Mach-Zehnder interferometry with interacting trapped Bose-Einstein condensates

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We theoretically analyze a Mach-Zehnder interferometer with trapped condensates and find that it is surprisingly stable against the nonlinearity induced by interparticle interactions. The phase sensitivity, which we study for number-squeezed input states, can overcome the shot noise limit and be increased up to the Heisenberg limit provided that a Bayesian or maximum-likelihood phase estimation strategy is used. We finally demonstrate the robustness of the Mach-Zehnder interferometer in the presence of interactions against condensate oscillations and a realistic atom-counting error.

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Atom interferometry [1] with trapped Bose-Einstein condensates (BECs) is a very promising tool for the most precise measurements. The nonlinearity of BECs makes it possible to create highly squeezed states, which should allow the classical shot noise limit for the phase sensitivity $\Delta\theta = 1/\sqrt{N}$ to be surpassed by a factor of $\sqrt{N}$ up to the Heisenberg limit (HL) $\Delta\theta = 1/N$ [2,3], where N is the number of atoms in the condensates.

Both atom chips [4] and dipole traps [5] allow for versatile control of trapped BECs, and coherent splitting and interference [6,7] have been demonstrated. The preparation of moderately number-squeezed states through splitting of a condensate by transforming a harmonic potential well into a double well [8] has been recently achieved experimentally [9,10], and it has been suggested to use optimal control strategies to create highly squeezed states at short time scales [11], exploiting the atom-atom interactions.

However, according to the current literature it is generally believed that interactions are detrimental for interferometry as they induce phase diffusion [12], thereby decreasing the phase coherence [9,10,13–16]. The proposed standard solution is to make the interactions small by employing Feshbach resonances [17,18] or by using state-selective potentials for internal degrees of freedom [15,16]. This is not always possible, and in many cases it is not desirable, because Feshbach tuning requires field-sensitive states which are, however, not ideal for precision interferometry. Moreover, residual interactions might still decrease the sensitivity.

In this paper, we analyze the *Mach-Zehnder* (MZ) interferometer for BECs trapped in a double-well potential in the presence of atom-atom interactions. We show that the sensitivity is not substantially degraded by the interactions, and Heisenberg scaling can be achieved with the resources of number-squeezed input states and atom-number measurements as the readout. Our scheme is robust against mechanical excitations of the BEC and finite atom-number detection efficiency.

The initial state of the interferometer sequence consists of two uncoupled, stationary BECs with number fluctuations ΔJ_z [19]. We resort here to a generic description characterized by two parameters, tunnel coupling Ω and interaction energy U_0 [8], and discuss a realistic model at the end of the paper. We first introduce the *ideal* (i.e., noninteracting) MZ interferometer as discussed in [20–22]. It consists of two *cold-atom beam splitters* (BSs) with Hamiltonian $H_t = -\Omega\hat{J}_x$, and in between a phase accumulation due to an energy difference ΔE between the two wells (with Hamiltonian $H_e = -\Delta E\hat{J}_z$). We visualize a typical interferometer sequence on the Bloch sphere [13] in Fig. 1(a). A BS corresponds to a $\pi/2$ rotation around the x axis during a time T_t. The first BS transforms the number-squeezed input state into a phase-squeezed one. The second BS transforms an accumulated phase $\theta = \Delta E T_e$ (a z rotation caused by an external potential during a time T_e) into a relative atom-number difference. The whole interferometer transformation can be written as $|\psi_{\rm OUT}^{(\theta)}\rangle = e^{-i\theta\hat{J}_y}|\psi_{\rm IN}\rangle$. A number-squeezed input state [with squeezing factor $\xi_N := \Delta J_z/(\sqrt{N}/2) < 1$] reduces the measurement uncertainty in the atom number of the final state [20,23].

Atom-atom interactions are described by the Hamiltonian $H_i = U_0\hat{J}_z^2$ [24], and the whole interferometer transformation reads now

$$\left|\psi_{\rm OUT}^{(\theta)}\right\rangle = e^{-i(H_t+H_i)T_t}e^{-i(H_e+H_i)T_e}e^{-i(H_t+H_i)T_t}|\psi_{\rm IN}\rangle\,. \quad (1)$$

Even for very small interactions ($U_0 N = 0.1$), the state gets distorted [Fig. 1(b)]. For larger interactions [Fig. 1(c)], the state covers almost the whole sphere. If we employ the usual parameter estimation based on the mean value of all the measurement results [2,20,23], the phase sensitivity is worse than shot noise.

