distance z_c does not depend on the sign of TOD which is also confirmed in our numerical simulations.

In the last section of the paper we formulate the criterion for the initial value $\varepsilon_0 = \varepsilon(0)$ of an introduced distance-dependent small dimensionless parameter $\varepsilon(z)$ of the theory providing high accuracy for analytical description of asymptotic of the power profiles, the chirp function of the pulses and the critical lengths. The high accuracy of our theory describing all features of TOD effects, including pulse breakup in the quasi-similariton regimes, has been confirmed by numerical simulations.

II. QUASIPARABOLIC PULSES IN FIBER AMPLIFIERS WITH TOD

In the presence of TOD the pulse propagation in fiber amplifiers can be described by the generalized NLS equation [31]

$$i\psi_z = \frac{\beta_2}{2}\psi_{\tau\tau} + i\frac{\beta_3}{6}\psi_{\tau\tau\tau} - \gamma|\psi|^2\psi + i\frac{g(z)}{2}\psi, \tag{1}$$

where $\psi(z,\tau)$ is the slowly varying pulse envelope in a comoving frame, β_2, β_3, and γ, respectively, are the second-order and third-order dispersion parameters and the nonlinearity coefficient, and $g(z)$ is the distributed gain along the fiber. Using the ansatz

$$\psi(z,\tau) = \exp\left(\tfrac{1}{2}G(z)\right)\tilde{\psi}(z,\tau), \tag{2}$$

with the definitions

$$G(z) = \int_0^z g(z')dz', \quad \Gamma(z) = \gamma\exp[G(z)], \tag{3}$$

we transform the generalized NLSE to NLSE without gain:

$$i\tilde{\psi}_z = \frac{\beta_2}{2}\tilde{\psi}_{\tau\tau} + i\frac{\beta_3}{6}\tilde{\psi}_{\tau\tau\tau} - \Gamma(z)|\tilde{\psi}|^2\tilde{\psi}. \tag{4}$$

The Eq. (4) with a complex function written in the form $\tilde{\psi}(z,\tau) = A(z,\tau)\exp[\Phi(z,\tau)]$ yields the system of equations for real amplitude $A(z,\tau)$ and the phase $\Phi(z,\tau)$ as

$$(A^2)_z = \beta_2\Phi_{\tau\tau}A^2 + \beta_2\Phi_\tau(A^2)_\tau - \frac{\beta_3}{2}(A^2)_\tau(\Phi_\tau)^2 - \beta_3A^2\Phi_\tau\Phi_{\tau\tau} + \frac{\beta_3}{3}AA_{\tau\tau\tau}, \tag{5}$$

$$\Phi_z = \Gamma A^2 + \frac{\beta_2}{2}(\Phi_\tau)^2 + \frac{\beta_3}{6}\Phi_{\tau\tau\tau} - \frac{\beta_3}{6}(\Phi_\tau)^3 - \frac{\beta_2}{2}\left(\frac{A_{\tau\tau}}{A}\right) + \frac{\beta_3}{2}\Phi_{\tau\tau}\left(\frac{A_\tau}{A}\right) + \frac{\beta_3}{2}\Phi_\tau\left(\frac{A_{\tau\tau}}{A}\right). \tag{6}$$

Because we search for an asymptotical solution of Eq. (4), some terms in this system of equations can be neglected since they decrease much faster than the other ones when $\xi = gz \gg 1$ in the asymptotical regime. We note that an asymptotical regime $\xi \gg 1$ takes place when the condition $\xi_c = gz_c \gg 1$ is satisfied. Here z_c is the distance where the pulse breaks up due to shock-type instabilities resulting from TOD. In particular, we may neglect in an asymptotical regime the last term ($\sim AA_{\tau\tau\tau}$) in the right-hand part of Eq. (5) and three last terms ($\sim A_{\tau\tau}/A$ and $\sim A_\tau/A$) in the right-hand part of Eq. (6).

This statement can be proved using the explicit expressions for the functions $A(z,\tau)$ and $\Phi(z,\tau)$ found in the next section. In the proof of this statement one can also use the equations:

$$\frac{A_\tau}{A} = \frac{A_T}{wA}, \quad \frac{A_{\tau\tau}}{A} = \frac{A_{TT}}{w^2A}, \quad AA_{\tau\tau\tau} = A^2\frac{A_{TTT}}{w^3A}, \tag{7}$$

where $w = w(z)$ is the effective width of the pulse and $T = \tau/w(z)$. We note that the effective width $w(z)$ (defined in the next section) coincides with the width of the parabolic pulses [7].

Using the definition $A(z,\tau)^2 = P(z,\tau)$ and the exchange $\beta_3 \to \epsilon\beta_3$ in the system of Eqs. (5) and (6), and neglecting the mentioned terms ($\sim A_\tau/A$, $\sim A_{\tau\tau}/A$, and $\sim AA_{\tau\tau\tau}$) we find the system of equations:

$$P_z = \beta_2\Phi_{\tau\tau}P + \beta_2\Phi_\tau P_\tau - \frac{\epsilon\beta_3}{2}(\Phi_\tau)^2P_\tau - \epsilon\beta_3\Phi_\tau\Phi_{\tau\tau}P, \tag{8}$$

$$\Phi_z = \Gamma P + \frac{\beta_2}{2}(\Phi_\tau)^2 + \frac{\epsilon\beta_3}{6}\Phi_{\tau\tau\tau} - \frac{\epsilon\beta_3}{6}(\Phi_\tau)^3. \tag{9}$$

Here ϵ is the formal parameter which is used to indicate the order of "small" dimensionless parameter of the theory. We do not introduce at this stage an explicit form of this small constant parameter, which we assume to be proportional to β_3. It is worth pointing out that in the final stage of the perturbation procedure the formal parameter must be taken as $\epsilon = 1$. One of the basic concepts of our perturbation method is the decomposition of the functions $P(z,\tau)$ and $\Phi(z,\tau)$ in the form:

$$P(z,\tau) = \alpha_0 - \epsilon\alpha_1\tau - \alpha_2\tau^2 - \epsilon\alpha_3\tau^3 - \epsilon^2\alpha_4\tau^4 - \cdots, \tag{10}$$

$$\Phi(z,\tau) = c_0 + \epsilon c_1\tau + c_2\tau^2 + \epsilon c_3\tau^3 + \epsilon^2c_4\tau^4 + \cdots. \tag{11}$$

The series here have a special form because we assume that at $\beta_3 = 0$ ($\epsilon = 0$) the solution of the Eq. (1) is a parabolic pulse [7,8]. We also emphasize that the power $P(z,\tau)$ of the pulse and the phase $\Phi(z,\tau)$ in Eqs. (10) and (11) are defined only in some interval $\tau_a < \tau < \tau_b$ of variable τ where $P(z,\tau)$ is positive and satisfies the condition $P(z,\tau_a) = P(z,\tau_b) = 0$.

Substitution of the series in Eqs. (10) and (11) into the system of Eqs. (8) and (9) and neglecting the terms which are proportional to ϵ^2 and higher order of ϵ leads to the following system of nonlinear differential equations for the functions $\alpha_n(z)$ and $c_n(z)$ ($n = 0,1,2,3$):

$$\frac{d\alpha_0}{dz} = 2\beta_2c_2\alpha_0, \tag{12}$$

$$\frac{d\alpha_1}{dz} = 4\beta_2c_2\alpha_1 - 6\beta_2c_3\alpha_0 + 2\beta_2c_1\alpha_2 + 4\beta_3c_2^2\alpha_0, \tag{13}$$

$$\frac{d\alpha_2}{dz} = 6\beta_2c_2\alpha_2, \tag{14}$$

$$\frac{d\alpha_3}{dz} = 8\beta_2c_2\alpha_3 + 12\beta_2c_3\alpha_2 - 8\beta_3c_2^2\alpha_2, \tag{15}$$

$$\frac{dc_0}{dz} = \Gamma\alpha_0, \tag{16}$$

$$\frac{dc_1}{dz} = 2\beta_2c_1c_2 - \Gamma\alpha_1, \tag{17}$$

$$\frac{dc_2}{dz} = 2\beta_2 c_2^2 - \Gamma\alpha_2, \tag{18}$$

$$\frac{dc_3}{dz} = 6\beta_2 c_2 c_3 - \frac{4}{3}\beta_3 c_2^3 - \Gamma\alpha_3. \tag{19}$$

To find a unique solution of the system of Eqs. (12)–(19) we formulate the boundary conditions:

$$\alpha_0,\alpha_1,\alpha_2,\alpha_3 \to 0 \quad \text{for} \quad z \to \infty. \tag{20}$$

These conditions follows from the fact that there is no gain term in Eq. (4) and hence $P(z,\tau) \to 0$ at $z \to \infty$. The boundary conditions in Eq. (20) are formal and they are not connected with the pulse breakup. We also require that the solution of Eq. (1) [which is connected with a solution of Eq. (4)] at $\beta_3 = 0$ transforms to the known similariton solution [7,8]. In the conclusion of this section we emphasize that the system of Eqs. (12)–(19) with $\Gamma(z) = \gamma \exp[G(z)]$ is valid for an arbitrary gain function $g(z)$ when some dimensionless parameter ϵ of the theory ($\sim\beta_3$) is small.

III. ANALYTICAL SOLUTION FOR FIBER AMPLIFIERS WITH TOD

We consider in this section, pulse propagation in fiber amplifiers with constant gain based on the nonlinear system of differential equations [Eqs. (12)–(19)] with the boundary condition given by Eq. (20). In this case the function Γ in the system of Eqs. (12)–(19) is $\Gamma(z) = \gamma e^{gz}$, and the solution of this system of nonlinear differential equations with boundary conditions Eq. (20) is given by

$$\alpha_0(z) = \Lambda e^{-gz/3}, \quad \alpha_1(z) = \frac{g\beta_3\Lambda}{18\beta_2^2}e^{-gz/3}, \tag{21}$$

$$\alpha_2(z) = \frac{g^2}{18\gamma\beta_2}e^{-gz}, \quad \alpha_3(z) = -\frac{2\beta_3 g^3}{243\gamma\beta_2^3}e^{-gz}, \tag{22}$$

$$c_0(z) = \phi_0 + \frac{3\gamma\Lambda}{2g}e^{2gz/3}, \quad c_1(z) = -\frac{\gamma\beta_3\Lambda}{18\beta_2^2}e^{2gz/3}, \tag{23}$$

$$c_2(z) = -\frac{g}{6\beta_2}, \quad c_3(z) = \frac{7\beta_3 g^2}{486\beta_2^3}, \tag{24}$$

where ϕ_0 and Λ are some constants. We show below that a positive constant Λ is defined by the input energy E_0 of the pulse. Hence, the solution given by Eqs. (21)–(24) is the unique solution of the system of Eqs. (12)–(19) for the boundary conditions given in Eq. (20).

