Entanglement dynamics of three interacting two-level atoms within a common structured environment

Nguyen Ba An[*]
Center for Theoretical Physics, Institute of Physics, 10 Dao Tan, Hanoi, Vietnam and School of Computational Sciences, Korea Institute for Advanced Study, 207-43 Cheongryangni 2-dong, Dongdaemun-gu, Seoul 130-722, Korea

Jaewan Kim[†]
School of Computational Sciences, Korea Institute for Advanced Study, 207-43 Cheongryangni 2-dong, Dongdaemun-gu, Seoul 130-722, Korea

Kisik Kim[‡]
Department of Physics, Inha University, 253 Yonghyun-Dong, Nam-Gu, Incheon 402-751, Korea
(Received 20 June 2011; published 24 August 2011)

We derive exact time evolution of three two-level atoms coupled to a common environment. The environment is structured and is modeled by a leaky cavity with Lorentzian spectral density. The atoms are initially prepared in a generalized W state and later on experience pairwise dipole-dipole interactions and couplings to the cavity. We study tripartite disentangling and entangling dynamics as well as protecting bipartite entanglement with both atom-atom interactions and atom-cavity couplings taken simultaneously into account.

DOI: 10.1103/PhysRevA.84.022329 PACS number(s): 03.67.Mn, 03.65.Ud, 03.65.Yz, 03.67.Bg

I. INTRODUCTION

The advantage of quantum over classical protocols for communicating and computing rests in the use of entanglement, which is the most characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought [1]. Fragility of entanglement, however, makes its utility very limited. Due to unavoidable interactions with surrounding dissipative environments, a pure entangled state inevitably becomes a mixed one with an amount of entanglement lower than that in the initial state. If at a given time the entanglement sustained is large enough, it can be distilled [2], otherwise it seems of little use for practical purposes. That fact encourages theoretical analysis of entanglement dynamics in various more or less realistic scenarios. There have been many papers on dynamics of bipartite entanglements. For multipartite entanglements the number of papers is fewer because of complexity in mathematical formulation for the whole system in general and lack of good entanglement measures for an interested subsystem in particular. Dynamics of three qubits in separated environments (see, e.g., [3–5]) has been investigated. Separated environments practically correspond to protocols which are performed by remote parties. Another realistic scenario arises when several qubits are stored in a quantum register or memory in which case one and the same environment influences all the qubits. Dynamics of three qubits in a common environment has also been studied (see e.g., [6–13]). Common environments may differ one from another by their nature: some of studied cases are, for example, fermionic symmetry-broken environment [6], quantum critical environment [7], dephasing environment [9], multimode electromagnetic field [8,11], quantum spin environment [10,13], and so on. The regime of qubit-environment coupling is usually treated as Markovian [8,12] (valid only for weak couplings, memoryless environments), but non-Markovian regime is also dealt with [11]. Most papers considered equal qubit-qubit interactions and regarded the qubits as an entire entity which is collectively coupled to the environment (i.e., each qubit experiences the environment in the same way or, in other words, the environment does not distinguish the qubits) [6–8,10,12,13]. As for entanglement measures negativities [14] are often resorted to, but the so-called lower bound of concurrence (LBC) [15] could serve as a reasonable approximation as well [4,5,11].

In this paper, since atoms have been regarded as a very good candidate for the content of a quantum register or memory [16], we shall concretize three qubits in terms of three two-level atoms which are immersed within a common structured environment modeled by a leaky multimode cavity with Lorentzian spectral density. As each atom possesses an electric dipole moment there are direct pairwise dipole-dipole interactions among them, which we shall take into account together with the atom-cavity coupling. At variance with previous investigations, here we regard the atoms as entities which are individually coupled to the cavity with possibly different strengths. After this Introduction, in Sec. II we formulate our problem and solve it for time evolution. Our solution is exact since it is derived rigorously without Born and Markovian approximations. In Secs. III and IV, based on the analytical result obtained in Sec. II, we study disentangling and entangling dynamics of the three atoms, employing LBC as an entanglement measure. Next, in Sec. V, we propose a method to protect bipartite entanglement by making use of tripartite dynamics. Finally, we draw the conclusion in Sec. VI.

II. THE MODEL

Consider three two-level atoms coupled to the same structured environment which we model as a leaky multimode

[*]nban@iop.vast.ac.vn
[†]jaewan@kias.re.kr
[‡]kisik@inha.ac.kr

cavity. As each atom is an electric dipole they interact with each other via dipole-dipole interactions. The Hamiltonian H of the total system contains three parts ($\hbar = 1$)

$$H = H_0 + H_1 + H_2, \tag{1}$$

with H_0 the free Hamiltonian, H_1 describing the atom-cavity coupling, and H_2 responsible for the dipole-dipole interaction,

$$H_0 = \sum_{n=1}^{3} \Omega \sigma_n^+ \sigma_n^- + \sum_j \omega_j a_j^\dagger a_j, \tag{2}$$

$$H_1 = \sum_{n=1}^{3} \sum_j \alpha_n (g_j \sigma_n^+ a_j + g_j^* \sigma_n^- a_j^\dagger), \tag{3}$$

$$\begin{aligned} H_2 = {} & D_1(\sigma_2^+\sigma_3^- + \sigma_3^+\sigma_2^-) + D_2(\sigma_3^+\sigma_1^- + \sigma_1^+\sigma_3^-) \\ & + D_3(\sigma_1^+\sigma_2^- + \sigma_2^+\sigma_1^-). \end{aligned} \tag{4}$$

In Eqs. (2)–(4) Ω and ω_j are the atomic transition frequency and frequency of the cavity mode j photon, whose creation (annihilation) operator is denoted by $\hat{a}_j^\dagger$ ($\hat{a}_j$); $\hat{\sigma}_n^+ = |1\rangle_{nn}\langle 0| = (\hat{\sigma}_n^-)^+$ with $|0\rangle_n$ ($|1\rangle_n$) the nth atom ground (excited) state, and the dipole-dipole interaction between different atoms m and n is described by $D_l = [\mathbf{d}\cdot\mathbf{d} - 3(\mathbf{d}\cdot\mathbf{r}_{mn})(\mathbf{d}\cdot\mathbf{r}_{mn})/r_{mn}^2]/r_{mn}^3$ with $\mathbf{d}$ the atom's electric dipole moment and r_{mn} the separation between the two atoms. As the coupling of an atom to the cavity depends on the value of the cavity field at the atom's position, we introduce dimensionless real constants α_n in Eq. (3) to individualize the atoms: the actual coupling strength between an nth atom and an jth mode photon is thus measured by $\alpha_n|g_j|$.

Suppose that initially the cavity is empty and the three atoms are in the state

$$\begin{aligned} |W(0)\rangle = {} & a_1(0)|100\rangle_{123} + a_2(0)|010\rangle_{123} + a_3(0)|001\rangle_{123}, \\ & \sum_{n=1}^{3} |a_n(0)|^2 = 1, \end{aligned} \tag{5}$$

which we call here generalized W state since it reduces to the W state [17] when all the coefficients are equal and to a three-atom separable state (a product of a two-atom entangled state and a single-atom state) when two (one) of the coefficients vanish(es). The total system state $|\Psi(0)\rangle = |W(0)\rangle|\bar{\mathbf{0}}\rangle$, where $|\bar{\mathbf{0}}\rangle = \otimes_j |0_j\rangle_c$ with $|0_j\rangle_c$ the cavity state containing zero photons of mode j, evolves at time $t > 0$ into

$$\begin{aligned} |\Psi(t)\rangle = {} & e^{-i\Omega t}[a_1(t)|100\rangle_{123} + a_2(t)|010\rangle_{123} + a_3(t)|001\rangle_{123}]|\bar{\mathbf{0}}\rangle \\ & + \sum_j b_j(t) e^{-i\omega_j t}|000\rangle_{123}|1_j\rangle_c. \end{aligned} \tag{6}$$