Contrary to the expectations of this estimation, we now show that interactions do not substantially limit interferometry. In a completely general fashion we use the *quantum Fisher information* $F_Q(|\psi_{\rm IN}\rangle) = 4(\Delta R)^2$, where $\hat{R}$ is the generator of the interferometer transformation [22,25], to compute the *Cramér-Rao lower bound* (CRLB), which determines the best possible phase sensitivity independent of the choice of the measured observable [26]. For the interferometer transformation Eq. (1), we find

$$\Delta\theta_{\rm CRLB} \geqslant \frac{1}{\sqrt{mF_Q(|\psi_{\rm IN}\rangle)}} = \frac{1}{\sqrt{m}2\Delta J_z(t=T_t)}\,, \quad (2)$$

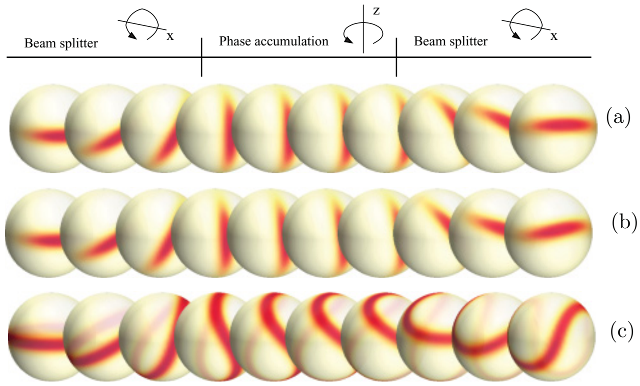

FIG. 1. (Color online) Mach-Zehnder interferometer sequence for a finite phase θ (a) in the absence and (b), (c) in the presence of interactions, visualized on the Bloch sphere ($T_t = T_e = 1$). (a) A number-squeezed initial state (small width along the z axis, number-squeezing factor $\xi_N = 0.2$) is transformed into a phase-squeezed one (small width along the equator) by a BS (rotation around the x axis). Next a phase is accumulated due to an external potential (rotation around the z axis). A second BS transforms the state such that the phase is mapped onto a number difference. (b) Even for very small interactions ($U_0 N = 0.1$), the number squeezing in the final state is lost. (c) For larger interactions ($U_0 N = 1$), the initial state (here a Fock state) gets strongly distorted and winds around the Bloch sphere.

i.e., it is given by the number fluctuations after the first nonlinear BS [27]; m denotes the number of independent measurements.

We start by analyzing a Fock input state $|\psi_{\rm IN}\rangle \propto (\hat a_R^\dagger)^{N/2}(\hat a_L^\dagger)^{N/2}|0\rangle$. From the scaling of $H = H_t + H_i$ with N we find $\Delta J_z(t = T_t) \approx \alpha N$ (with constant α). Thus, we expect Heisenberg scaling $\propto 1/N$ whenever $U_0 N$ is constant for increasing N [28].

Now we have to clarify whether one can indeed achieve a sensitivity close to the Heisenberg limit if one is restricted to a number measurement as in experiments. The classical Fisher information (CFI) [25]

$$F(\theta,|\psi_{\rm IN}\rangle) = \int dn \frac{1}{P(n|\theta)}\left(\frac{\partial P(n|\theta)}{\partial\theta}\right)^2 \qquad (3)$$

allows one to estimate a lower bound of $\Delta\theta = 1/\sqrt{mF(\theta)}$ for this specific type of measurement (we consider $\theta \ll 1$). Hereby, $P(n|\theta) = |\langle n|\psi_{\rm OUT}^{(\theta)}\rangle|^2$ is the conditional probability that an atom-number difference n is measured for phase θ. Below we choose a constant $U_0 N = 1$ and vary the BS and accumulation times T_t and T_e (with $\Omega T_t = \pi/2$ fixed). The influence of larger interactions can then be extracted through simple rescaling.