Neglecting the higher-order terms ($\sim\epsilon^n$ with $n > 1$) in Eq. (10) and using Eqs. (21) and (22) we find

$$A(z,\tau)^2 = \Lambda e^{-gz/3}\left[1 - \varepsilon(z)\left(\frac{\tau}{w(z)}\right) - \left(\frac{\tau}{w(z)}\right)^2 + \frac{8}{3}\varepsilon(z)\left(\frac{\tau}{w(z)}\right)^3\right]\mathcal{I}(z,\tau), \tag{25}$$

where the effective width $w(z)$ and the parameter $\varepsilon(z)$ are

$$w(z) = \frac{3}{g}\sqrt{2\gamma\beta_2\Lambda}\exp\left(\frac{1}{3}gz\right), \tag{26}$$

$$\varepsilon(z) = \frac{\beta_3}{6\beta_2^2}\sqrt{2\gamma\beta_2\Lambda}\exp\left(\frac{1}{3}gz\right). \tag{27}$$

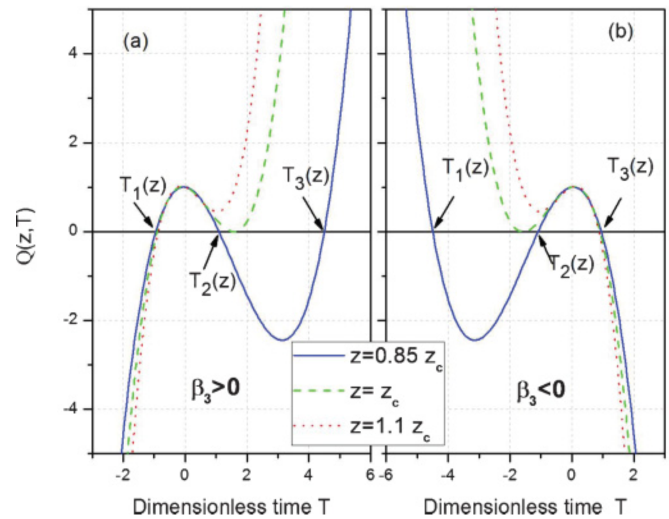

FIG. 1. (Color online) Polynomial $Q(z,T)$ for different propagation distances: (a) $\beta_3 > 0$ and (b) $\beta_3 < 0$.

Thus, we define here a small dimensionless parameter $\varepsilon(z)$, which is distance dependent and hence leads to non-self-similar propagation of the quasi-similaritons. However, we emphasize that the perturbation theory developed in the paper is based on a constant formal parameter ϵ which is proportional to β_3. We also note that the constant parameter of the perturbation theory can be defined as $\varepsilon_0 = \varepsilon(0)$.

The rectangular function $\mathcal{I}(z,\tau)$ defines here the region of variable τ where the polynomial

$$Q(z,T) = 1 - \varepsilon(z)T - T^2 + \tfrac{8}{3}\varepsilon(z)T^3, \tag{28}$$

with $T = \tau/w(z)$ is positive and Eq. (25) describes the power of a bounded pulse with finite support. To define this region of τ we should consider the roots of the polynomial $Q(z,T)$ (see Fig. 1). From Eq. (28) it follows that the three roots of an equation $Q(z,T_k) = 0$ are real only when $z \leqslant z_c$ (for some critical distance z_c), and when $z < z_c$ all of them differ and can be ordered as $T_1(z) < T_2(z) < T_3(z)$. A critical parameter $\varepsilon_c = \varepsilon(z_c)$ can be calculated using the condition that at $z = z_c$ the two roots of the polynomial $Q(z,T)$ are equal: $T_2(z_c) = T_3(z_c)$ or $T_1(z_c) = T_2(z_c)$ for positive or negative β_3, respectively. This condition yields the critical parameter ε_c by an equation $32\varepsilon_c^4 - 429\varepsilon_c^2 + 12 = 0$, and hence $\varepsilon_c = \pm 0.167$ for positive and negative β_3, respectively. We note that for $z > z_c$ the polynomial $Q(z,T)$ has only one real root, which is unphysical because the pulse must be bounded. The fact that the power of the pulse is not a bounded function of τ and the pulse does not have a finite support in the region $z > z_c$ is also demonstrated in Fig. 1. The origin of this unphysical behavior of the solution is connected with instabilities which arise for propagating lengths $z > z_c$. This fact allows us to identify z_c with the distance for which the pulse breaks up due to the shock-type instabilities [27].

Thus the bounded pulse with finite support can be defined in the region $z \leqslant z_c$ by the rectangular function:

$$\mathcal{I}(z,\tau) = \begin{cases} 1 & \text{if} \quad w(z)T_k(z) \leqslant \tau \leqslant w(z)T_{k+1}(z) \\ 0 & \text{otherwise} \end{cases}, \tag{29}$$

where $k = 1$ for $\beta_3 > 0$ and $k = 2$ for $\beta_3 < 0$, respectively.

We have found above the limitation on the parameter $\varepsilon(z)$: $|\varepsilon(z)| \leqslant |\varepsilon_c| = 0.167$. Hence the function $\varepsilon(z)$ is a small dimensionless distance-dependent parameter in this theory and in the case when $\beta_3 = 0$ we have $\varepsilon(z) = 0$, and the pulse becomes parabolic.

We can write the solution of Eq. (1) in the form $\psi(z,\tau) = U(z,\tau)\exp[i\Phi(z,\tau)]$ where $U(z,\tau)$ is the positive amplitude and $\Phi(z,\tau)$ is the phase, then for $g = \text{const.}$ from Eq. (2) follows that:

$$U(z,\tau) = |\psi(z,\tau)| = A(z,\tau)\exp\left(\tfrac{1}{2}gz\right). \tag{30}$$

Equations (25) and (30) yield the amplitude $U(z,\tau)$:

$$U(z,\tau) = e^{gz/3}\sqrt{\Lambda Q(z,\tau/w(z))}\mathcal{I}(z,\tau). \tag{31}$$

Using Eq. (31) one can find the energy of the pulse as

$$E(z) = \int_{-\infty}^{+\infty} |\psi(z,\tau)|^2 d\tau = \frac{4}{3}\Lambda w(z)\eta_k(z)e^{2gz/3}, \tag{32}$$

where the functions $\eta_k(z)$ for $k = 1,2$ are

$$\eta_k(z) = \frac{3}{4}[T_{k+1}(z) - T_k(z)] - \frac{3\varepsilon(z)}{8}\left[T_{k+1}^2(z) - T_k^2(z)\right] - \frac{1}{4}\left[T_{k+1}^3(z) - T_k^3(z)\right] + \frac{\varepsilon(z)}{2}\left[T_{k+1}^4(z) - T_k^4(z)\right], \tag{33}$$

with $k = 1$ for $\beta_3 > 0$ and $k = 2$ for $\beta_3 < 0$, respectively.

The first-order perturbation theory using the small parameter ϵ yields the following equations for the roots of the polynomial $Q(z,T)$:

$$T_k(z) = -1 + \tfrac{5}{6}\varepsilon(z), \quad T_{k+1}(z) = 1 + \tfrac{5}{6}\varepsilon(z), \tag{34}$$

where $k = 1$ for $\beta_3 > 0$ and $k = 2$ for $\beta_3 < 0$, respectively. The substitution of these roots in Eq. (33) leads in the first order in the small parameter ϵ to the equation $\eta_k(z) = 1$ for $k = 1,2$. Thus, to first order in ϵ the energy of the pulse is

$$E(z) = \frac{4}{3}\Lambda w(z)e^{2gz/3}, \quad E_0 = \frac{4}{g}\sqrt{2\gamma\beta_2\Lambda^3}, \tag{35}$$

where $E_0 = E(0)$. From Eq. (35) follows that the parameter Λ is given by

$$\Lambda = \frac{1}{4}\left(\frac{2g^2E_0^2}{\gamma\beta_2}\right)^{1/3}. \tag{36}$$

This equation allows us to present the solution of Eq. (1) with $g = \text{const.}$ as a function of input energy E_0. Using Eq. (36) we find the amplitude of the pulse given by Eq. (31) as

$$U(z,\tau) = \left(\frac{3E(z)}{4w(z)}\right)^{1/2}\sqrt{\mathcal{Q}(z,\tau)}\mathcal{I}(z,\tau), \tag{37}$$

where $E(z) = E_0e^{gz}$ and the function $\mathcal{Q}(z,\tau)$ is

$$\mathcal{Q}(z,\tau) = 1 - \varepsilon(z)\left(\frac{\tau}{w(z)}\right) - \left(\frac{\tau}{w(z)}\right)^2 + \frac{8}{3}\varepsilon(z)\left(\frac{\tau}{w(z)}\right)^3. \tag{38}$$

Here the effective width $w(z)$ of the pulse and the small dimensionless parameter $\varepsilon(z)$ are

$$w(z) = 3\left(\frac{\gamma\beta_2E_0}{2g^2}\right)^{1/3}e^{gz/3}, \tag{39}$$

$$\varepsilon(z) = \frac{\beta_3 g}{18\beta_2^2}w(z) = \frac{\beta_3}{6\beta_2^2}\left(\frac{1}{2}\gamma\beta_2 gE_0\right)^{1/3}e^{gz/3}. \tag{40}$$

We note that the effective width $w(z)$ defined by Eq. (39) coincides with the effective width of the parabolic pulses [7]. It is follows from Eqs. (29) and (34) that in the first order of small parameter ϵ the duration of the pulse is $w(z)[T_{k+1}(z) - T_k(z)] = 2w(z)$. We also note that in the first order of small parameter ϵ the full width at half maximum for quasiparabolic pulses is $\sqrt{2}w(z)$.

The rectangular function $\mathcal{I}(z,\tau)$ is defined here by Eq. (29) for $z \leqslant z_c$, where the critical length z_c is given by $\varepsilon_c = \varepsilon(z_c)$. Because $\varepsilon(z_c) = \varepsilon_0 e^{gz_c/3}$, the critical length is $z_c = (3/g)\ln(|\varepsilon_c|/|\varepsilon_0|)$, where $\varepsilon_0 = \varepsilon(0) = \beta_3 g w(0)/(18\beta_2^2)$.

The phase of the pulse in first order in ϵ follows from Eq. (11) and Eqs. (23), (24), and (36):

$$\Phi(z,\tau) = \phi_0 + \frac{3}{8}\left(\frac{2\gamma^2E_0^2}{\beta_2 g}\right)^{1/3}e^{2gz/3} - \frac{\beta_3 g}{72\beta_2^2}\left(\frac{2\gamma^2E_0^2}{\beta_2 g}\right)^{1/3}e^{2gz/3}\tau - \left(\frac{g}{6\beta_2}\right)\tau^2 + \left(\frac{7\beta_3 g^2}{486\beta_2^3}\right)\tau^3. \tag{41}$$

Hence the chirp function $\Omega(z,\tau) = -\Phi_\tau(z,\tau)$ of the pulse is quadratic in τ:

$$\Omega(z,\tau) = \frac{\beta_3 g}{72\beta_2^2}\left(\frac{2\gamma^2E_0^2}{\beta_2 g}\right)^{1/3}e^{2gz/3} + \left(\frac{g}{3\beta_2}\right)\tau - \left(\frac{7\beta_3 g^2}{162\beta_2^3}\right)\tau^2.$$

The solution given by Eqs. (37)–(41) is based on statement that in an asymptotical regime $\xi = gz \gg 1$ we can neglect the last term ($\sim AA_{\tau\tau\tau}$) in the right-hand part of Eq. (5) and also we can neglect three last terms ($\sim A_{\tau\tau}/A$ and $\sim A_\tau/A$) in the right-hand part of Eq. (6). This statement, which is also confirmed by our numerical simulations, can be proved using the solution found above together with Eq. (7). We do not present here this tedious but technically simple proof. The asymptotical regime for $\xi \gg 1$ applies only when the condition $\xi_c = gz_c \gg 1$ is satisfied. Numerical simulations (see below) show that the condition $\xi_c \gg 1$ is important in this theory, providing a highly accurate analytical description of propagating quasi-similaritons and critical lengths z_c in fiber amplifiers with TOD.