In the above equation $|1_j\rangle_c$ denotes the cavity state containing one photon of mode j and the time-dependent coefficients $a_n(t)$, $b_j(t)$ are to be determined from the equation of motion $i d|\Psi(t)\rangle/dt = (H_1 + H_2)|\Psi(t)\rangle$. Using Eqs. (3), (4), and (6) yields

$$i\frac{da_1(t)}{dt} = \alpha_1 \sum_j g_j e^{-i(\omega_j - \Omega)t} b_j(t) + D_3 a_2(t) + D_2 a_3(t), \tag{7}$$

$$i\frac{da_2(t)}{dt} = \alpha_2 \sum_j g_j e^{-i(\omega_j - \Omega)t} b_j(t) + D_3 a_1(t) + D_1 a_3(t), \tag{8}$$

$$i\frac{da_3(t)}{dt} = \alpha_3 \sum_j g_j e^{-i(\omega_j - \Omega)t} b_j(t) + D_2 a_1(t) + D_1 a_2(t), \tag{9}$$

$$i\frac{db_j(t)}{dt} = g_j^* e^{i(\omega_j - \Omega)t} \sum_{n=1}^{3} \alpha_n a_n(t). \tag{10}$$

Integrating Eq. (10) with the initial condition $b_j(0) = 0$ gives

$$b_j(t) = -i \int_0^t dt' g_j^* e^{i(\omega_j - \Omega)t'} \sum_{n=1}^{3} \alpha_n a_n(t'). \tag{11}$$

Then, substituting Eq. (11) into Eqs. (7)–(9) we get for $a_{1,2,3}(t)$

$$\begin{aligned} \frac{da_1(t)}{dt} = {} & -\alpha_1 \int_0^t dt' \sum_j |g_j|^2 e^{-i(\omega_j - \Omega)(t-t')} \\ & \times \sum_{n=1}^{3} \alpha_n a_n(t') - i D_3 a_2(t) - i D_2 a_3(t), \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{da_2(t)}{dt} = {} & -\alpha_2 \int_0^t dt' \sum_j |g_j|^2 e^{-i(\omega_j - \Omega)(t-t')} \\ & \times \sum_{n=1}^{3} \alpha_n a_n(t') - i D_3 a_1(t) - i D_1 a_3(t), \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{da_3(t)}{dt} = {} & -\alpha_3 \int_0^t dt' \sum_j |g_j|^2 e^{-i(\omega_j - \Omega)(t-t')} \\ & \times \sum_{n=1}^{3} \alpha_n a_n(t') - i D_2 a_1(t) - i D_1 a_2(t). \end{aligned} \tag{14}$$

In the limit of large number of cavity modes the sum $\sum_j |g_j|^2 e^{-i(\omega_j - \Omega)(t-t')}$ can be replaced by an integral $\int d\omega J(\omega) e^{-i(\omega - \Omega)(t-t')}$ with $J(\omega)$ the spectral density specifying the cavity structure. Equations (12)–(14) thus take the form

$$\begin{aligned} \frac{da_1(t)}{dt} = {} & -\alpha_1 \int_0^t dt' \sum_{n=1}^{3} \alpha_n a_n(t') \int d\omega J(\omega) e^{-i(\omega - \Omega)(t-t')} \\ & - i D_3 a_2(t) - i D_2 a_3(t), \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{da_2(t)}{dt} = {} & -\alpha_2 \int_0^t dt' \sum_{n=1}^{3} \alpha_n a_n(t') \int d\omega J(\omega) e^{-i(\omega - \Omega)(t-t')} \\ & - i D_3 a_1(t) - i D_1 a_3(t), \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{da_3(t)}{dt} = {} & -\alpha_3 \int_0^t dt' \sum_{n=1}^{3} \alpha_n a_n(t') \int d\omega J(\omega) e^{-i(\omega - \Omega)(t-t')} \\ & - i D_2 a_1(t) - i D_1 a_2(t). \end{aligned} \tag{17}$$

In the following we are concerned with a realistic cavity whose photons can be leaked out through its nonperfect mirrors. The spectrum of the field in such a cavity displays a Lorentzian broadening with

$$J(\omega) = \frac{R^2}{\pi}\frac{\Gamma}{(\omega-\omega_c)^2+\Gamma^2}, \tag{18}$$

where ω_c is frequency of the cavity-supported mode, Γ is half-width at half-height of the field spectrum profile inside the cavity, and R specifies the atom-cavity coupling. The cavity correlation time is characterized by $T_c = \Gamma^{-1}$, while the qubit relaxation time by $T_q = (2R\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2})^{-1}$ [11]. The weak (strong) coupling regime corresponding to $T_c < T_q$ ($T_c > T_q$) is called Markovian (non-Markovian). In the Markovian (non-Markovian) regime the history is forgotten (memorized).

Taking the Laplace transform of both sides of Eqs. (15)–(17) with $J(\omega)$ given by Eq. (18) we arrive at a set of equations for $\{\widetilde{a}_n(z);\ n = 1,2,3\}$. Here we use the notation $\widetilde{f}(z) = \mathcal{L}[f(\tau)] = \int_0^\infty f(\tau)e^{-z\tau}d\tau$. Solving that equation set we obtain for $n = 1,2,3$

$$\widetilde{a}_n(z) = \frac{\sum_{m=1}^{3} A_{nm}(z)a_m(0)}{\sum_{m=0}^{4} B_m z^m}, \tag{19}$$

where for $l,m,n \in \{1,2,3\}$ and $l \neq m \neq n \neq l$

$$A_{nn}(z) = \left(z^2+d_n^2\right)(z+1+i\delta)+\frac{G^2}{4}\left[\left(1-r_n^2\right)z-2id_nr_lr_m\right], \tag{20}$$

$$A_{mn}(z) = A_{nm}(z) = -(z+1+i\delta)(d_md_n+izd_l) + \frac{iG^2}{4}[(d_mr_m+d_nr_n-d_lr_l)+ir_mr_nz], \tag{21}$$

and

$$B_0 = \frac{G^2}{4}\left[d_1^2r_1^2+d_2^2r_2^2+d_3^2r_3^2-2(d_1d_2r_1r_2+d_1d_3r_1r_3 + d_2d_3r_2r_3)\right]-2id_1d_2d_3(1+i\delta), \tag{22}$$

$$B_1 = -\frac{iG^2}{2}(d_1r_2r_3+d_2r_3r_1+d_3r_1r_2) + \left(d_1^2+d_2^2+d_3^2\right)(1+i\delta)-2id_1d_2d_3, \tag{23}$$

$$B_2 = \frac{G^2}{4}+d_1^2+d_2^2+d_3^2,\quad B_3 = (1+i\delta),\quad B_4 = 1. \tag{24}$$

In the above formulas the following dimensionless quantities have been introduced for convenience:

$$d_l = \frac{D_l}{\Gamma},\quad \delta = \frac{\omega_c-\Omega}{\Gamma},\quad \alpha = \sqrt{\sum_{n=1}^{3}\alpha_n^2},\quad r_n = \frac{\alpha_n}{\alpha},\quad G = \frac{2R\alpha}{\Gamma}. \tag{25}$$

Note that $\sum_{n=1}^{3} r_n^2 = 1$ by definition. The time dependence can then be obtained by applying the inverse Laplace transform on Eq. (19), which means

$$a_n(\tau) = \sum_j \lim_{z\to z_j}(z-z_j)\widetilde{a}_n(z)e^{z_j\tau}, \tag{26}$$

with $\tau = \Gamma t$ and z_j a pole of $\widetilde{a}_n(z)$.