Heisenberg scaling $\Delta\theta = \beta/N$ persists in the presence of interactions also for a number measurement; see Fig. 2(a). The sensitivity is degraded only by an almost N-independent prefactor β, which varies with T_t as is shown in Fig. 2(b) (dark solid line). Fast BSs [$T_t \sim 1/(U_0 N)$] give rise to a prefactor of ~ 1, but also for slower BSs we can exploit quantum correlations for MZ interferometry, which is relevant for relatively large interactions ($U_0 N = 10$) [13].

The number readout works because interactions transform the conditional probability distributions $P(n|\theta)$, which have

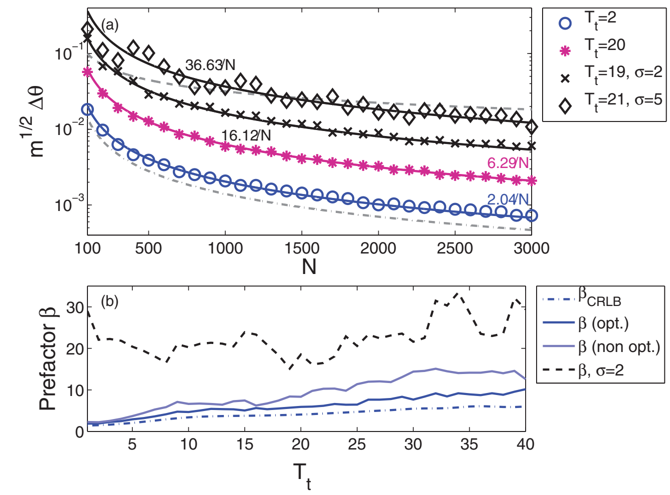

FIG. 2. (Color online) (a) Scaling of $\sqrt{m}\Delta\theta$ with N for $T_t = 2$ (circles), $T_t = 20$ (stars), and finite detection error ± 2 (crosses, for $T_t = 19$) and ± 5 (diamonds, $T_t = 21$) ($U_0 N = 1$ and T_e fixed), compared to shot noise (dashed line) and the $U_0 = 0$ HL $\sqrt{m}\Delta\theta = 1.4\sqrt{m}/N$ (dash-dotted line). (b) The prefactor β (obtained from fitting) is shown for optimized $T_e < 40$ (dark solid line), compared to $\sqrt{m}\Delta\theta_{\rm CRLB} N$ (dash-dotted line) and $\sqrt{m}\Delta\theta N$ for nonoptimized values of $T_e < 40$ (shaded solid line). Also results for a finite detection error of ± 2 are shown (dashed line).

for the ideal MZ and $\theta = 0$ a single peak with width 1, into a complicated pattern with substructures of the same width [see Figs. 3(c), 3(f), 3(i), and 3(l)]. These serve as the measurement stick and determine the smallest phase which can be resolved [22]. The patterns vary with T_e, such that some of them show more distinct $1/N$-sized peaks [blue line in Fig. 3(c)], maximizing the CFI of Eq. (3) better than others (bright red line).