It follows from Eqs. (32), (36), and (39) that the energy of the pulse is $E(z) = E_0\eta_k(z)\exp(gz)$, where the functions $\eta_k(z)$ $(k = 1,2)$ are given by Eq. (33). To demonstrate the dependence of the functions $\eta_k(z)$ on the propagation distances z we plot in the Fig. 2 the function $\eta_1(z)$ for the case $\beta_3 > 0$, and the function $\eta_2(z)$ for the case $\beta_3 < 0$ with the same input energies. It is seen in the graph that $\eta_1(z) = \eta_2(z)$ and the functions $\eta_k(z)$ are very close to 1 for all values of $\xi \leqslant \xi_c$

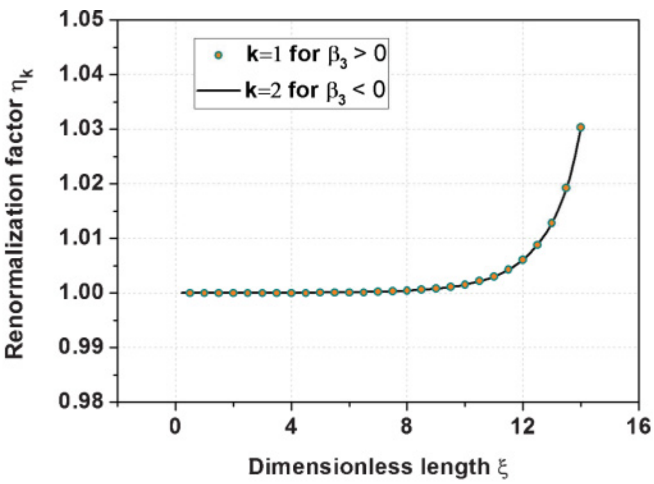

FIG. 2. (Color online) Renormalization functions $\eta_1(z)$ for $\beta_3 > 0$ ($\kappa = 3.018 \times 10^{-2}$) and $\eta_2(z)$ for $\beta_3 < 0$ ($\kappa = -3.018 \times 10^{-2}$) with input dimensionless energy $\mathcal{V}_0 = 3.922 \times 10^{-2}$.

leading to a small deviations from 1 in the vicinity of $\xi = \xi_c = 14.09$. We use in Fig. 2 the dimensionless parameters for normalized distance, the energy, and TOD as $\xi = gz$, $\mathcal{V}_0 = \gamma E_0/\sqrt{g\beta_2}$, and $\kappa = \beta_3\sqrt{g/\beta_2^3}$, respectively.

Since $\eta_k(z) = 1$ ($k = 1,2$) in the first order in ϵ, the energy of the propagating pulses in the first order of the perturbation theory is $E(z) = E_0 e^{gz}$. It is also easy to see that, in the limit $\beta_3 \to 0$, the solution for propagating pulses in fiber amplifiers with TOD [Eqs. (37)–(41)] reduces to a parabolic solution [7,8]. Because the small function $\varepsilon(z)$ of the theory is bounded by the condition $|\varepsilon(z)| \leqslant |\varepsilon_c| = 0.167$ ($|\varepsilon(z)| \ll 1$), the solution [see Eqs. (37)–(41)] describes the pulses which we may interpret as quasi-similaritons that retain their integrity, although the pulse shape deviates from parabolic and the chirp becomes nonlinear. From the solution above it also follows that the critical length $z_c = (3/g)\ln(|\varepsilon_c|/|\varepsilon_0|)$ can be identified with the critical length at which the TOD generates pulse breakup because at $z > z_c$ the pulse becomes unbounded which is unphysical. We consider this issue in more detail in Sec. VI.

Finally, we note that the phase given by Eq. (41) can also be written as

$$\Phi(z,\tau) = \phi_0 + \left(\frac{g}{12\beta_2}\right) w(z)^2 - \left(\frac{g}{18\beta_2}\right)\varepsilon(z)w(z)\tau - \left(\frac{g}{6\beta_2}\right)\tau^2 + \left(\frac{7g}{27\beta_2}\right)\frac{\varepsilon(z)}{w(z)}\tau^3. \quad (42)$$

Hence the phase is presented here in an explicit form as a function of $w(z)$ and $\varepsilon(z)$ which is important for the renormalization procedure developed below.

IV. RENORMALIZATION PROCEDURE AND SOLUTION FOR DISPERSION-DECREASING FIBERS

It was found above that the energy of the propagating pulses (quasi-similaritons) coincides with the exact energy $E(z) = E_0 e^{gz}$ of the pulses only in the first order in the small constant parameter ϵ. We formulate in this section a renormalization procedure for the solution given by Eqs. (37)–(41) which yields the exact energy with distance of the pulse and takes into account higher orders of parameter ϵ ($\epsilon \sim \beta_3$) in the perturbation theory for the effective width of the pulse.

We formulate the generalized renormalization procedure of the solution by the exchange $E(z) \to E(z)/\eta_k(z)$ and $w(z) \to w_k^{(n)}(z)$ in Eqs. (37), (38) and (29), where $w_k^{(n)}(z)$ is

$$w_k^{(n)}(z) = \frac{w(z)}{\eta_k(z)^n} = 3\left(\frac{\gamma\beta_2 E_0}{2g^2}\right)^{1/3}\frac{e^{gz/3}}{\eta_k(z)^n}. \quad (43)$$

Here the functions $\eta_k(z)$ are given by Eq. (33) with $k = 1$ for $\beta_3 > 0$ and $k = 2$ for $\beta_3 < 0$, respectively, and $w(z)$ is defined by Eq. (39). We emphasize that the exchange $E(z) \to E(z)/\eta_k(z)$ in Eq. (37) yields the exact energy with distance of the pulse and the exchange $w(z) \to w_k^{(n)}(z)$ in Eqs. (37), (38), and (29) yields the renormalization of the pulse width. Here n is a renormalization parameter (some real quantity) which can be found analytically by a variational procedure or numerically. We have found (see Sec. VII) that with a high accuracy this parameter can be chosen as an integer number $n = 3$. The renormalization parameter is equal to $n = 1$ and the phase is nonrenormalized in the previous solution [27].

Using the procedure described above we can present the renormalized amplitude $U_R(z,\tau)$ and polynomial $\mathcal{Q}_R(z,\tau)$ as

$$U_R(z,\tau) = \left(\frac{3E(z)}{4\eta_k(z)w_k^{(n)}(z)}\right)^{1/2}\sqrt{\mathcal{Q}_R(z,\tau)}\mathcal{I}_R(z,\tau), \quad (44)$$

$$\mathcal{Q}_R(z,\tau) = 1 - \varepsilon(z)\left(\frac{\tau}{w_k^{(n)}(z)}\right) - \left(\frac{\tau}{w_k^{(n)}(z)}\right)^2 + \frac{8}{3}\varepsilon(z)\left(\frac{\tau}{w_k^{(n)}(z)}\right)^3, \quad (45)$$

where $E(z) = E_0 e^{gz}$ and small parameter (function) $\varepsilon(z)$ is given by Eq. (40).

The exchange $w(z) \to w_k^{(n)}(z)$ also yields a new rectangular function $\mathcal{I}_R(z,\tau)$ in Eq. (44) given by

$$\mathcal{I}_R(z,\tau) = \begin{cases} 1 & \text{if } w_k^{(n)}(z)T_k(z) \leqslant \tau \leqslant w_k^{(n)}(z)T_{k+1}(z) \\ 0 & \text{otherwise} \end{cases}, \quad (46)$$

with $k = 1$ for $\beta_3 > 0$ and $k = 2$ for $\beta_3 < 0$. New renormalized phase follows by the exchange $w(z) \to w_k^{(n)}(z)$ in Eq. (42):

$$\Phi_R(z,\tau) = \phi_0 + \left(\frac{g}{12\beta_2}\right) w_k^{(n)}(z)^2 - \left(\frac{g}{18\beta_2}\right)\varepsilon(z)w_k^{(n)}(z)\tau - \left(\frac{g}{6\beta_2}\right)\tau^2 + \left(\frac{7g}{27\beta_2}\right)\frac{\varepsilon(z)}{w_k^{(n)}(z)}\tau^3. \quad (47)$$

This yields a renormalized chirp given by

$$\Omega_R(z,\tau) = \frac{\beta_3 g}{72\beta_2^2}\left(\frac{2\gamma^2 E_0^2}{\beta_2 g}\right)^{1/3}\frac{e^{2gz/3}}{\eta_k(z)^n} + \left(\frac{g}{3\beta_2}\right)\tau - \left(\frac{7\beta_3 g^2}{162\beta_2^3}\right)\eta_k(z)^n\tau^2. \quad (48)$$

Using Eq. (44) for the renormalized amplitude $U_R(z,\tau)$ we may calculate the energy of the pulse $E(z) = \int_{-\infty}^{+\infty} U_R(z,\tau)^2 d\tau = E_0 e^{gz}$. Thus the energy of the pulse with distance for the renormalized solution is described by an exact equation. We emphasize that the renormalization procedure yields only a small correction to the solution because for all distances $z \leqslant z_c$ the parameter $\varepsilon(z)$ is small. Our simulations show (see Fig. 2) that the functions $\eta_k(z)$ $(k = 1,2)$ are very close to the value 1 for $z \leqslant z_c$ and hence the difference between the functions $w(z)$ and $w_k^{(n)}(z)$ is small. Such corrections are noticeable only in the vicinity of the critical distance $z = z_c$.

In the conclusion of this section we consider the renormalization functions $\eta_1(z)$ and $\eta_2(z)$ for the cases which differ only with the parameter β_3: (1) $\beta_3 = \beta_3^+ > 0$ and (2) $\beta_3 = \beta_3^- < 0$, where $\beta_3^- = -\beta_3^+$. The ordered roots of the polynomial $Q(z,T)$ in Eq. (28) for these two cases are connected as $T_1^-(z) = -T_3^+(z)$, $T_2^-(z) = -T_2^+(z)$, $T_3^-(z) = -T_1^+(z)$, where the indexes $\pm$ indicate two different cases (1) and (2), respectively. Using this result and Eqs. (33) and (43) we can prove that $\eta_1(z) = \eta_2(z)$ and hence $w_1^{(n)}(z) = w_2^{(n)}(z)$ where the functions $\eta_1(z)$, $w_1^{(n)}(z)$, and $\eta_2(z)$, $w_2^{(n)}(z)$ are defined here for the cases (1) and (2), respectively. This result is also confirmed in Fig. 2 where we present the numerical simulations for the functions $\eta_1(z)$ and $\eta_2(z)$ with some fixed parameters for fiber amplifier.