Defining the single-photon collective normalized state of the cavity field as

$$|\overline{1}\rangle = \frac{e^{i\omega_c t}}{b(\tau)}\sum_j b_j(\tau)e^{-i\omega_j t}|1_j\rangle_c, \tag{27}$$

with

$$b(\tau) = \sqrt{1-\sum_{n=1}^{3}|a_n(\tau)|^2}, \tag{28}$$

we have explicitly

$$|\Psi(\tau)\rangle = e^{-i\overline{\Omega}\tau}[a_1(\tau)|100\rangle_{123}+a_2(\tau)|010\rangle_{123} + a_3(\tau)|001\rangle_{123}]|\overline{0}\rangle + e^{-i\overline{\omega}_c\tau}b(\tau)|000\rangle_{123}|\overline{1}\rangle, \tag{29}$$

where $\overline{\Omega} = \Omega/\Gamma$ and $\overline{\omega}_c = \omega_c/\Gamma$.

III. DISENTANGLING DYNAMICS

Suppose that initially the atoms are prepared in state $|W(0)\rangle$ with all the coefficients $a_n(0)$ being nonzero, that is, they possess some amount of tripartite entanglement, which can be quantified by the three-qubit concurrence $\mathcal{C}_3$. For a general three-qubit pure state $|\Phi\rangle = \sum_{i,j,k=0}^{1} x_{ijk}|ijk\rangle_{123}$ with $\sum_{i,j,k=0}^{1}|x_{ijk}|^2 = 1$, $\mathcal{C}_3$ is defined as [18]

$$\mathcal{C}_3(|\Phi\rangle) = \sqrt{\frac{1}{3}\sum_{i,j,k,p,q,m=0}^{1}(|x_{ijk}x_{pqm}-x_{pjk}x_{iqm}|^2+|x_{ijk}x_{pqm}-x_{iqk}x_{pjm}|^2+|x_{ijk}x_{pqm}-x_{ijm}x_{pqk}|^2)}. \tag{30}$$

As the system evolves the interactions H_1 and H_2 are "switched on." If there are no direct interactions between the atoms ($H_2 = 0$) their entanglement inevitably degrades approaching a stationary value [11]. On the other side, if there is no cavity ($H_1 = 0$) the atoms' entanglement exhibits undamped oscillation. What is the interplay of both H_1 and H_2 in the atoms' disentangling dynamics? To elucidate this question we need know how to quantify tripartite entanglement at time $\tau > 0$ at which the atomic state becomes mixed, that is, $|W(0)\rangle \Rightarrow \rho(\tau) = \mathrm{Tr}_{\mathrm{cavity}}|\Psi(\tau)\rangle\langle\Psi(\tau)|$,

$$\rho(\tau) = |W(\tau)\rangle_{123123}\langle W(\tau)| + |b(\tau)|^2|000\rangle_{123123}\langle 000|. \tag{31}$$

Precisely, the concurrence for any mixed three-qubit state ϱ is defined by the convex roof [19]

$$\mathcal{C}_3(\varrho) = \min_{\{p_j,\Phi_j\}} \sum_j p_j \mathcal{C}_3(|\Phi_j\rangle), \tag{32}$$

where $\{|\Phi_j\rangle\}$ includes all the possible pure states into which ϱ can be decomposed, that is, $\varrho = \sum_j p_j |\Phi_j\rangle\langle\Phi_j|$ with $p_j > 0$ and $\sum_j p_j = 1$. The minimization procedure in Eq. (32) is formidable and generally does not guarantee a global optimization. Therefore, to be able to assess entanglement analytically, we resort to an approximation in terms of LBC, $\underline{\mathcal{C}}_3(\varrho)$, whose expression is given by [15]

$$\underline{\mathcal{C}}_3(\varrho) = \sqrt{\frac{1}{3}\sum_{j=1}^{6}\left\{\left[C_j^{12|3}(\varrho)\right]^2 + \left[C_j^{23|1}(\varrho)\right]^2 + \left[C_j^{31|2}(\varrho)\right]^2\right\}}, \tag{33}$$

where

$$C_j^{mn|l}(\varrho) = \max\left\{0, \sqrt{\lambda_{j,1}^{mn|l}} - \sum_{q>1}\sqrt{\lambda_{j,q}^{mn|l}}\right\}, \tag{34}$$

with $\lambda_{j,q}^{mn|l}$ the eigenvalues, in decreasing order, of the matrix $\widetilde{\varrho} = \varrho(L_j^{mn} \otimes \sigma_y^l)\varrho^*(L_j^{mn} \otimes \sigma_y^l)$, L_j^{mn} $(j = 1,2,\ldots,6)$ the six generators of group SO(4) [20] acting on qubits m, n, and σ_y^l the y-Pauli matrix acting on qubit l. Surely, $\underline{\mathcal{C}}_3(\varrho) > 0$ signifies that ϱ is entangled and a separable state ϱ always has $\underline{\mathcal{C}}_3(\varrho) = 0$. Yet, $\underline{\mathcal{C}}_3(\varrho) = 0$ does not necessarily imply separability of ϱ. The LBC of three-qubit X states was analyzed in Ref. [4] to demonstrate when it goes to zero. Numerical calculations in Ref. [5] showed coincidence between the LBC and the corresponding convex roof (32) for some density matrices with ranks not greater than 4. For the density matrix of our concern, $\rho(\tau)$ in Eq. (31), which is a rank 4 one, its LBC can be derived as [11]

$$\underline{\mathcal{C}}_3(\rho(\tau)) = \sqrt{\frac{8}{3}\sum_{m,n=1;\, m<n}^{3} |a_m(\tau)a_n(\tau)|^2}. \tag{35}$$

Since the time-dependent coefficients $a_n(\tau)$ were obtained (in the previous section) without approximations imposed on the involved parameters, Eq. (35) is valid in both Markovian ($T_c < T_q$, or the same, $G = T_c/T_q < 1$) and non-Markovian ($T_c > T_q$, or the same, $G = T_c/T_q > 1$) regime, without ($d_m = 0$) or with ($d_m \neq 0$) dipole-dipole interactions.

As is obvious from Eq. (26), the atoms' dynamics is governed by the poles $\{z_j\}$ of $\widetilde{a}_n(z)$. A transparent distinction between presence and absence of the dipole-dipole interaction can be recognized from Eqs. (19)–(24). When there are no dipole-dipole interactions B_0 and B_1 [see Eqs. (22) and (23)] vanish and $\widetilde{a}_n(z)$ has three poles, one of which is always zero and the other two are complex with a negative real part, thus always rendering existence of stationary solutions [11]. The analysis in the presence of dipole-dipole interactions is complicated depending on the atoms' configurations through $d_{1,2,3}$ and $r_{1,2,3}$. For clarity we shall distinguish between three cases regarding the pairwise dipole-dipole interactions. Case 1 implies equal pairwise interactions: $d_1 = d_2 = d_3 = d$. In case 2 two of the three interactions are equal but different from the third: $d_m = d_n \neq d_l$. And, case 3 means all the pairwise dipole-dipole interactions are different: $d_1 \neq d_2 \neq d_3 \neq d_1$.

