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Tripartite nonlocality and continuous-variable entanglement in thermal states of trapped ions

Jie Li,[1] Thomás Fogarty,[2] Cecilia Cormick,[3] John Goold,[2,4] Thomas Busch,[2] and Mauro Paternostro[1]
[1]*School of Mathematics and Physics, Queen's University, Belfast BT7 1NN, United Kingdom*
[2]*Physics Department, University College Cork, Cork, Ireland*
[3]*Theoretische Physik, Universität des Saarlandes, D-66041 Saarbrücken, Germany*
[4]*Clarendon Laboratory, University of Oxford, United Kingdom*
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We study a system of three trapped ions in an anisotropic bidimensional trap. By focusing on the transverse modes of the ions, we show that the mutual ion-ion Coulomb interactions set entanglement of a genuine tripartite nature, to some extent persistent to the thermal nature of the vibronic modes. We tackle this issue by addressing a nonlocality test in the phase space of the ionic system and quantifying the genuine residual tripartite entanglement in the continuous variable state of the transverse modes.

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I. INTRODUCTION

The progress recently accomplished in ion-trap technology has pushed our ability to manage systems of a few interacting ions up to the point of demonstrating multipartite entanglement of the particles' internal degrees of freedom in the form of so-called W and Dicke states [1,2] or of performing high-fidelity three-qubit gates that are universal for quantum computation [3]. Moreover, both the direct interaction between the vibrational states of two trapped ions and their entanglement achieved through the use of their internal states have recently been experimentally demonstrated [4,5]. It is thus clear that such recent advances are putting the harnessing of multiparticle systems fully within the grasp of our experimental capabilities. Such exciting perspectives have paved the way for theoretical studies on the various facets of quantum statistical properties of systems of trapped particles. Considerable attention has been devoted to the structural changes undergone by an array of trapped ions under increasingly anisotropic trapping potentials [6], while an extensive program of simulation of quantum field theory via quantum control over the parameters of a system of trapped particles has been conducted [7,8]. In particular, Retzker *et al.* [9] have investigated the distribution of entanglement among the elements of an n-element ionic chain prepared in its ground state. On the other hand, it has been theoretically proven that, by allowing for some temporal control over the trapping frequencies of an array of ions, one can engineer a whole set of inter-ion operations mimicking the toolbox typical of linear photonics [10] and arrange for entanglement distribution across the array itself.

It is thus clear that the interest in achieving, controlling, and manipulating entanglement in many-ion settings lays at the basis of current efforts at all levels. Therefore, it is important to extend our comprehension of the structure of multipartite entanglement in simple configurations of a prompt experimental implementation, potentially exploring the possibility to achieve and maintain quantum correlations in conditions that are far from ideal. This latter aspect is particularly important from a fundamental viewpoint, too. The astonishing degree of control that is allowed in systems of trapped ions makes them the ideal platform to investigate and simulate complex situations that incorporate explicitly finite-temperature effects or the coupling with an environment. This would indeed allow us to explore in a controlled way the boundary between the classical and the quantum world.

This is precisely the task of the present work. We consider a simple yet interesting system consisting of three trapped ions in a linear configuration. We show that the Coulomb ion-ion coupling sets multipartite entanglement among the transversal modes of vibration and that, without the need for the time control over the trapping frequencies of the setup, a nonlocal state of the three vibrational modes is achieved, even at nonzero temperature of the modes themselves. Moreover, although tripartite nonlocality is not shown in our model due to the partial degree of symmetry enjoyed by the system, we demonstrate that genuine tripartite entanglement is established robustly, to some extent, against incoherent thermal effects. Our work thus sheds light on the quantum correlation properties of a system endowed with a low number of degrees of freedom, yet complex enough to give rise to interesting features of multipartite entanglement.

The remainder of the paper is organized as follows. In Sec. II we describe the system we address and provide the details of its solution, leading toward the determination of the covariance matrix describing the vibrational state of the transverse modes of the ions. Section III addresses the problem of revealing the nonlocality of such a vibronic state. We show that a tripartite Mermin-Klyshko (MK) inequality is violated by the state of the three ions for various configurations of masses and trapping frequencies and for temperatures up to $\sim 10^{-6}$ K. Section IV is devoted to the analysis of tripartite entanglement at both zero and nonzero temperature, while we draw our conclusions and open up a few questions that remain to be addressed in Sec. V. Finally, Appendix some technical details of our analysis.

II. THE MODEL

We consider a chain of three ions trapped by a common harmonic potential along the longitudinal z direction of a reference frame. We explicitly allow for differences in the mass of each particle, therefore including the possibility that different singly charged ion species are trapped, along the lines of Ref. [11]. In particular, we take the mass of the two end-chain ions, which we label as 1 and 3, to be m while the mass

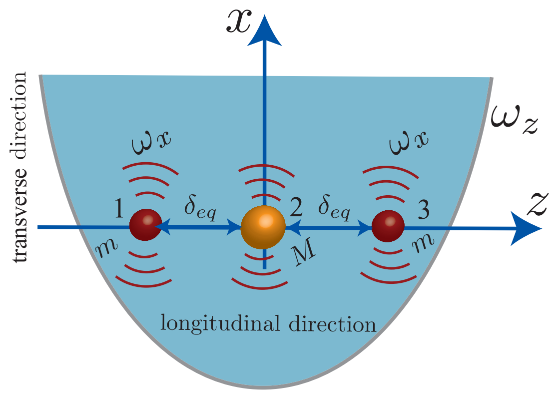

FIG. 1. (Color online) Scheme of the physical system at hand. We consider three ions in a parabolic potential along the longitudinal direction (the trapping potential ω_z is the same for all the ions). Ions 1 and 3 have the same mass m, while ion 2 (which occupies the position $z = 0$ in our reference frame) has mass M. The ions are also confined in the transverse direction with the two end-chain ions having trapping frequency ω_x. The transverse trapping frequency scales inversely with the mass of the ion, so it will be different for the central one. The equilibrium distance between an ion and its nearest neighbor is $\delta_{\rm eq}$ (see text for details).