A. Pulse propagation in fibers with decreasing dispersions

The generalized NLS, Eq. (1), can be transformed to the equation describing the propagation of the pulses in a fiber with decreasing second-order and third-order dispersion parameters. Such a transformation applied to the solution given by Eqs. (37)–(41) leads to the solution of the NLSE describing pulse propagation in a fiber with decreasing dispersion parameters. In the case when in Eq. (1) $g =$ const. the transformation is $\bar{\psi}(s,\tau) = \tilde{\psi}(z,\tau)$ where $s = (e^{gz} - 1)/g$. This transformation applied to Eq. (4) yields the equation for a fiber with decreasing second-order and third-order dispersions as

$$i\bar{\psi}_s = \frac{\bar{\beta}_2(s)}{2}\bar{\psi}_{\tau\tau} + i\frac{\bar{\beta}_3(s)}{6}\bar{\psi}_{\tau\tau\tau} - \gamma|\bar{\psi}|^2\bar{\psi}, \quad (49)$$

$$\bar{\beta}_2(s) = \frac{\beta_2}{1+gs}, \quad \bar{\beta}_3(s) = \frac{\beta_3}{1+gs}. \quad (50)$$

We may consider the variable s in Eq. (49) as a propagation distance in a fiber with decreasing dispersion parameters $\bar{\beta}_2(s)$ and $\bar{\beta}_3(s)$, and the parameter g here is a factor describing decreasing dispersion parameters. In the case of $g =$ const., Eq. (2) has the form $\tilde{\psi}(z,\tau) = e^{-gz/2}\psi(z,\tau)$ and the transformation for solution in a fiber with decreasing second-order and third-order dispersions can be written as

$$\bar{\psi}(s,\tau) = \frac{1}{\sqrt{1+gs}}\psi(z,\tau), \quad z = \frac{1}{g}\ln(1+gs). \quad (51)$$

We note that a similar transformation [30] has been applied to a parabolic solution in the case $\beta_3 = 0$. Using the ansatz $\bar{\psi}(s,\tau) = \bar{U}_R(s,\tau)\exp[i\bar{\Phi}_R(s,\tau)]$ and Eq. (51) we can transform the solution given by Eqs. (43)–(48) to the solution of the Eq. (49) with decreasing second-order and third-order dispersions given by Eq. (50). This transformation yields the amplitude $\bar{U}_R(s,\tau)$ in the form:

$$\bar{U}_R(s,\tau) = \frac{1}{2}\left(\frac{2g^2E_0^2}{\gamma\beta_2}\right)^{1/6}\bar{\eta}_k(s)^{(n-1)/2}(1+gs)^{-1/6} \times\sqrt{\bar{Q}_R(s,\tau)}\bar{\mathcal{I}}_R(s,\tau), \quad (52)$$

$$\bar{Q}_R(s,\tau) = 1 - \bar{\varepsilon}(s)\left(\frac{\tau}{\bar{w}_k^{(n)}(s)}\right) - \left(\frac{\tau}{\bar{w}_k^{(n)}(s)}\right)^2 + \frac{8}{3}\bar{\varepsilon}(s)\left(\frac{\tau}{\bar{w}_k^{(n)}(s)}\right)^3. \quad (53)$$

$$\bar{\varepsilon}(s) = \frac{\beta_3}{6\beta_2^2}\left(\frac{1}{2}\gamma\beta_2 g E_0\right)^{1/3}(1+gs)^{1/3}. \quad (54)$$

$$\bar{w}_k^{(n)}(s) = \frac{\bar{w}(s)}{\bar{\eta}_k(s)^n} = 3\left(\frac{\gamma\beta_2E_0}{2g^2}\right)^{1/3}\frac{(1+gs)^{1/3}}{\bar{\eta}_k(s)^n}. \quad (55)$$

The functions $\bar{\eta}_k(s)$ ($k = 1$ for $\beta_3 > 0$ and $k = 2$ for $\beta_3 < 0$, respectively) are connected to the renormalization procedure and they are given by

$$\bar{\eta}_k(s) = \frac{3}{4}[\bar{T}_{k+1}(s) - \bar{T}_k(s)] - \frac{3\bar{\varepsilon}(s)}{8}[\bar{T}_{k+1}^2(s) - \bar{T}_k^2(s)] - \frac{1}{4}[\bar{T}_{k+1}^3(s) - \bar{T}_k^3(s)] + \frac{\bar{\varepsilon}(s)}{2}[\bar{T}_{k+1}^4(s) - \bar{T}_k^4(s)], \quad (56)$$

where $\bar{T}_k(s)$ are the roots of the polynomial $\bar{Q}(s,\bar{T})$:

$$\bar{Q}(s,\bar{T}) = 1 - \bar{\varepsilon}(s)\bar{T} - \bar{T}^2 + \tfrac{8}{3}\bar{\varepsilon}(s)\bar{T}^3, \quad (57)$$

with $\bar{T} = \tau/\bar{w}_k^{(n)}(s)$. We assume here that $s \leqslant s_c$, where $s_c = g^{-1}(e^{gz_c} - 1)$ is the critical distance, and all roots $\bar{T}_k(s)$ are real and ordered: $\bar{T}_1(s) < \bar{T}_2(s) < \bar{T}_3(s)$ for $s < s_c$. The rectangular function $\bar{\mathcal{I}}_R(s,\tau)$ is defined as

$$\bar{\mathcal{I}}_R(s,\tau) = \begin{cases} 1 & \text{if} \quad \bar{w}_k^{(n)}(z)\bar{T}_k(z) \leqslant \tau \leqslant \bar{w}_k^{(n)}(z)\bar{T}_{k+1}(z) \\ 0 & \text{otherwise} \end{cases}, \quad (58)$$

with $k = 1$ for $\beta_3 > 0$ and $k = 2$ for $\beta_3 < 0$, respectively.

The phase of the propagating pulses in a fiber with decreasing second and third-order dispersion parameters given by Eqs. (49) and (50) has the form:

$$\bar{\Phi}_R(s,\tau) = \phi_0 + \frac{3}{8}\left(\frac{2\gamma^2E_0^2}{\beta_2 g}\right)^{1/3}\bar{\eta}_k(s)^{-2n}(1+gs)^{2/3} - \frac{\beta_3 g}{72\beta_2^2}\left(\frac{2\gamma^2E_0^2}{\beta_2 g}\right)^{1/3}\bar{\eta}_k(s)^{-n}(1+gs)^{2/3}\tau - \left(\frac{g}{6\beta_2}\right)\tau^2 + \left(\frac{7\beta_3 g^2}{486\beta_2^3}\right)\bar{\eta}_k(s)^n\tau^3. \quad (59)$$

Hence the chirp function $\bar{\Omega}(s,\tau) = -\bar{\Phi}_\tau(s,\tau)$ of the pulse is a quadratic in τ:

$$\bar{\Omega}_R(s,\tau) = \frac{\beta_3 g}{72\beta_2^2}\left(\frac{2\gamma^2 E_0^2}{\beta_2 g}\right)^{1/3}\bar{\eta}_k(s)^{-n}(1+gs)^{2/3} + \left(\frac{g}{3\beta_2}\right)\tau - \left(\frac{7\beta_3 g^2}{162\beta_2^3}\right)\bar{\eta}_k(s)^n\tau^2. \quad (60)$$

We emphasize that the renormalization procedure used in this solution for propagating pulses in a fiber with decreasing second- and third-order dispersions leads to conservation of the pulse energy: $E(z) = \text{const.} = E_0$. However, the deviation of the functions $\bar{\eta}_k(s)$ from 1 for distances $s \leqslant s_c$ is very small and hence this procedure leads to small corrections. Hence, one can take in the above equations $\bar{\eta}_k(s) = 1$, which considerably simplifies the solution, although in this case the pulse energy differs slightly along the fiber from input energy E_0.

V. AMPLIFIERS WITH VARYING GAIN AND DISPERSION-DECREASING FIBERS

In this section we consider the general case when the quasiparabolic pulses propagate in optical fiber amplifiers with TOD and varying gain function $g(z)$. We describe the pulse propagation in fiber amplifiers by the nonlinear system of differential equations Eqs. (12)–(19) with the boundary condition given by Eq. (20). In this section we consider a restricted class of varying gain functions for which the asymptotic behavior of propagating pulses depends only on the input energy. This remarkable property is connected with an existence of a global attractor [10] of the nonlinear system of differential equations, Eqs. (12)–(19). In this case pulses with different initial conditions are attracted to the same structurally stable asymptotic solution.

The solution of the Eqs. (12) and (14) and Eqs. (16) and (18) for the functions $\alpha_0(z)$,$\alpha_2(z)$ and $c_0(z)$,$c_2(z)$ can be presented in the form:

$$\alpha_0(z) = \frac{3E_0}{4w(z)}, \quad \alpha_2(z) = \frac{3E_0}{4w(z)^3}, \quad (61)$$

$$c_0(z) = \phi_0 + \frac{3\gamma}{4}\int_0^z \frac{E(z')}{w(z')}dz',$$
$$c_2(z) = -\left[\frac{1}{2\beta_2 w(z)}\right]\frac{dw(z)}{dz}. \quad (62)$$

Here $E(z) = E_0\exp(G(z))$ is the energy of the pulse for distributed gain and the effective width $w(z)$ is given by the following equation:

$$\frac{d^2w(z)}{dz^2} = \left(\frac{3\gamma\beta_2}{2}\right)\frac{E(z)}{w(z)^2}. \quad (63)$$

We note that this equation is the same as the equation for the case $\beta_3 = 0$ [8]. The solution of this equation does not depend on the boundary conditions for a restricted class of varying gain functions mentioned above. Thus, in this case, pulses with different initial conditions have the same structurally stable asymptotic solution.

Neglecting the higher-order terms ($\sim\epsilon^n$ with $n > 1$) in Eq. (10) and using Eq. (61) we find the amplitude of the pulse in this general case as

$$U(z,\tau) = \left(\frac{3E(z)}{4w(z)}\right)^{1/2}\sqrt{\hat{\mathcal{Q}}(z,\tau)}\hat{\mathcal{I}}(z,\tau), \quad (64)$$

$$\hat{\mathcal{Q}}(z,\tau) = 1 - \varepsilon_1(z)\left(\frac{\tau}{w(z)}\right) - \left(\frac{\tau}{w(z)}\right)^2 + \varepsilon_2(z)\left(\frac{\tau}{w(z)}\right)^3, \quad (65)$$

where $\varepsilon_1(z)$ and $\varepsilon_1(z)$ are two small dimensionless distance-dependent parameters:

$$\varepsilon_1(z) = \frac{\alpha_1(z)w(z)}{\alpha_0(z)} = \frac{4\alpha_1(z)w(z)^2}{3E_0}, \quad (66)$$

$$\varepsilon_2(z) = -\frac{\alpha_3(z)w(z)^3}{\alpha_0(z)} = -\frac{4\alpha_3(z)w(z)^4}{3E_0}. \quad (67)$$

We emphasize that the functions $\varepsilon_1(z)$ and $\varepsilon_2(z)$ are both proportional to β_3. Moreover, it can be shown that in the general case, when $g \neq \text{const.}$, the functions $\varepsilon_1(z)$ and $\varepsilon_2(z)$ essentially differ: $\varepsilon_1(z)$ is not proportional to $\varepsilon_2(z)$. However, if $g = \text{const.}$ we have found above that $\varepsilon_2(z) = (8/3)\varepsilon(z)$ and $\varepsilon_1(z) = \varepsilon(z)$.