Let us consider case 1 which corresponds to a configuration when the atoms sit on the vertices of an equilateral triangle which can however be positioned anywhere within the cavity. In Fig. 1 we plot the LBC $\underline{\mathcal{C}}_3$ versus τ for the atoms initially prepared in the state $|W(0)\rangle = \frac{1}{2}|100\rangle + \frac{1}{2}|010\rangle + \frac{1}{\sqrt{2}}|001\rangle)$ which was used as the quantum channel for perfect two-party teleportation [21]. For the other parameters we take $r_1 = 1/\sqrt{2}$, $r_2 = 1/\sqrt{3}$, $r_3 = 1/\sqrt{6}$, $\delta = -1 < 0$, and $G = 0.8$ (Markovian regime) or $G = 8$ (non-Markovian regime). Note that dipole-dipole interactions induce oscillations for whatever strength of atom-cavity couplings. This fact explains the presence of oscillations even in the Markovian regime in Fig. 1(a) when $d > 0$ (solid curves), as opposed to the absence of oscillations when $d = 0$ (dashed curve). The role played by the dipole-dipole interaction is simple in Fig. 1: in both the Markovian and non-Markovian regime it slows down the entanglement degradation. The stronger the dipole-dipole interaction the slower the rate of disentangling. However, such a simple behavior does not always happen. To verify this we plot in Fig. 2 the evolution of $\underline{\mathcal{C}}_3(\tau)$ for the same set of parameters as in Fig. 1, except δ which is now positive. In this case the dipole-dipole interaction manifests its role more complexly than in Fig. 1. As seen from Fig. 2, no matter the atom-cavity coupling is weak [Fig. 2(a)] or strong [Fig.2(b)], an initial increase in d (from $d = 0$ up to a certain value) speeds up the decay of entanglement. But, a further increase in d (after a certain value) changes its influence abruptly: it turns out to slowing down, instead of speeding up, the atoms' disentangling process.

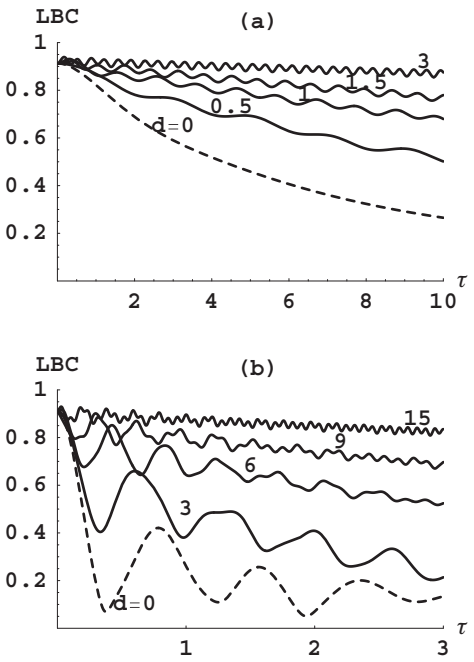

FIG. 1. The lower bound of concurrence LBC $= \underline{\mathcal{C}}_3$ as a function of τ for $a_1(0) = a_2(0) = 1/2, a_3(0) = 1/\sqrt{2}, r_1 = 1/\sqrt{2}, r_2 = 1/\sqrt{3}$, $r_3 = 1/\sqrt{6}$, and $\delta = -1$ in (a) the Markovian regime with $G = 0.8$ or (b) the non-Markovian regime with $G = 8$. The different values of d are indicated near each curve.

The distinct influences of the dipole-dipole interaction on the disentangling dynamics of the atoms shown in Figs. 1 and 2 can be explained analytically when $r_1 = r_2 = r_3 \ (= 1/\sqrt{3})$, in which situation Eq. (19) gets an explicit form

$$\widetilde{a}_n(z) = \frac{3ia_n(0)[G^2 + 4(z+2id)(z+1+i\delta)] - i[G^2 + 12id(1+z+i\delta)]\sum_{m=1}^{3} a_m(0)}{3(d+iz)[G^2 + 4(z+2id)(1+z+i\delta)]}. \tag{36}$$

The inverse Laplace transform of the r.h.s. of Eq. (36) can be taken analytically yielding

$$a_n(\tau) = e^{id\tau}\left\{a_n(0) - \frac{1}{3}[1 - Q(\tau)]\sum_{m=1}^{3} a_m(0)\right\}, \tag{37}$$

where

$$Q(\tau) = e^{-[1+i(\delta+4d)]\tau/2}\left[\cosh\left(\frac{\Theta\tau}{2}\right) + \frac{1+i(\delta-2d)}{\Theta} \times \sinh\left(\frac{\Theta\tau}{2}\right)\right] \tag{38}$$

with

$$\Theta = \sqrt{[1+i(\delta-2d)]^2 - G^2}. \tag{39}$$

Evidently from Eq. (36), $\widetilde{a}_n(z)$ has three poles of the forms

$$z_1 = id, \quad z_{2,3} = -\frac{1}{2}[1 + i(\delta+2d) \mp \Theta]. \tag{40}$$

Because $z_1 = id$ is purely imaginary the evolution of $a_n(\tau)$ is permanently oscillatory, in sharp contrast to the case of $d = 0$ [11] in which a stationary/equilibrium solution exists. Expressing the poles of $\widetilde{a}_n(z)$ as $z_j = i\Omega_j - \lambda_j$, with Ω_j an oscillation frequency and $\lambda_j > 0$ a damping rate, we obtain from Eqs. (40)

$$\Omega_1 = d, \quad \Omega_{2,3} = -\tfrac{1}{2}[(\delta+2d) \mp \mathrm{Im}(\Theta)] \tag{41}$$

and

$$\gamma_1 = 0, \quad \gamma_{2,3} = \tfrac{1}{2}[1 \mp \mathrm{Re}(\Theta)]. \tag{42}$$

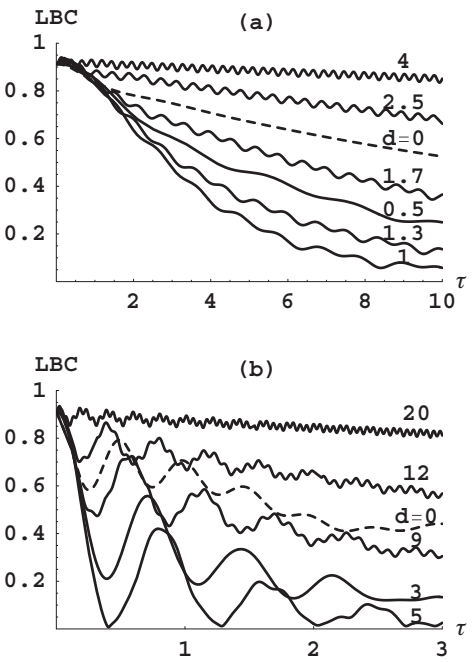

FIG. 2. The lower bound of concurrence LBC $= \mathcal{C}_3$ as a function of τ for $a_1(0) = a_2(0) = 1/2, a_3(0) = 1/\sqrt{2}, r_1 = 1/\sqrt{2}, r_2 = 1/\sqrt{3}$, and $r_3 = 1/\sqrt{6}$ in (a) the Markovian regime with $G = 0.8$ and $\delta = 2$ or (b) the non-Markovian regime with $G = 8$ and $\delta = 10$. The used values of d are indicated near each curve.