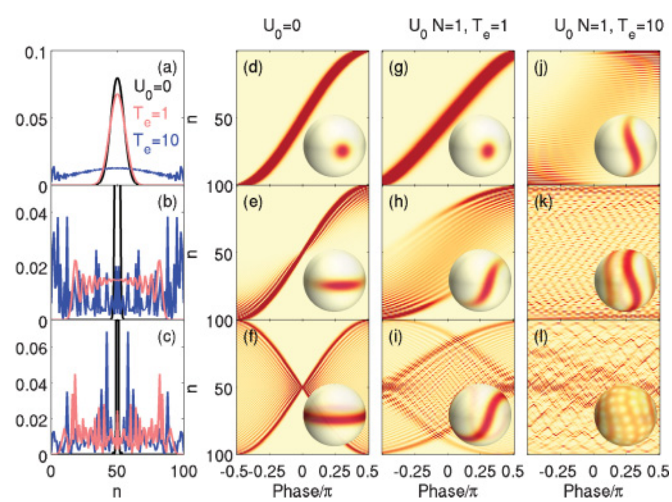

FIG. 3. (Color online) Probabilities $P(n|\theta) = |\langle n|\psi_{\rm OUT}^{(\theta)}\rangle|^2$ for (a), (d), (g), and (j) binomial, (b), (e), (h), and (k) moderately number squeezed ($\xi_N = 0.2$), and (c), (f), (i), and (l) Fock states for $N = 100$, $U_0 N = 1$, and $T_t = 1$; (d)–(f) the ideal case (no interactions), (g)–(i) the interacting case for $T_e = 1$, and (j)–(l) the interacting case for $T_e = 10$. The states are also shown on the Bloch sphere. (a)–(c) $P(n|\theta = 0)$.

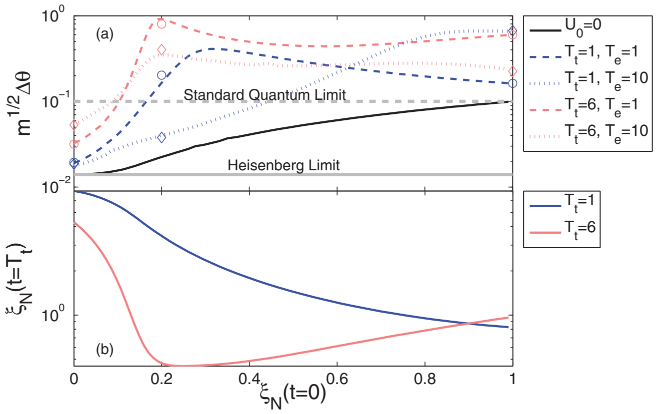

FIG. 4. (Color online) (a) Phase sensitivity $\sqrt{m}\Delta\theta$ for different interaction strengths vs the initial number fluctuations $\xi_N(t=0)$ for $U_0N=1$ ($N=100$). The symbols show results for a simulated Bayesian phase estimation (circles, $T_e=1$; diamonds, $T_e=10$). (b) Number fluctuations after the first BS, $\xi_N(t=T_t)$, which determine the CRLB, Eq. (2).

Thus, the number measurement is not the "optimal" measurement for all values of T_e [26]. We compare the maximal and minimal prefactors, which can be obtained by varying T_e [shaded and dark solid lines, respectively, in Fig. 2(b)]. The latter lies close to the CRLB (dash-dotted line). Most importantly, there is no upper limit to T_e, which allows, in principle, signals to be accumulated for a very long time.

In many experimental situations only input states with finite number squeezing $\xi_N<1$ are available. For the ideal MZ the sensitivity increases monotonously with number squeezing [black line in Fig. 4(a)], up to the HL $\sqrt{m}\Delta\theta = 1.4\sqrt{m}/N$.

We start with analyzing the case of *long BS times* T_t (bright red lines). We find a transition of the phase sensitivity as a function of ξ_N: Starting from $\xi_N=1$ (binomial state), the sensitivity first decreases up to a point, say around $\xi_N=0.2$–0.3. Then it becomes better again and finally approaches the HL for very small values of ξ_N. Also the CRLB, Eq. (2), which is a strict lower bound to $\Delta\theta$, shows a transition. The reason is an absence of number fluctuations after the first BS whenever the input state is only moderately number squeezed [red line in Fig. 4(b)].

For *short BS times* T_t (blue lines), we find a transition only for short phase accumulation time T_e (blue dashed line). In contrast, a longer T_e gives a monotonous behavior (blue dotted line) similar to the CRLB [blue line in Fig. 4(b)].

We can get insight into this behavior from the conditional probabilities $P(n|\theta)$ for different input states in Fig. 3. For binomial and moderately number-squeezed states [Figs. 3(d) and 3(e)], they are close to Gaussian shape [black lines in Figs. 3(a) and 3(b)]. Interactions wash out the structure of the squeezed state [Fig. 3(h)] and increase the variance in the final atom-number distribution. Thereby the coherent spin squeezing of the initial state is decreased to values even worse as compared to the more robust binomial initial state [14]. For longer T_e, interactions induce substructures [Fig. 3(k)]. In contrast, for a Fock input state a complex pattern emerges even for very small T_e [Fig. 3(i)], whereas a binomial input state does not build up any substructure at all [Figs. 3(g) and 3(j)]. Visualized on the Bloch sphere, $P(n|\theta)$ shows an interference pattern whenever a state winds around for a long enough time such that it becomes a superposition of different phase components [Figs. 3(k), 3(i), and 3(l)].