The rectangular function $\hat{\mathcal{I}}(z,\tau)$ in Eq. (64) has the same form as in Eq. (29), where the function $w(z)$ is given here by Eq. (63) and the functions $T_k(z)$ are the roots of the polynomial $\hat{Q}(z,T) = 1 - \varepsilon_1(z)T - T^2 + \varepsilon_2(z)T^3$ [$T = \tau/w(z)$]. All roots of the polynomial $\hat{Q}(z,T)$ are real when $z \leqslant z_c$ and they are ordered: $T_1(z) < T_2(z) < T_3(z)$ for $z < z_c$. We define the critical length z_c as the distance at which two real roots of the polynomial $\hat{Q}(z,T)$ are equal: $T_2(z_c) = T_3(z_c)$ or $T_1(z_c) = T_2(z_c)$ for positive or negative β_3, respectively. This definition is sensible because for $z > z_c$ the polynomial $\hat{Q}(z,T)$ has only one real root, which is unphysical (because the pulse must be bounded) and hence the length z_c can be identified with pulse breakup.

The phase of the pulse for the general case (distributed gain and TOD) in the first order in ϵ follows from Eq. (11) and Eq. (62):

$$\Phi(z,\tau) = \phi_0 + \frac{3\gamma}{4}\int_0^z \frac{E(z')}{w(z')}dz' + c_1(z)\tau - \frac{1}{2\beta_2 w(z)}\left(\frac{dw(z)}{dz}\right)\tau^2 + c_3(z)\tau^3. \quad (68)$$

In the Eqs. (66)–(68), the functions $\alpha_3(z)$ and $c_3(z)$ are the solutions of Eqs. (15) and (19) and the functions $\alpha_1(z)$ and $c_1(z)$ are the solutions of Eqs. (13) and (17). The analytical solution of Eq. (63) as well as Eqs. (15) and (19) and Eqs. (13) and (17) is possible only for some particular gain functions $g(z)$ because these equations depend on an arbitrary increasing positive function $E(z)$ [in Eqs. (16)–(19) $\Gamma(z) = (\gamma/E_0)E(z)$]. The solutions of Eq. (63) with some particular distributed gain functions have been given elsewhere [10]. The renormalization procedure for the general case [Eqs. (63)–(68)] is similar to the procedure described in the previous section (see the Appendix).

In the case $\beta_3 = 0$ the Eqs. (64), (65) and (68) describe the propagation of the parabolic pulses in the fiber amplifiers with distributed gain [8]:

$$U(z,\tau) = \left(\frac{3E(z)}{4w(z)}\right)^{1/2}\left(1 - \frac{\tau^2}{w(z)^2}\right)^{1/2}\theta(w(z) - |\tau|), \quad (69)$$

$$\Phi(z,\tau) = \phi_0 + \frac{3\gamma}{4}\int_0^z \frac{E(z')}{w(z')}dz' - \frac{1}{2\beta_2 w(z)}\left(\frac{dw(z)}{dz}\right)\tau^2, \quad (70)$$

where the width $w(z)$ is given by Eq. (63) and $\theta(x)$ is the step function: $\theta(x) = 1$ for $x \geqslant 0$ and $\theta(x) = 0$ otherwise. If $g =$ const. one can find the solution of Eq. (63) in the form $w(z) = w_0 e^{\lambda z}$ which yields $\lambda = g/3$ and hence the width $w(z)$ is given by Eq. (39). Thus the Eqs. (69) and (70) reduce in this case to the known parabolic solution [7].

A. Fibers with decreasing second- and third-order dispersions

In the general case ($g \neq$ const.) the Eq. (4) can be transformed to the equation describing the propagation of the pulses in a fiber with decreasing second-order and third-order dispersion parameters. This transformation is given by

$$\bar{\psi}(s,\tau) = \tilde{\psi}(z,\tau), \quad s = F(z) \equiv \int_0^z e^{G(z')}dz', \quad (71)$$

and the equation describing the propagation of the pulses in a fiber with decreasing dispersion parameters is

$$i\bar{\psi}_s = \frac{\bar{\beta}_2(s)}{2}\bar{\psi}_{\tau\tau} + i\frac{\bar{\beta}_3(s)}{6}\bar{\psi}_{\tau\tau\tau} - \gamma|\bar{\psi}|^2\bar{\psi}. \quad (72)$$

Here decreasing dispersion parameters $\bar{\beta}_2(s)$ and $\bar{\beta}_3(s)$ are

$$\bar{\beta}_2(s) = \beta_2 \exp[-G(R(s))], \quad \bar{\beta}_3(s) = \beta_3 \exp[-G(R(s))], \quad (73)$$

where $z = R(s)$ is the inverse function: $z = R(F(z))$. Using the ansatz $\bar{\psi}(s,\tau) = \bar{U}(s,\tau)\exp[i\bar{\Phi}(s,\tau)]$ we can find the amplitude $\bar{U}(s,\tau)$ and the phase $\bar{\Phi}(s,\tau)$ of the propagating pulses in the form:

$$\bar{U}(s,\tau) = U(R(s),\tau)\exp\left[-\tfrac{1}{2}G(R(s))\right], \quad (74)$$

$$\bar{\Phi}(s,\tau) = \Phi(R(s),\tau). \quad (75)$$

where $U(z,\tau)$ and $\Phi(z,\tau)$ are given by Eqs. (64) and (68). The general renormalization procedure is described in the Appendix and leads to both renormalized functions $U(z,\tau)$ and $\bar{U}(s,\tau)$.

VI. BREAKUP AND CRITICAL DISTANCES FOR PROPAGATING PULSES

The breakup of the pulses propagating in fiber amplifiers is connected with generation of the optical shock-type instabilities due to TOD [25,26]. The numerical simulations of Eq. (1) demonstrate that these instabilities can be identified with some distances $z = z_c$ for which the shape of the pulses becomes "noisy."

We have also shown in Sec. IV that the transformation given by Eq. (51) maps the solution of the NLS onto the solution for fiber with decreasing second-order and third-order dispersions. Hence the same mechanism (the optical shock-type instability due to TOD) leads to breakup of the pulses propagating in fiber with decreasing second-order and third-order dispersions.

It was shown in Sec. III that the critical length z_c and the critical parameter $\varepsilon_c = \varepsilon(z_c)$ ($|\varepsilon(z)| \leqslant |\varepsilon_c|$) can be calculated analytically using the condition that, at $z = z_c$, the two roots of the polynomial $Q(z,T)$ given by Eq. (28) are equal: $T_2(z_c) = T_3(z_c)$ or $T_1(z_c) = T_2(z_c)$ for positive and negative β_3, respectively. For distances $z > z_c$, Eqs. (37)–(42) breaks down, because in this case the polynomial $Q(z,T)$ has only one real root which is unphysical (the pulse must be bounded). Thus we can identify the critical distance z_c with the distance at which the TOD generate pulse breakup [27].

The condition $T_2(z_c) = T_3(z_c)$ [or $T_1(z_c) = T_2(z_c)$] yields the critical parameter ε_c as

$$32\varepsilon_c^4 - 429\varepsilon_c^2 + 12 = 0. \quad (76)$$

The solution of Eq. (76) is $\varepsilon_c = \pm 0.167$ for positive and negative β_3, respectively. The critical distance z_c is defined by equation $\varepsilon(z_c) = \varepsilon_c$ which yields

$$z_c = \frac{3}{g}\ln\frac{|\varepsilon_c|}{|\varepsilon_0|}, \quad \varepsilon_0 = \frac{\beta_3}{6\beta_2^2}\left(\frac{1}{2}\gamma\beta_2 g E_0\right)^{1/3}, \quad (77)$$

with $\varepsilon_0 = \varepsilon(0)$. Hence the critical distance at which the pulse breaks up is

$$z_c = \frac{1}{g}\ln\left(\frac{\sigma\beta_2^5}{\gamma g E_0|\beta_3|^3}\right), \quad \sigma = 432|\varepsilon_c|^3, \quad (78)$$

where $\sigma = 2.01$ for $\varepsilon_c = \pm 0.167$.

In the case of pulse propagation in fiber with decreasing dispersions from Eq. (51) follows that $s_c = g^{-1}(e^{g z_c} - 1)$. Thus [see Eqs. (49) and (50)] the critical length s_c in a fiber at which the pulse breaks up is

$$s_c = \frac{1}{g}\left(\frac{\sigma\beta_2^5}{\gamma g E_0|\beta_3|^3} - 1\right). \quad (79)$$

The critical distance z_c for amplifiers with TOD and distributed gain $g(z)$ follows from the condition that two roots of the polynomial $\hat{Q}(z,T) = 1 - \varepsilon_1(z)T - T^2 + \varepsilon_2(z)T^3$ are equal: $T_2(z_c) = T_3(z_c)$ or $T_1(z_c) = T_2(z_c)$ for positive and negative β_3, respectively (all roots are real for $z \leqslant z_c$). This condition yields the equation for critical distance z_c as

$$4\varepsilon_2(z_c)\varepsilon_1(z_c)^3 + \varepsilon_1(z_c)^2 + 18\varepsilon_1(z_c)\varepsilon_2(z_c) - 27\varepsilon_2(z_c)^2 + 4 = 0. \quad (80)$$

The Eq. (80) reduces to Eq. (76) when $g =$ const. because in this case $\varepsilon_1(z) = \varepsilon(z)$ and $\varepsilon_2(z) = (8/3)\varepsilon(z)$. We note that Eqs. (12)–(19) are invariant under exchange $\beta_3 \to -\beta_3$ because the functions $\alpha_1(z),\alpha_3(z)$ and $c_1(z),c_3(z)$ are proportional to β_3. The Eq. (80) is also invariant under the exchange $\beta_3 \to -\beta_3$ because the functions $\varepsilon_1(z)$ and $\varepsilon_2(z)$ are proportional to $\alpha_1(z)$ and $\alpha_3(z)$ [see Eqs. (66) and (67)]. Thus we may assert that the critical distance z_c does not depend on sign of β_3. In the particular case $g =$ const. this statement directly follows from Eq. (78). In the general case of pulse propagation in fiber with

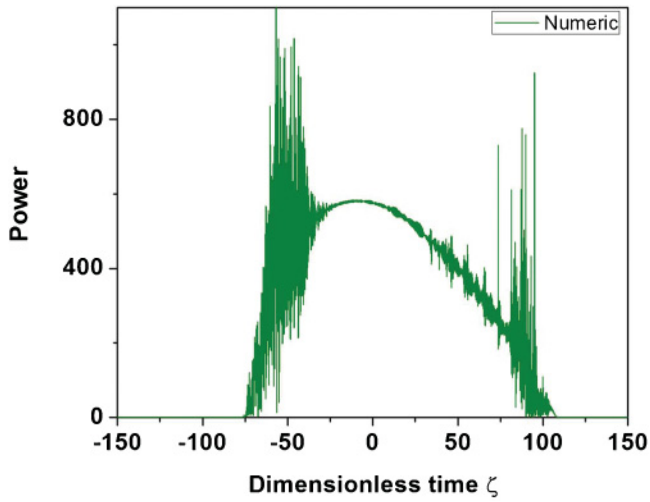

FIG. 3. (Color online) Numerical pulse profile for dimensionless distance $\xi = 14.43$ with $\kappa = 3.018 \times 10^{-2}$ and $\mathcal{V}_0 = 3.922 \times 10^{-2}$.

decreasing dispersions [see Eqs. (72) and (73)] the critical length s_c in fiber is defined by equation $s_c = \int_0^{z_c} e^{G(z')}dz'$, where z_c is given by Eq. (80).