The damping rates γ_2 and γ_3 as a function of δ and d are displayed for the Markovian regime in Figs. 3(a) and 3(b) or the non-Markovian regime in Figs. 3(c) and 3(d). Since $\gamma_2 < \gamma_3$ the term proportional to $e^{z_3\tau}$ in $a_n(\tau)$ damps out quickly. Hence, the disentangling rate is mainly dictated by γ_2. We observe that the way γ_2 depends on d is sensitive to the sign of δ. For nonpositive detunings $\delta \leqslant 0$ the damping rate γ_2 decreases monotonously with increasing d and becomes negligible when d gets sufficiently large. Interestingly, for positive detunings $\delta > 0$ such a simple γ_2-d dependence no longer holds. Now in the course of increasing d the damping rate γ_2 is first increasing too, but after reaching a maximum it will be decreasing down to zero, as can be clearly seen from Figs. 3(a) and 3(c). In the situation of unequal r_n the purely imaginary pole at id remains, but one more complex pole appears, which does not change the qualitative picture mentioned above. Numerical simulations for the situation of unequal r_n show dynamical behaviors similar to that with equal r_n, as is visible from Fig. 1 (for $\delta < 0$) and Fig. 2 (for $\delta > 0$). It is worth noticing that the sign of detuning does not play a role in the absence of dipole-dipole interactions ($d = 0$): just $|\delta|$ is important as discovered in Ref. [11] as well as seen from the plane $d = 0$ of Fig. 3 here. Generally, there is a competition between the dipole-dipole interaction (in terms of d) and atom-cavity coupling (in terms of G). When $\delta \leqslant 0$ the disentangling rate decreases with d for a fixed value of G, but increases with G for a fixed value of d [see Fig. 4(a)]. That is to say, the dipole-dipole interaction tends to protect entanglement, while the atom-cavity coupling tends to spoil it. However, when $\delta > 0$ the role of the atom-cavity coupling remains the same as in the case of $\delta \leqslant 0$, but that of the dipole-dipole interaction does not. More concretely, for a fixed value of G there exists a critical value d_c such that the disentangling occurs faster (slower) with increasing d when $d < d_c$ ($d > d_c$) [see Fig. 4(b)]. Figure 5 is a 3D view of the LBC with respect to both δ and d at a fixed moment of time in the Markovian [see Fig. 5(a)] or non-Markovian [see

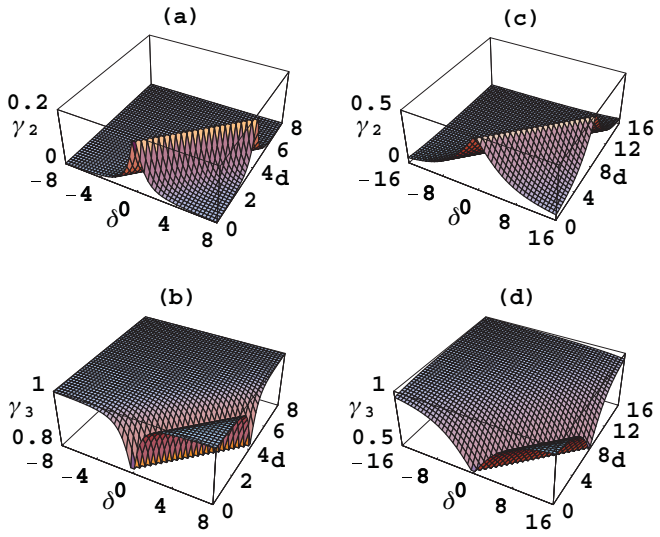

FIG. 3. (Color online) Dependence of the decay rates $\gamma_{2,3}$ [Eq. (42)] on δ and d for (a) and (b) $G = 0.8$ (Markovian regime) or (c) and (d) $G = 8$ (non-Markovian regime).

Fig. 5(b)] regime. The valley in Fig. 5 is in correspondence with the peak region in Figs. 3(a) and 3(c). The analytic expression of $a_n(\tau)$ in Eq. (37) also indicates that, under the most symmetric configuration (MSC) with $d_1 = d_2 = d_3$ and $r_1 = r_2 = r_3$, the LBC of the so-called "zero-sum amplitude states" [22], that is, those with $\sum_{n=1}^{3} a_n(0) = 0$, is preserved over times independent of other parameters. This information is helpful in engineering decoherence-free entangled states.

In case 2 ($d_m = d_n \neq d_l = d$), $\widetilde{a}_n(z)$ has four poles. If the r_n are not the same all the poles are complex with

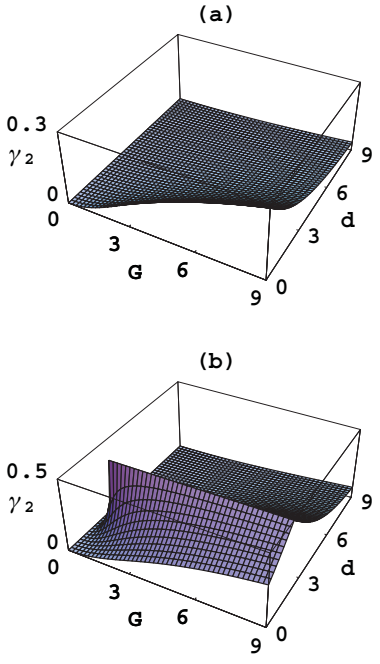

FIG. 4. (Color online) Dependence of the decay rate γ_2 [Eq. (42)] on G and d for (a) $\delta = -4$ or (b) $\delta = 4$.

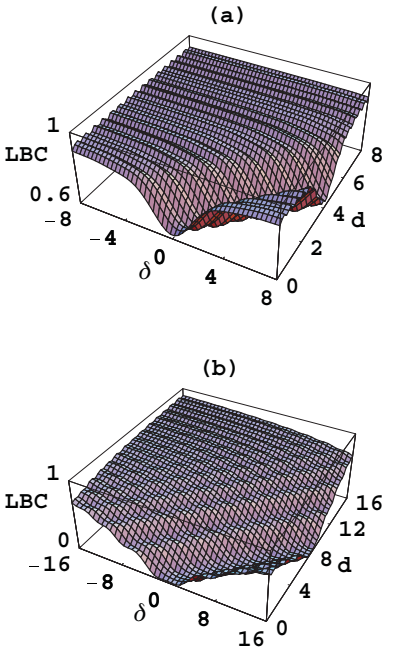

FIG. 5. (Color online) A 3D view of the lower bound of concurrence LBC with respect to δ and d at a fixed moment of time $\tau = 2$ for $r_1 = r_2 = r_3 = 1/\sqrt{3}$ and (a) $G = 0.8$ (Markovian regime) or (b) $G = 8$ (non-Markovian regime).

a negative real part, but if $r_1 = r_2 = r_3$ one of the poles is at id, that is, purely imaginary. In case 3 ($d_1 \neq d_2 \neq d_3 \neq d_1$), the four poles of $\widetilde{a}_n(z)$ are always complex with a negative real part, regardless of r_n. As can be inferred from above, existence of a purely imaginary pole depends primarily on the relation between d_n, but the relation between r_n also plays a role. Namely, the equality of all the pairwise dipole-dipole interactions (i.e., case 1) guarantees its existence for whatever r_n. But when only two among d_1, d_2, and d_3 are equal (i.e., case 2) it requires full symmetry of the atom-cavity couplings: $r_1 = r_2 = r_3$. Finally, no purely imaginary poles exist for full asymmetry of the dipole-dipole interactions: $d_1 \neq d_2 \neq d_3 \neq d_1$ (i.e., case 3), independent of r_n. Nonexistence of a purely imaginary pole means that all the coefficients $a_n(\tau)$ will vanish independently of their initial values. As a consequence, all the atoms' entanglement will disappear in the long-time limit since the eventual state of the atoms will be $|000\rangle_{123}$. On the contrary, when there exists a purely imaginary pole, oscillations of the coefficients $a_n(\tau)$ persist forever as $a_m(\tau \to \infty) = e^{id\tau} \lim_{z \to id}(z - id)\widetilde{a}_m(z)$. However, the LBC, which depends on absolute values of $a_n(\tau)$, will approach a "stationary" value determined by Eq. (35) with $|a_m(\tau \to \infty)| = |\lim_{z \to id}(z - id)\widetilde{a}_m(z)|$.

At this moment one may ask a question: "whether can one establish an analytic expression for the critical value d_c of the dipole-dipole interaction for the case of $\delta > 0$?" It turns out that it is possible in case 1 in which the damping rates were derived analytically in terms of Eqs. (42). The value of d_c is determined from the condition $\frac{\partial \gamma_2}{\partial d}|_{d=d_c} = 0$, which, due to Eqs. (42), is equivalent to $\frac{\partial \mathrm{Re}\Theta}{\partial d}|_{d=d_c} = 0$.