In real experiments, $\Delta\theta$ as calculated from the CFI can be obtained by using a Bayesian (or alternatively maximum-likelihood) phase estimation protocol [29]. Thereby a series of m measurements is performed, and the atom-number difference of each measurement is used for the phase estimation. We find that such a protocol gives sensitivity in accordance with the more general lower bounds as reported by the symbols in Fig. 4(a) (for $m=20$).

The MZ interferometer is robust against shot-to-shot fluctuations in the atom number or nonlinearity [30]. A finite atom-counting error has the effect of broadening the substructures in the probability distributions as $\tilde{P}(n|\theta) \propto \sum_k P^{\text{error}}(n|k)P(k|\theta)$, where $P^{\text{error}}(n|k)$ is the error probability for measuring k atoms instead of n. In Fig. 2 we show that a binomial error probability with width $\sigma=2$ gives rise to just another prefactor, because a constant detection error is less important for larger N. Even for a detection error $\sigma=5$ [9,31], subshot noise can be found for $N>2000$.

Implementing the interferometer with trapped condensates, one achieves the BS by lowering the barrier between two split condensates, thereby introducing tunneling. The full two-mode physics including the spatial dynamics can be accounted for by the *multiconfigurational time-dependent Hartree for bosons* (MCTDHB) method [32], which represents a framework using time-dependent mode functions. For a typical trapping geometry on atom chips with $\omega_\perp = 2\pi\times 2$ kHz transverse frequency, we find that for tunnel pulses on the order of several milliseconds, rapid oscillations are induced in the condensates, which lead to unwanted excitations [33]. In our earlier work [11,13] we have developed and demonstrated optimal control [34,35] within the MCTDHB method. This allows us to design controls for fast BS operations without exciting the condensates, which is achieved by *trapping* the condensates in stationary states after each of the two BSs, while at the same time achieving appropriate tunnel pulses.

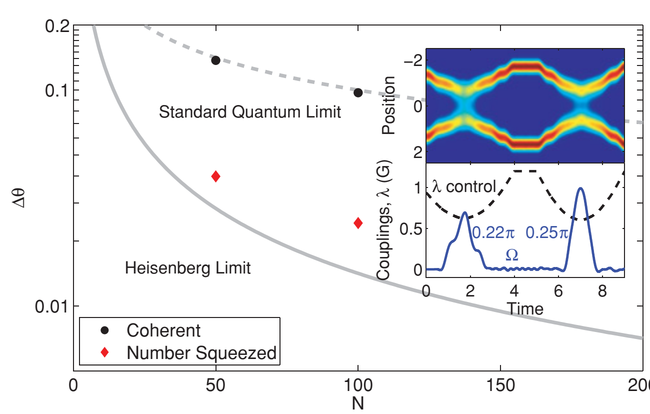

FIG. 5. (Color online) $\Delta\theta$ for realistic control sequences calculated with MCTDHB method for $U_0N=0.1$ and a binomial state with $T_t=8$, and $U_0N=1$ and a highly number-squeezed state with $\xi_N(t=0)=0.05$ and $T_t=4$. The insets show the density (top), and the optimal control and tunnel coupling (bottom) for the squeezed state and $N=100$.

An approximately $\pi/4$ tunnel pulse [36] is achieved for $T_t = 4$ and highly number-squeezed input states. In Fig. 5 we show results for binomial and number-squeezed input states with a phase sensitivity close to the HL.

To summarize, we analyzed the phase sensitivity of a trapped BEC Mach-Zehnder interferometer in the presence of interactions. Heisenberg scaling can be achieved for an atom-number measurement, and there is no upper limit to the phase accumulation time. For finitely number-squeezed input states the phase sensitivity is characterized by a transition. We demonstrated robustness against condensate oscillations and finite detection error, and thus our results can be compared to current experiments.

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