The numerical simulations show that for sufficiently long propagation distances [about $z = z_c$ in Eq. (78)], side peaks gradually develop on the main pulse leading to total breakup of the pulses. Such an example is shown in Fig. 3 with dimensionless time $\zeta = \sqrt{g/\beta_2}\tau$ and power $u^2 = (\gamma/g)|\psi|^2$.

To verify the accuracy of Eq. (78) in numerical simulations we should define some visual criterion for breakup of the pulses. We note that different parameters σ in Eq. (78) yield different critical distances for breakup of the pulses. For an example, we may define the critical distance z_c for breakup of the pulse as the propagation length for which a side peak arises near the pulse with the same maximum power as in the main pulse. Such a case is demonstrated in Fig. 4 for the distance $\xi = \xi_c = 14.09$. Our simulations show that this definition for critical length yields $\sigma = 1.4$ and hence we find that $\varepsilon_c = \varepsilon(z_c) = \pm 0.148$ for positive and negative β_3, respectively, because $\sigma = 432|\varepsilon_c|^3$. This means that for this definition of the critical length, the pulse breaks up slightly before the two real roots of the polynomial $Q(z,T)$ are equal.

Using the above criterion for the breakup of the pulse it is easy to recognize the critical distances in numerical simulations. This criterion is demonstrated in Fig. 4 with dimensionless time $\zeta = \sqrt{g/\beta_2}\tau$ and power $u^2 = (\gamma/g)|\psi|^2$. We present the comparison of the numerical results and analytical prediction for critical distances [with $\sigma = 2.01$ and $\sigma = 1.4$ in Eq. (78)] in the next section.

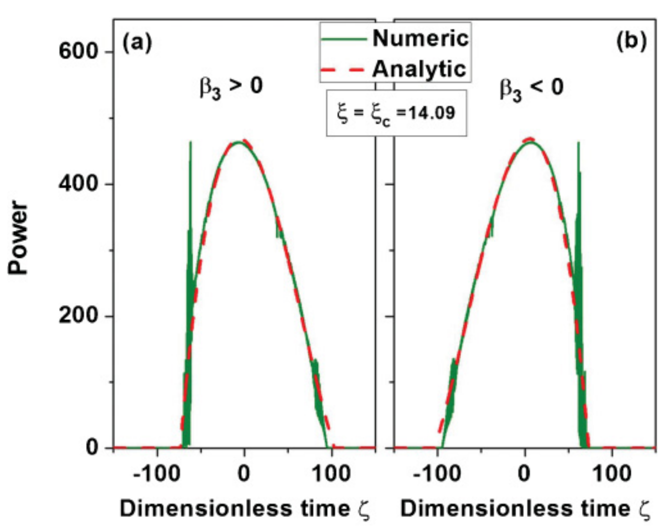

FIG. 4. (Color online) Numerical and analytical (renormalization parameter $n = 3$) pulse profiles for dimensionless distance $\xi = 14.09$ and $\mathcal{V}_0 = 3.922 \times 10^{-2}$; (a) $\kappa = 3.018 \times 10^{-2}$ and (b) $\kappa = -3.018 \times 10^{-2}$.

VII. CRITERION FOR ASYMPTOTICAL EVOLUTION AND NUMERICAL SIMULATIONS

It is useful to define in the case $g = \text{const.}$ for numerical simulations the dimensionless variables $\xi = gz$ and $\zeta = \tau/\tau_g$, where $\tau_g = \sqrt{\beta_2/g}$. In this case Eq. (1) has the dimensionless form:

$$iu_\xi = \frac{1}{2}u_{\zeta\zeta} + \frac{i\kappa}{6}u_{\zeta\zeta\zeta} - |u|^2u + \frac{i}{2}u, \tag{81}$$

where

$$u(\xi,\zeta) = \sqrt{\frac{\gamma}{g}}\psi(z,\tau), \quad \kappa = \frac{\beta_3}{\beta_2}\sqrt{\frac{g}{\beta_2}}. \tag{82}$$

Equation (78) for the critical distance can also be written in dimensionless form as

$$\xi_c \equiv gz_c = \ln\left(\frac{\sigma}{\mathcal{V}_0|\kappa|^3}\right), \tag{83}$$

where $\mathcal{V}_0 = \int_{-\infty}^{+\infty}|u(0,\zeta)|^2 d\zeta = \gamma E_0/\sqrt{g\beta_2}$ is the dimensionless input pulse energy.

The numerical simulations show that for a given initial pulse the asymptotical regime is described by the theory with high accuracy for long distances $\xi \gg 1$. From this condition it follows that the inequality $\xi_c \geqslant \xi_a$ must be satisfied for some fixed dimensionless parameter ξ_a ($\xi_a \gg 1$) because $\xi \leqslant \xi_c$. Thus, using Eq. (77) we find that an asymptotical quasi-similariton pulse evolution regime is described by this theory when

$$|\varepsilon_0| = \frac{|\beta_3|}{6\beta_2^2}\left(\frac{1}{2}\gamma\beta_2 g E_0\right)^{1/3} \leqslant |\varepsilon_c|e^{-\xi_a/3}, \tag{84}$$

with the proper value for the parameter ξ_a. The parameter ξ_a in this equation defines the accuracy of the asymptotical theory.

We also may introduce the dimensionless parameter δ of the theory [27]:

$$\delta = \mathcal{V}_0|\kappa|^3 = \frac{\gamma g E_0|\beta_3|^3}{\beta_2^5}. \tag{85}$$

Because $\delta = 432|\varepsilon_0|^3$, the condition given by Eq. (84) is equivalent to the condition $\delta \leqslant \sigma e^{-\xi_a}$. As an example, using

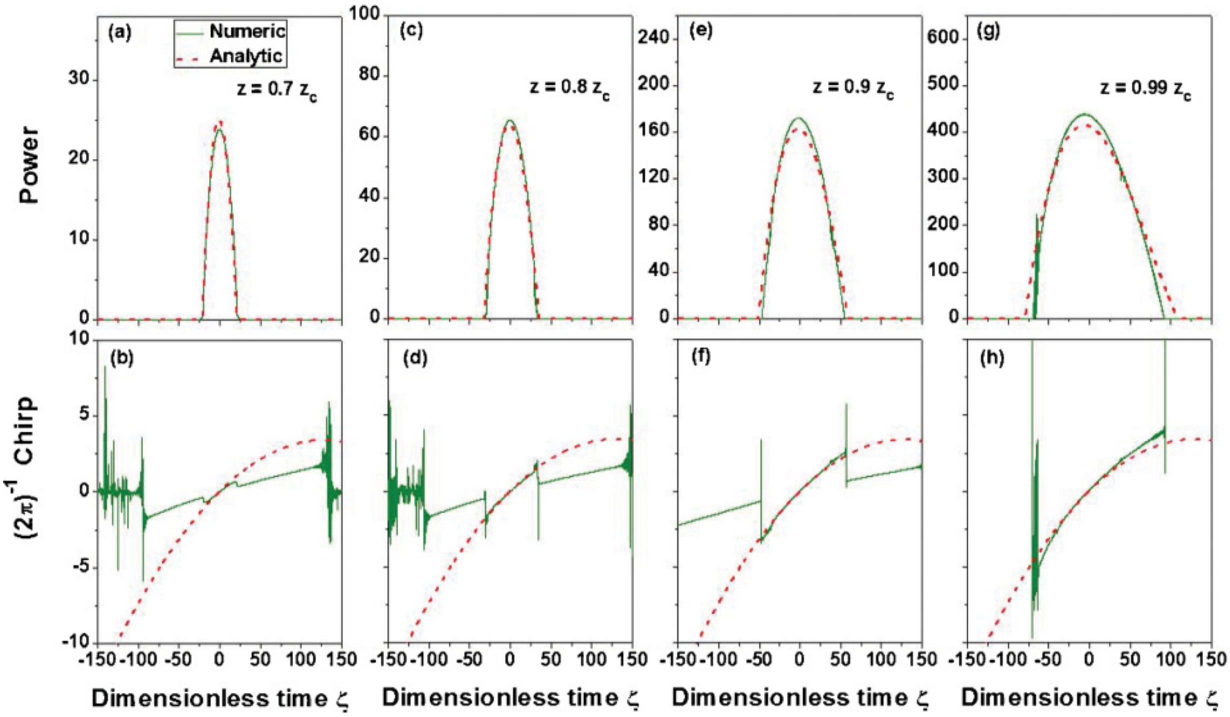

FIG. 5. (Color online) Pulse power profile and chirp of the numerical and analytical (nonrenormalized) solutions for propagating distances $z = 0.7z_c$ [(a) and (b)], $z = 0.8z_c$ [(c) and (d)], $z = 0.9z_c$ [(e) and (f)], and $z = 0.99z_c$ [(g) and (h)] with $\mathcal{V}_0 = 3.922 \times 10^{-2}$ and $\kappa = 3.018 \times 10^{-2}$.

$\xi_a \simeq 10$ we have the condition $\delta \leqslant 0.9 \times 10^{-4}$ which is equivalent to the condition $|\varepsilon_0| \leqslant 0.6 \times 10^{-2}$, providing high accuracy for the analytical solution.