From Eq. (39),

$$\mathrm{Re}\Theta = [4\widetilde{d}^2 + (1 - G^2 - \widetilde{d}^2)^2]^{1/4} \times \cos\left[\frac{1}{2}\arctan\left(\frac{2\widetilde{d}}{1 - G^2 - \widetilde{d}^2}\right)\right], \quad (43)$$

where $\widetilde{d} = \delta - 2d$, so

$$\frac{\partial \mathrm{Re}\Theta}{\partial \widetilde{d}} = \frac{1}{[4\widetilde{d}^2 + (1 - G^2 - \widetilde{d}^2)^2]^{3/4}}\{\widetilde{d}(1 + G^2 + \widetilde{d}^2) \times \cos\left[\frac{1}{2}\arctan\left(\frac{2\widetilde{d}}{1 - G^2 - \widetilde{d}^2}\right)\right] - (1 - G^2 + \widetilde{d}^2) \times \sin\left[\frac{1}{2}\arctan\left(\frac{2\widetilde{d}}{1 - G^2 - \widetilde{d}^2}\right)\right]\}. \quad (44)$$

The solution $\widetilde{d}_c$ satisfying $\frac{\partial \mathrm{Re}\Theta}{\partial \widetilde{d}}|_{\widetilde{d}=\widetilde{d}_c} = 0$ is simply $\widetilde{d}_c = 0$, independent of G, that is,

$$d_c = \frac{\delta}{2}, \quad (45)$$

which is relevant (irrelevant) for $\delta > 0$ ($\delta \leqslant 0$) in agreement with Figs. 2(a), 2(b), and 4(b).

Having understood the roles played by different atom-atom interactions as well as by different atom-cavity couplings, in the next sections, unless otherwise specified, we shall mainly deal with the case of MSC, to fully exploit the analytical result.

IV. ENTANGLING DYNAMICS

The characteristic feature of a common environment contrasts very well with that of separated environments. In separated (dissipative) environments any entanglement will inevitably be lost, either asymptotically or suddenly (see, e.g., [23]), and separable subsystems remain unentangled all the times because entanglement cannot be created locally. Things change dramatically, however, if the subsystems are coupled to one and the same common environment. When environment is common, depending on the case as discussed in the previous section, an initially stored entanglement may approach a finite amount for time tending to infinity and separable noninteracting qubits may become entangled during the course of evolution [11]. In this section, we shall take direct interactions between the qubits (here dipole-dipole interactions between two-level atoms) into account in studying the atoms' entangling dynamics.

Suppose that at time $t = 0$ the atoms are in a product state $|100\rangle$. For a later time $t > 0$ the state becomes mixed of the form (31) whose LBC $\underline{\mathcal{C}}_3(\tau)$ is evaluated in terms of the coefficients $a_n(\tau)$ through Eq. (35). For MSC $a_n(\tau)$ have been derived analytically in Eq. (37) and we can simply use it with $a_1(0) = 1$ and $a_2(0) = a_3(0) = 0$. So

$$a_1(\tau) = \tfrac{1}{3}e^{id\tau}[Q(\tau) + 2]$$

and

$$a_2(\tau) = a_3(\tau) = \tfrac{1}{3}e^{id\tau}[Q(\tau) - 1], \quad (46)$$

with $Q(\tau)$ given by Eq. (38). As predicted theoretically for the situation with equal r_n and confirmed graphically in Figs. 6(a) and 6(b) also for the situation with unequal r_n, when time goes to infinity the LBC $\underline{\mathcal{C}}_3(\tau \to \infty)$ tends to a fixed value equal

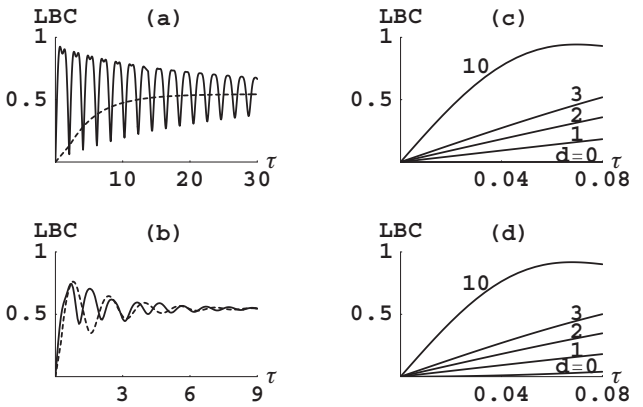

FIG. 6. Left: Evolution of the lower bound of concurrence LBC for $a_1(0) = 1, a_2(0) = a_3(0) = 0, r_1 = 1/\sqrt{2}, r_2 = 1/\sqrt{3}, r_3 = 1/\sqrt{6}$, and $\delta = 0$. In (a) $G = 0.8$ (Markovian regime) and $d = 0, 1$ (dashed line, solid line). In (b) $G = 8$ (non-Markovian regime) and $d = 0, 1$ (dashed line, solid line). Right: The evolution during a short initial period of time in dependence on d for (c) $G = 0.8$ (Markovian regime) or (d) $G = 8$ (non-Markovian regime). The values of d are indicated near the curves.

to $(2/3)^{3/2} \approx 0.5443$ for whatever values of the atom-cavity coupling G, the atom-atom interaction d, and the detuning δ. However, in the course of evolution the amount of created entanglement at certain moments can be much larger in the presence of the dipole-dipole interaction than in the absence of it, especially in the Markovian regime [see Fig. 6(a)]. Besides the long-term behavior, of interest is also the "immediate" rate of entangling process in dependence on the dipole-dipole interaction d. From Eqs. (37) and (35) it derives the short-term behavior of $\underline{\mathcal{C}}_3(\tau)$ as

$$\underline{\mathcal{C}}_3(\tau \ll 1) = \frac{G^2}{6\sqrt{3}}\tau^2 + O(\tau^3) \quad (47)$$

in the absence of the dipole-dipole interaction and

$$\underline{\mathcal{C}}_3(\tau \ll 1) = \frac{4d}{\sqrt{3}}\tau + O(\tau^2) \quad (48)$$

in the presence of the dipole-dipole interaction. These two expressions reveal that the dipole-dipole interaction greatly favors entanglement creation right after the system starts evolving since in leading order $\underline{\mathcal{C}}_3(\tau \ll 1) \propto \tau^2$ for $d = 0$, but $\underline{\mathcal{C}}_3(\tau \ll 1) \propto \tau$ for $d > 0$. Also, as indicated by Eq. (48), the stronger the atom-atom interaction the larger the amount of immediately created entanglement. Although the scalings (47) and (48) are rigorous only for MSC, they do reflect the tendency also for the situation when $r_1 \neq r_2 \neq r_3 \neq r_1$, as illustrated in Figs. 6(c) and 6(d). Interestingly to observe from Figs. 6(c) and 6(d) that a huge amount of entanglement can be obtained after a very short time when d is large enough, while it is negligible when $d = 0$. For instance, the value of $d = 10$ yields $\underline{\mathcal{C}}_3(\tau = 0.07) > 0.9 > \underline{\mathcal{C}}_3(\tau \to \infty)$, while that of $d = 0$ just gives $\underline{\mathcal{C}}_3(\tau = 0.07) \simeq 10^{-4}$ $(10^{-2}) < \underline{\mathcal{C}}_3(\tau \to \infty)$ for $G = 0.8$ $(G = 8)$. Nevertheless, the amount of created entanglement depends on the actual moment of evolution. Generally, at a given moment a greater value of d may generate a higher, an equal or a lower amount of entanglement as compared with that generated by a smaller value of d. This

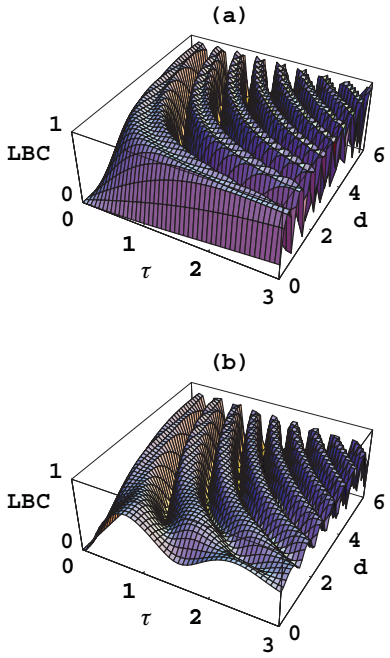

FIG. 7. (Color online) The lower bound of concurrence LBC = $\underline{\mathcal{C}}_3$ as a function of τ and d for $\delta = 0$, $r_1 = r_2 = r_3 = 1/\sqrt{3}$ and (a) $G = 0.8$ (Markovian regime) or (b) $G = 8$ (non-Markovian regime).

is demonstrated by 3D graphics in Fig. 7 which represents dependence of the LBC on both the dipole-dipole interaction and time.