A. Numerical simulations

In the numerical simulations we use Eq. (81) with a Gaussian initial pulse: $u(0,\zeta) = \pi^{-1/4}\sqrt{\mathcal{V}_0}\exp(-\zeta^2/2)$. Let

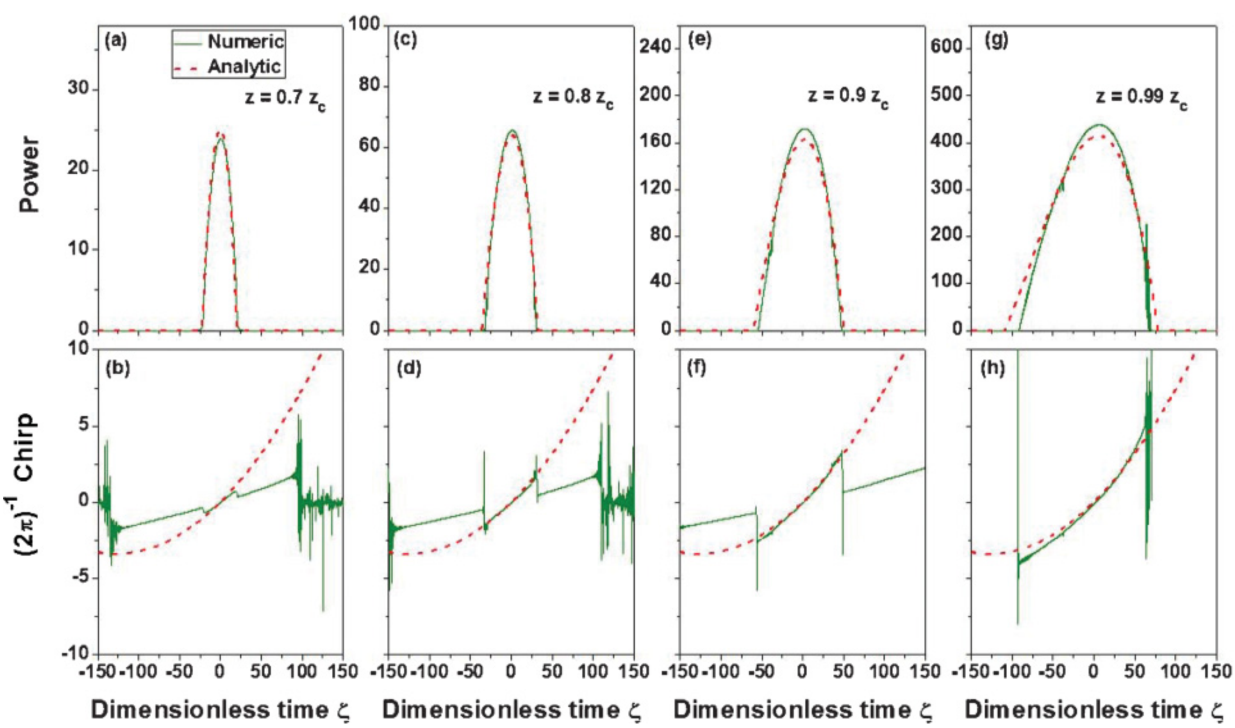

FIG. 6. (Color online) Pulse power profile and chirp of the numerical and analytical (nonrenormalized) solutions for propagating distances $z = 0.7z_c$ [(a) and (b)], $z = 0.8z_c$ [(c) and (d)], $z = 0.9z_c$ [(e) and (f)], and $z = 0.99z_c$ [(g) and (h)] with $\mathcal{V}_0 = 3.922 \times 10^{-2}$ and $\kappa = -3.018 \times 10^{-2}$.

us use the fiber amplifier parameters as an example ($\beta_2 = 0.13$ ps^2 m^{-1}, $\beta_3 = \pm 10^{-3}$ ps^3 m^{-1}, $\gamma = 2 \times 10^{-3}$ W^{-1} m^{-1}, $g = 2$ m^{-1}) and the input energy of the pulse ($E_0 = 10$ pJ). In this case we find $\varepsilon_0 = \pm 1.356 \times 10^{-3}$ and the condition $|\varepsilon_0| \leqslant 0.6 \times 10^{-2}$ is satisfied. We note that in this case the dimensionless energy of the initial pulse is $\mathcal{V}_0 = 3.922 \times 10^{-2}$ and the absolute value of the parameter $\kappa = \pm 3.018 \times 10^{-2}$ is small. The dimensionless parameter of the theory given by Eq. (85) in this case is $\delta = 1.078 \times 10^{-6}$.

Using these parameters of the fiber amplifier, we present in Fig. 5 ($\beta_3 > 0$) and Fig. 6 ($\beta_3 < 0$) the temporal profile and chirp of the pulses for analytical solution given by Eqs. (37)–(41) (dotted curves) and numerical solution (solid curves) for different propagating distances. The chirp of the analytic solution matches well the chirp of the numerical pulse for a wide range of distances. A small mismatch in the profile of the pulses can be observed for both a positive and a negative β_3 for distances close to z_c.

The agreement between the numerical and analytical solution can be improved for both the chirp and the temporal profile of the pulses using the renormalized analytical solution given by Eqs. (43)–(48). The good matching of the curves in Figs. 7 and 8 is obtained using the renormalization parameter $n = 3$. This parameter can be found both analytically by a variational procedure or numerically.

The power profile and the chirp of the pulses for the above parameters of the fiber amplifier and input energy ($\kappa = 3.018 \times 10^{-2}$ and $\mathcal{V}_0 = 3.922 \times 10^{-2}$) are shown in Fig. 7 for analytical (dotted curves) and numerical (solid curves) solutions for propagating distances $z = 0.7z_c$, $z = 0.8z_c$, $z = 0.9z_c$, and $z = 0.99z_c$, respectively. In the case of negative TOD ($\kappa = -3.018 \times 10^{-2}$) and the same input energy the power profile and the chirp of the pulses of fiber amplifier are shown in Fig. 8.

We also illustrate in Fig. 9 the accuracy of the critical length calculated by Eq. (78) or Eq. (83) at which the pulse breaks up. An error about 2.4% can be observed between the numerical (triangles) and analytical (solid) curves with $\kappa = 0.03018$ and $\sigma = 2.01$ when the condition $|\varepsilon_0| \leqslant 0.13 \times 10^{-2}$ is satisfied. Such an error can be explained by the fact that we have used in numerical simulations the criterion of breakup of the pulses as it was defined in Sec. VI (this criterion is demonstrated in Fig. 4 as well). It is evident that different criterions of breakup of the pulses lead to different critical distances. Hence, to make consistent the particular criterion of breakup of the pulses with the critical length given by Eq. (78) or Eq. (83) one should choose an appropriate parameter σ in these equations. We note that the parameter $\sigma = 2.01$ was found from the condition that two roots of the polynomial $Q(z,T)$ given by Eq. (28) are equal. The numerical simulations in Fig. 9 show that this parameter should be adjusted (for the criterion defined in Sec. VI).

Much better precision for critical length z_c with error $\leqslant 0.2\%$ can be obtained using the numerical parameter $\sigma = 1.4$ in Eq. (78) and Eq. (83), which corresponds to the criterion for breakup of the pulses defined in Sec. VI (equal height of the side peak and main peak). The dependence of the dimensionless critical length $\xi_c = gz_c$ on the dimensionless

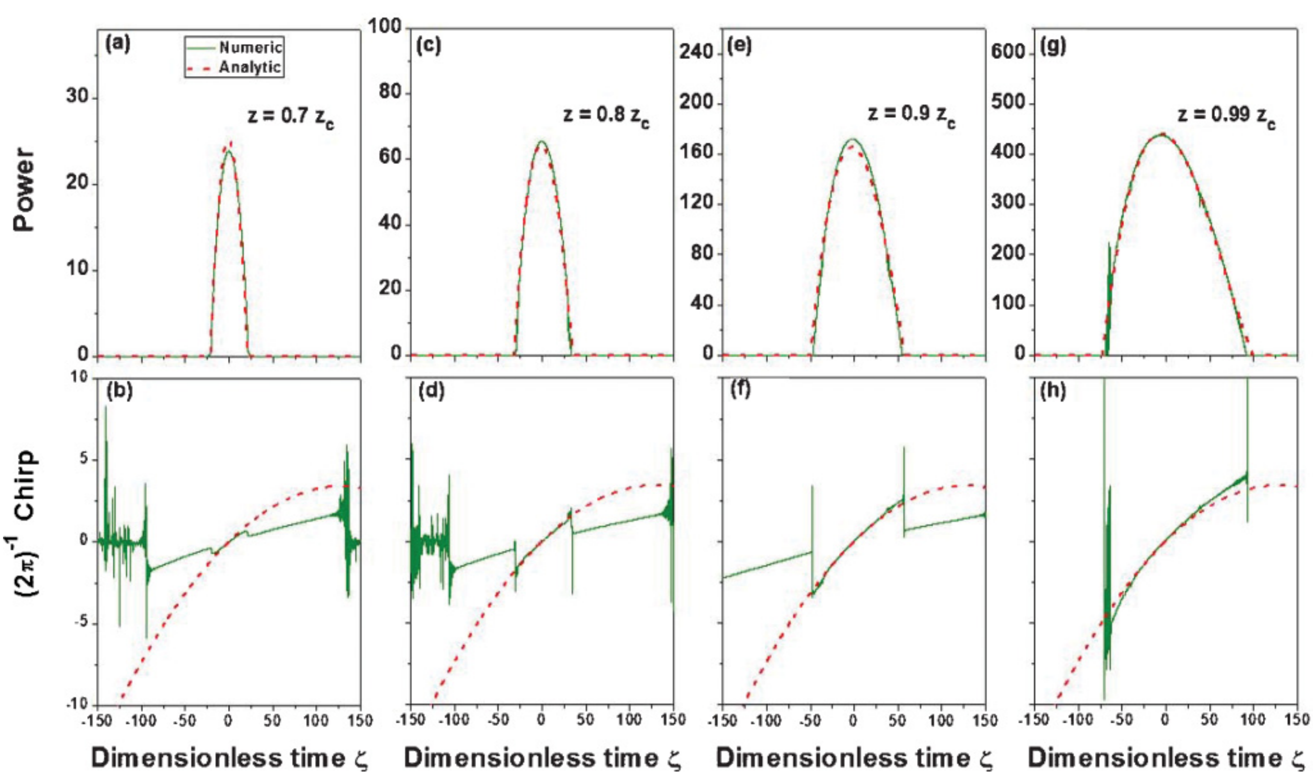

FIG. 7. (Color online) Pulse power profile and chirp of the numerical and analytical renormalized ($n = 3$) solutions for propagating distances $z = 0.7z_c$ [(a) and (b)], $z = 0.80z_c$ [(c) and (d)], $z = 0.90z_c$ [(e) and (f)], and $z = 0.99z_c$ [(g) and (h)] with $\mathcal{V}_0 = 3.922 \times 10^{-2}$ and $\kappa = 3.018 \times 10^{-2}$.

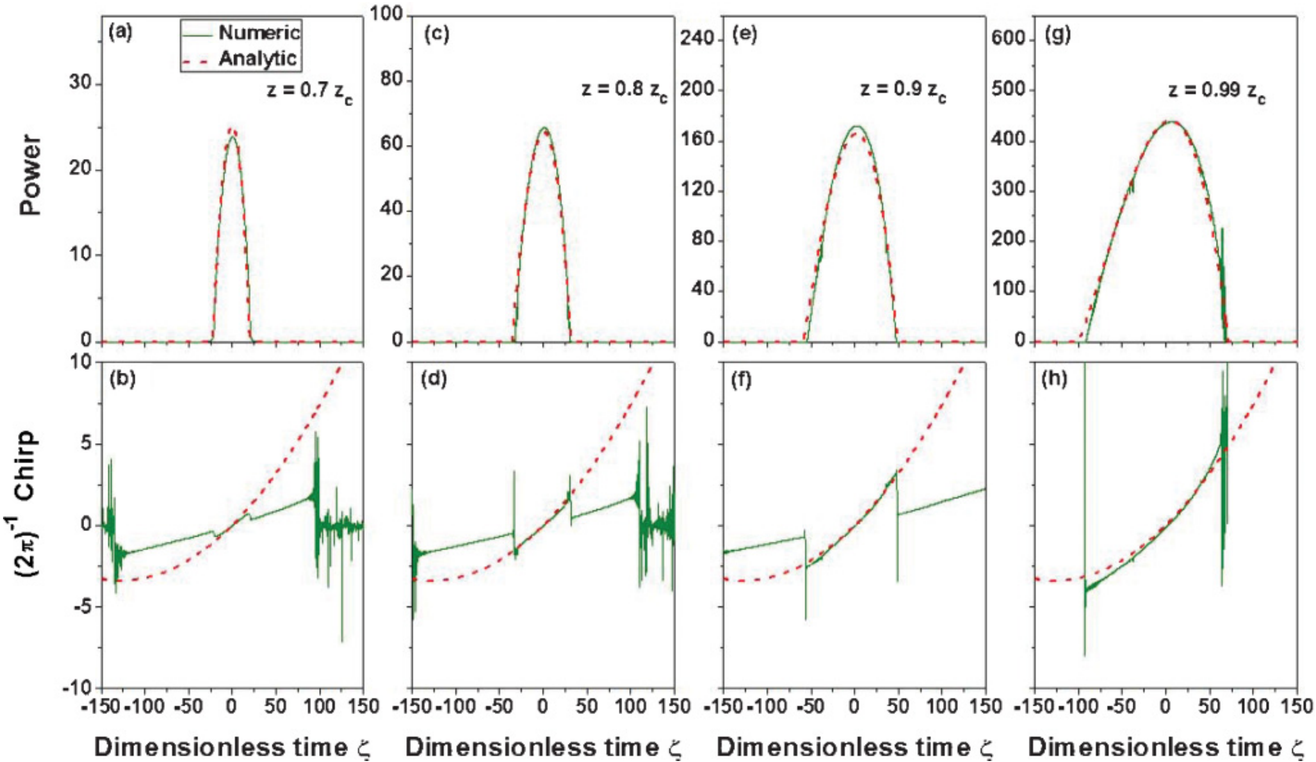

FIG. 8. (Color online) Pulse power profile and chirp of the numerical and analytical renormalized ($n = 3$) solutions for propagating distances $z = 0.7z_c$ [(a) and (b)], $z = 0.80z_c$ [(c) and (d)], $z = 0.90z_c$ [(e) and (f)], and $z = 0.99z_c$ [(g) and (h)] with $\mathcal{V}_0 = 3.922 \times 10^{-2}$ and $\kappa = -3.018 \times 10^{-2}$.

input pulse energy $\mathcal{V}_0$ with the parameter $\sigma = 1.4$ is shown in Fig. 9 (broken curve).

For the numerical parameter $\sigma = 1.4$ one can find that $\varepsilon_c = \pm 0.148$ ($\sigma = 432|\varepsilon_c|^3$). Hence, we have the next bound for small dimensionless parameter $\varepsilon(z)$: $|\varepsilon(z)| \leqslant 0.148$. The numerical simulations have been performed using a standard split-step Fourier method [31] to Eq. (81) with a step size $\Delta\xi = 10^{-4}$.

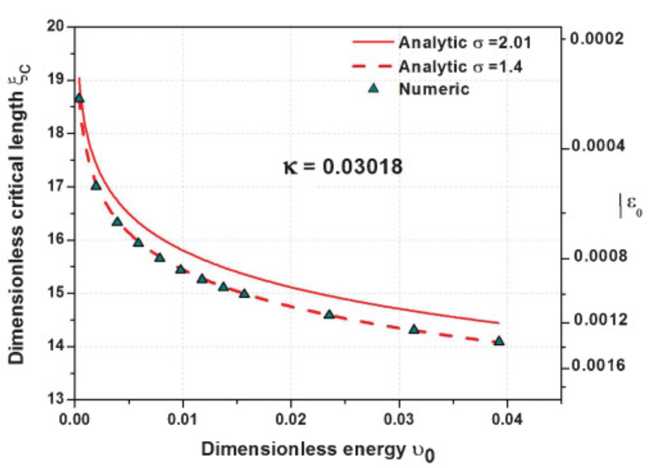

FIG. 9. (Color online) Numerical and analytical (for $\sigma = 2.01$ and $\sigma = 1.4$) dependence of the dimensionless critical length $\xi_c = gz_c$ on the dimensionless input pulse energy $\mathcal{V}_0$.

VIII. CONCLUSIONS

In this paper we have developed an analytical theory for propagating pulses in normal-dispersion fiber amplifiers with TOD, and for a dispersion-decreasing fiber with TOD. The analytical solution of the generalized nonlinear Schrödinger equation developed here is based on asymptotical methods, first-order perturbation theory, and a renormalization procedure. We have also formulated a renormalization procedure for the solution which yields the exact energy with distance of the pulse and takes into account higher orders of small parameter in the perturbation theory for the effective width of the pulse. We have also found the critical length z_c at which the TOD generates pulse breakup in the case $g = \text{const}$..

In the general case for fiber amplifiers with TOD and varying gain function we have proved in the first-order perturbation theory that the critical distance z_c does not depend on the sign of TOD.

It has been shown that there is a limitation on the input value ε_0 of the distance dependent small dimensionless parameter $\varepsilon(z)$. We have found a criterion which ensure the accuracy of the asymptotic solutions. This criterion is confirmed numerically showing that the analytical description of the pulses and the critical length formulas developed here for fiber amplifiers and dispersion-decreasing fibers with third-order dispersion are very accurate. This theory should prove valuable in the design of high-power optical fiber amplifiers and lasers where the performance can be seriously affected by third-order dispersion in the gain medium.

APPENDIX: GENERAL RENORMALIZATION PROCEDURE

The renormalization procedure in general case ($g \neq$ const.) is also based on the exchange $E(z) \to E(z)/\eta_k(z)$ and $w(z) \to w_k^{(n)}(z)$ in Eqs. (64), (65) and (68), where $w_k^{(n)}(z)$ is a new effective width:

$$w_k^{(n)}(z) = \frac{w(z)}{\eta_k(z)^n}. \quad \text{(A1)}$$

Here $w(z)$ is the width of the parabolic pulse satisfying to Eq. (63) and the functions $\eta_k(z)$ $(k = 1,2)$ are given by

$$\eta_k(z) = \frac{3}{4}[T_{k+1}(z) - T_k(z)] - \frac{3\varepsilon_1(z)}{8}\left[T_{k+1}^2(z) - T_k^2(z)\right] - \frac{1}{4}\left[T_{k+1}^3(z) - T_k^3(z)\right] + \frac{3\varepsilon_2(z)}{16}\left[T_{k+1}^4(z) - T_k^4(z)\right], \quad \text{(A2)}$$

where $k = 1$ for $\beta_3 > 0$ and $k = 2$ for $\beta_3 < 0$, respectively. The functions $T_k(z)$ are the roots of the polynomial:

$$\hat{Q}(z,T) = 1 - \varepsilon_1(z)T - T^2 + \varepsilon_2(z)T^3, \quad \text{(A3)}$$

with $T = \tau/w_k^{(n)}(z)$. The critical distance z_c for amplifiers with TOD and distributed gain $g(z)$ follows from the condition that two roots of the polynomial $\hat{Q}(z,T)$ are equal: $T_2(z_c) = T_3(z_c)$ or $T_1(z_c) = T_2(z_c)$ for positive and negative β_3, respectively. All roots $T_k(z)$ $(k = 1,2,3)$ are real in the region $z \leqslant z_c$ and we assume that the roots are ordered $T_1(z) < T_2(z) < T_3(z)$ for $z < z_c$.

Thus the exchange $E(z) \to E(z)/\eta_k(z)$ and $w(z) \to w_k^{(n)}(z)$ in Eqs. (64) and (65) yields the renormalized amplitude of the pulse in this general case as

$$U_R(z,\tau) = \left(\frac{3E(z)}{4\eta_k(z)w_k^{(n)}(z)}\right)^{1/2} \sqrt{\hat{\mathcal{Q}}_R(z,\tau)}\hat{\mathcal{I}}_R(z,\tau), \quad \text{(A4)}$$

where the polynomial $\hat{\mathcal{Q}}_R(z,\tau)$ is

$$\hat{\mathcal{Q}}_R(z,\tau) = 1 - \varepsilon_1(z)\left(\frac{\tau}{w_k^{(n)}(z)}\right) - \left(\frac{\tau}{w_k^{(n)}(z)}\right)^2 + \varepsilon_2(z)\left(\frac{\tau}{w_k^{(n)}(z)}\right)^3. \quad \text{(A5)}$$

Here $\varepsilon_1(z)$ and $\varepsilon_1(z)$ are two small dimensionless distance-dependent parameters given by Eqs. (66) and (67).

The exchange $w(z) \to w_k^{(n)}(z)$ in Eq. (65) also yields a new rectangular function $\hat{\mathcal{I}}_R(z,\tau)$ in Eq. (A4). This new rectangular function $\hat{\mathcal{I}}_R(z,\tau)$ is given by Eq. (46), where $w_k^{(n)}(z)$ is defined by Eqs. (A1) and (A2) and $T_k(z)$ are the roots of the polynomial in Eq. (A3).

Using new functions $\hat{\mathcal{Q}}_R(z,\tau)$ and $\hat{\mathcal{I}}_R(z,\tau)$ in Eq. (A4) we may calculate the energy of the pulse for renormalized solution:

$$E(z) = \int_{-\infty}^{+\infty} |\psi(z,\tau)|^2 d\tau = E_0 \exp[G(z)]. \quad \text{(A6)}$$

This equation demonstrates that the energy of the pulse with distance for the renormalized solution is described by an exact equation. However, this renormalization procedure yields a small correction to the solution because for all distances $z \leqslant z_c$ the parameters of the theory $\varepsilon_1(z)$ and $\varepsilon_2(z)$ are small and the functions $\eta_k(z)$ $(k = 1,2)$ are close to 1. Note that the presented renormalization procedure can be interpreted as a perturbation procedure for the effective width with all orders of constant parameter ϵ $(\epsilon \sim \beta_3)$ of the perturbation theory.

In conclusion, we consider the functions $\eta_1(z)$ and $\eta_2(z)$ for two different cases which differ only with the parameter β_3: (1) $\beta_3 = \beta_3^+ > 0$ and (2) $\beta_3 = \beta_3^- < 0$, where $\beta_3^- = -\beta_3^+$. The ordered roots of the polynomial in Eq. (A3) for these two cases are connected as: $T_1^-(z) = -T_3^+(z)$, $T_2^-(z) = -T_2^+(z)$, $T_3^-(z) = -T_1^+(z)$, where the indexes $\pm$ indicate two different cases, (1) and (2), respectively. Using this result and Eq. (A2), we can prove that $\eta_1(z) = \eta_2(z)$ and hence $w_1^{(n)}(z) = w_2^{(n)}(z)$, where the functions $\eta_1(z)$, $w_1^{(n)}(z)$ and $\eta_2(z)$, $w_2^{(n)}(z)$ are defined by Eqs. (A1) and (A2) for cases (1) and (2), respectively.

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