V. PROTECTING ENTANGLEMENT

Robustness of an entangled state can be assessed by how long it takes to disentangle. Protecting entanglement, which aims at prolonging the disentanglement period, is significant to boost the usage of entanglement as a resource for quantum information processing and quantum computing. There have been various methods to do that, for example, by means of quantum error correction codes [24], by using dynamical decoupling pulse sequences [25] or external driving fields [26], and so on. In the context of three qubits in a common cavity several ways to control tripartite entanglement have also been suggested in Ref. [11]. Our purpose in this section is to make use of the evolution of three qubits to protect bipartite entanglement.

First, let us consider a cavity containing only two two-level atoms (atom 1 and atom 2) being at $\tau = 0$ in the entangled state $|\psi^{(2)}(0)\rangle_{12} = a_1(0)|10\rangle_{12} + a_2(0)|01\rangle_{12}$, with $|a_1(0)|^2 + |a_2(0)|^2 = 1$. For $\tau > 0$ it becomes mixed and its entanglement can be measured by the bipartite concurrence [27] which in this case is given by

$$\mathcal{C}_2(\tau) = 2|a_1(\tau)a_2(\tau)|. \tag{49}$$

The time dependence of the coefficients $a_{1,2}(\tau)$ was derived in Ref. [28]. In the long-time limit

$$a_1(\tau \to \infty) = \tfrac{1}{2}e^{id}[a_1(0) - a_2(0)] \tag{50}$$

and

$$a_2(\tau \to \infty) = \tfrac{1}{2}e^{id}[a_2(0) - a_1(0)]. \tag{51}$$

This results in the asymptotic concurrence

$$\mathcal{C}_2^{(2)}(\tau \to \infty) = \tfrac{1}{2}|a_1(0) - a_2(0)|^2, \tag{52}$$

where the superindex "(2)″ in the l.h.s. of Eq. (52) distinguishes the two-atom case from the three-atom case (see below) in which it is "(3)″.

Next, we consider a scenario in which at the beginning one more atom (atom 3) is added to but not entangled with atoms 1 and 2. For example, the initial three-atom state can be of the form (5) with $a_3(0) = 0$, that is, the three-atom initial state is $|\psi^{(3)}(0)\rangle_{123} = |\psi^{(2)}(0)\rangle_{12}|0\rangle_3 = a_1(0)|100\rangle_{123} + a_2(0)|010\rangle_{123}$. Now we let the three atoms evolve together but we are interested only in the entanglement of atoms 1 and 2. That is to say, we need trace out both the cavity modes and the atom 3 to have the reduced density matrix of the two interested atoms. Their concurrence is calculated by the same formula (49), but in this (three-atom) case the time-dependent coefficients $a_{1,2,3}(\tau)$ are determined by our Eq. (37), according to which the long-time limit solutions are

$$a_1(\tau \to \infty) = \tfrac{1}{3}e^{id}[2a_1(0) - a_2(0)], \tag{53}$$

$$a_2(\tau \to \infty) = \tfrac{1}{3}e^{id}[2a_2(0) - a_1(0)], \tag{54}$$

and

$$a_3(\tau \to \infty) = -\tfrac{1}{3}e^{id}[a_1(0) + a_2(0)], \tag{55}$$

which yield

$$\mathcal{C}_2^{(3)}(\tau \to \infty) = \tfrac{2}{9}|[2a_1(0) - a_2(0)][2a_2(0) - a_1(0)]|. \tag{56}$$

The expressions (52) and (56) for the concurrence clearly reveal the "decoherence-free" phenomenon pointed out in Refs. [28,29] for the two-qubit case and here (see Sec. III above) for the three-qubit case that entanglement of the Bell state $|\Psi^-\rangle = (|10\rangle - |01\rangle)_{12}/\sqrt{2}$ is invariant over time:

$$\mathcal{C}_2^{(2,3)}(\tau = 0) = \mathcal{C}_2^{(2,3)}(\tau \to \infty) = 1. \tag{57}$$

Moreover, as is somewhat surprised, for another Bell state $|\Psi^+\rangle = (|10\rangle + |01\rangle)_{12}/\sqrt{2}$ we get from Eqs. (52) and (56)

$$\mathcal{C}_2^{(2)}(\tau = 0) = \mathcal{C}_2^{(3)}(\tau = 0) = 1, \tag{58}$$

but

$$\mathcal{C}_2^{(3)}(\tau \to \infty) = \tfrac{1}{9} > \mathcal{C}_2^{(2)}(\tau \to \infty) = 0. \tag{59}$$

This means that while the state $|\Psi^+\rangle_{12}$ will lose all its entanglement if atoms 1 and 2 evolve alone, it will retain a finite amount of entanglement if the two atoms evolve together with a third one (atom 3) which was initially in the ground state $|0\rangle_3$. The asymptotic values $C_2^{(2,3)}(\tau \to \infty)$ of the bipartite concurrence do not depend on concrete values of d, G, and δ. The whole time evolution, however, does, as plotted in Fig. 8. In the Markovian regime, $C_2^{(3)}(\tau)$ exceeds $C_2^{(2)}(\tau)$ for all $\tau > 0$ when there are no dipole-dipole interactions [see Fig. 8(a)]. In the presence of such interactions $C_2^{(3)}(\tau)$ displays pronounced large-amplitude oscillations which last long before reaching the asymptotic value, whereas $C_2^{(2)}(\tau)$ quickly dies [see Fig. 8(c)]. In the non-Markovian regime, due to the memory effect, both $C_2^{(2)}(\tau)$ and $C_2^{(3)}(\tau)$ oscillate no matter whether the dipole-dipole interaction is present or not.

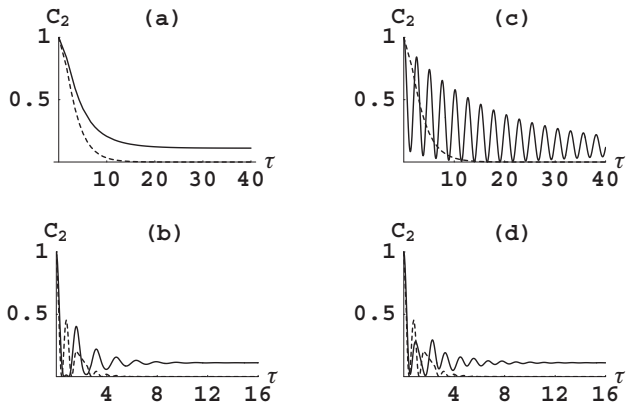

FIG. 8. Time dependence of the bipartite concurrence C_2 of atoms 1 and 2 when they evolve alone from the initial state $|\Psi^+\rangle_{12}$ (dashed curves) and when they evolve together with atom 3 from the initial state $|\Psi^+\rangle_{12}|0\rangle_3$ (solid curves). The parameters used are (a) $d = 0$, $G = 0.8$, (b) $d = 0$, $G = 8$, (c) $d = 0.8$, $G = 0.8$, and (d) $d = 0.8$, $G = 8$.

However, the oscillation of $C_2^{(2)}(\tau)$ is quickly perished, while that of $C_2^{(3)}(\tau)$ will reach the "stationary" value after a transient period [see Figs. 8(b) and 8(d)]. A similar idea for protecting bipartite entanglement was touched upon in a recent paper [12] which concerns another physical context and is applicable only to the Markovian regime since the dynamics was derived there as approximate solutions of Lindblad-type master equations, but not as exact solutions like in the present paper which is applicable to both Markovian and non-Markovian regime.

VI. CONCLUSION

In conclusion, we have extended the model in Ref. [11] by including direct dipole-dipole interactions between three two-level atoms inside a common structured cavity with Lorentzian spectral density. Unlike in other works [6–8,10,12,13], here we treat the atoms at an individual level. Each atom experiences the same cavity but with its own coupling constant measured by the dimensionless parameter r_n. Different pairs of atoms may also have different strengths of pairwise dipole-dipole interactions measured by the dimensionless parameter d_n. We have exactly solved the equations of motion (7)–(10) for the system wave function (6) and found out that entanglement dynamics of the atoms is sensitive to the symmetry between d_n as well as between r_n. If the d_n are all different, then the atoms will eventually relax to the state $|000\rangle$, independent of r_n. When two among the three d_n are equal, relaxation to $|000\rangle$ happens also if the r_n are not identical, but does not if $r_1 = r_2 = r_3$. In the latter situation the coefficients $a_n(t)$ in Eq. (6) contain a undamped oscillatory component, rendering a finite amount of entanglement (in terms of LBC [15]) in the long-time limit because the LBC is determined by $|a_m(t)||a_n(t)|$ as in Eq. (35). If the d_n are all equal, that is, $d_1 = d_2 = d_3 = d$, then the LBC always tends to a finite value for whatever the values of r_n. However, as opposed to the case without atom-atom interactions [11], the entanglement dynamics turns out to depend on the sign of δ, the detuning between the atom's and the cavity-supported mode's frequencies [see Eq. (25)]. For $\delta \leqslant 0$ the average disentangling rate decreases with increasing d, but for $\delta > 0$ it first increases and then, after reaching a maximum, decreases with increasing d. In the case of MSC ($d_1 = d_2 = d_3$ and $r_1 = r_2 = r_3$) explicit expressions for the time-dependent coefficients $a_n(\tau)$ have been derived in Eq. (37), allowing us to explore the entangling dynamics analytically. If the atoms are at $\tau = 0$ in either of the states $|100\rangle$, $|010\rangle$, or $|001\rangle$, then as $\tau \to \infty$ their LBC tends to a fixed value equal to $(2/3)^{3/2}$ and for $\tau \ll 1$ it scales with τ like LBC $\propto d\tau + G^2\tau^2 + O(\tau^3)$, implying an immediate enhancement of entanglement creation thanks to the dipole-dipole interaction. Finally, we have also proposed a method to protect entanglement between two atoms by exploiting evolution of three atoms. Namely, we have demonstrated that, by adding a third atom in the state $|0\rangle_3$ to the two-atom Bell state $|\Psi^+\rangle_{12} = (|10\rangle + |01\rangle)_{12}/\sqrt{2}$ and letting all the three evolve together, the entanglement (in terms of concurrence [27]) between atoms 1 and 2 is sustained even in the long-time limit, while it is lost completely if the two atoms evolve alone without the third.

ACKNOWLEDGMENTS

N.B.A. acknowledges support from the Vietnam National Foundation for Science and Technology Development (NAFOSTED).

[1] E. Schrödinger, Naturwissenschaften **23**, 807 (1935).
[2] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).
[3] M. Ge, L. F. Zhu, and L. Qiu, Commun. Theor. Phys. **49**, 1443 (2008); Z. X. Man, Y. J. Xia, and N. B. An, J. Phys. B: At. Mol. Opt. Phys. **41**, 155501 (2008); J. Mod. Opt. **56**, 1022 (2009).
[4] Y. S. Weinstein, Phys. Rev. A **82**, 032326 (2010).
[5] M. Siomau and S. Fritzsche, Eur. Phys. J. D **60**, 397 (2010).
[6] X. S. Ma, A. M. Wang, X. D. Yang, and H. You, J. Phys. A **38**, 2761 (2005).
[7] X. S. Ma, A. M. Wang, and Y. Cao, Phys. Rev. B **76**, 155327 (2007).
[8] J.-H. An, S.-J. Wang, and H.-G. Luo, Physica A: Statistical Mechanics and its Applications **382**, 753 (2007).
[9] Y. S. Weinstein, Phys. Rev. A **79**, 012318 (2009).
[10] J. L. Guo, J. S. Jin, and H. S. Song, Physica A **388**, 3667 (2009).
[11] N. B. An, J. Kim, and K. Kim, Phys. Rev. A **82**, 032316 (2010).
[12] F. Benatti and A. Nagy, Ann. Phys. **326**, 740 (2011).
[13] X. S. Ma, G. S. Liu, and A. M. Wang, Int. J. Quant. Inf. **9**, 791 (2011).
[14] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[15] X. H. Gao, S. M. Fei, and K. Wu, Phys. Rev. A **74**, 050303(R) (2006); X. H. Gao and S. M. Fei, Eur. Phys. J. Special Topics **159**, 71 (2008); M. Li, S. M. Fei, and Z. X. Wang, J. Phys. A **42**, 145303 (2009).
[16] C. Langer *et al.*, Phys. Rev. Lett. **95**, 060502 (2005).

[17] W. Dur, G. Vidal, and J. I. Cirac, Phys. Rev. A. **62**, 062314 (2000).
[18] S. Albeverio and S.-M. Fei, J. Opt. B: Quantum Semiclass. Opt. **3**, 223 (2001).
[19] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[20] Z. Q. Ma and X. Y. Gu, *Problems & Solutions in Group Theory for Pysicists* (World Scientific, Singapore, 2004).
[21] P. Agrawal and A. Pati, Phys. Rev. A **74**, 062320 (2006).
[22] A. K. Pati, Pramana: J. Phys. **59**, 217 (2002); N. B. An, Phys. Rev. A **69**, 022315 (2004).
[23] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004); **97**, 140403 (2006); L. Aolita, R. Chaves, D. Cavalcanti, A. Acin, and L. Davidovich, *ibid.* **100**, 080501 (2008); N. B. An and J. Kim, Phys. Rev. A **79**, 022303 (2009).
[24] A. R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098 (1996); A. M. Steane, Phys. Rev. Lett. **77**, 793 (1996); E. Knill, R. Laflamme, and W. H. Zurik, Science **279**, 342 (1998).
[25] K. Khodjasteh and D. A. Lidar, Phys. Rev. Lett. **95**, 180501 (2005); G. S. Uhrig, *ibid.* **98**, 100504 (2007); B. Lee, W. M. Witzel, and S. Das Sarma, *ibid.* **100**, 160505 (2008).
[26] J. S. Zhang, J. B. Xu, and Q. Lin, Eur. Phys. J. D **51**, 283 (2009); J. Zhang, R. B. Wu, C. W. Li, and T. J. Tarn, J. Phys. A **42**, 035304 (2009).
[27] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[28] Y. Li, J. Zhou and, H. Guo, Phys. Rev. A **79**, 012309 (2009).
[29] S. Maniscalco, F. Francica, R. L. Zaffino, N. LoGullo, and F. Plastina, Phys. Rev. Lett. **100**, 090503 (2008).