of the central one (which is particle 2) is $M \neq m$ (see Fig. 1). The ions are assumed to be tightly confined in the x-z plane (direction x being dubbed, from now on, as the transverse one). The trapping frequency for particles 1 and 3 along axis x is ω_x, while we call ω_z the trapping frequency along the z direction. The mass ratio between the two ionic species and the ratio between transverse and longitudinal trapping frequencies will be free parameters in our model, whose Hamiltonian reads [11]

$$\hat{\mathcal{H}} = \sum_{j=1}^{3}\left(\frac{\hat{p}_j^2}{2m_j} + \frac{1}{2}\frac{m^2}{m_j}\omega_x^2\hat{x}_j^2\right) + \hat{\mathcal{H}}_z + \sum_{i>j}^{3}\frac{e^2}{4\pi\epsilon_0\sqrt{(\hat{x}_i-\hat{x}_j)^2+(\hat{z}_i-\hat{z}_j)^2}}, \tag{1}$$

where $m_1 = m_3 = m$ $(m_2 = M)$, $\hat{p}_j$ $(\hat{x}_j)$ is the momentum operator (position operator for the motion along x) of ion j. Finally, e is the electric charge (we consider single-charge ion species), ϵ_0 is the vacuum permittivity, and $\hat{\mathcal{H}}_z$ is the term describing the potential along the z direction. Clearly, the second line in Eq. (1) encompasses the Coulomb repulsive interaction among the ions in the system, which allocate themselves in the x-z plane in an equilibrium configuration that critically depends on the chosen working point [6]. In particular, here we shall consider trapping frequencies that keep the system well away from the linear-to-zigzag phase transition. For sufficiently tight transverse trapping, the equilibrium configuration is a linear array along the longitudinal axis with the central ion occupying the position $z_2 = 0$ and the end-chain one being at $z_1 = -z_3 = \delta_{\rm eq}$ with $\delta_{\rm eq}^3 = 5e^2/(16\pi m\omega_z^2\epsilon_0)$. Equation (1) can be simplified by considering small transversal deviations from the linear equilibrium configuration. We thus take a series expansion of $\hat{\mathcal{H}}$ around the equilibrium, neglecting any order higher than the second (along the lines of the study by Retzker *et al.* [7] and Serafini *et al.* [10], where it is shown that orders higher than the harmonic order are negligible away from the linear-to-zigzag transition). Therefore, hereafter we consider the model

$$\hat{\mathcal{H}}_2 = \sum_{j=1}^{3}\frac{\hat{p}_j^2}{2m_j} + \frac{1}{2}\sum_{i,j=1}^{3}V_{ij}\hat{x}_i\hat{x}_j, \tag{2}$$

where $V_{ii} = (m^2/m_i)\omega_x^2 - \sum_{j\neq i} e^2/(2\pi\epsilon_0|z_i - z_j|^3)$ and $V_{ij} = e^2/(2\pi\epsilon_0|z_i - z_j|^3)$ are the elements of the effective potential matrix

$$\boldsymbol{V} = \frac{m\omega_z^2}{5}\begin{pmatrix} 5\alpha^2-9 & 8 & 1 \\ 8 & 5\alpha^2\mu^2-16 & 8 \\ 1 & 8 & 5\alpha^2-9\end{pmatrix}. \tag{3}$$

We have introduced the mass ratio $\mu = \sqrt{m/M}$ and the ratio between the trapping frequencies $\alpha = \omega_x/\omega_z$. In the remainder of this paper, we have ensured that the working points chosen in our simulations are all such that the harmonic approximation is valid. In order to get rid of the mass difference in the kinetic term of the central ion, so as to ease the diagonalization of $\hat{\mathcal{H}}_2$, we rescale the operators associated with the central ion as $p_2' = \mu p_2$, $x_2' = x_2/\mu$, so that

$$\hat{\mathcal{H}}_2 = \hat{\boldsymbol{Q}}'(\boldsymbol{V}' \oplus \boldsymbol{K}')\hat{\boldsymbol{Q}}'^{t}, \tag{4}$$

with $\hat{\boldsymbol{Q}}' = (\hat{x}_1,\hat{x}_2',\hat{x}_3,\hat{p}_1,\hat{p}_2',\hat{p}_3)$ and the block matrices

$$\boldsymbol{V}' = \frac{1}{2}\begin{pmatrix} V_{11} & \mu V_{12} & V_{13} \\ \mu V_{12} & \mu^2 V_{22} & \mu V_{23} \\ V_{13} & \mu V_{23} & V_{11}\end{pmatrix}, \quad \boldsymbol{K}' = \frac{\mathbb{1}_3}{2m}. \tag{5}$$

To aid diagonalization we introduce the new set of quadratures $x_\pm = (x_1 \pm x_3)/\sqrt{2}$, and $p_\pm = (p_1 \pm p_3)/\sqrt{2}$ and use the fact that, by symmetry, $V_{12} = V_{23}$. The mode $(\hat{x}_-,\hat{p}_-)$, having frequency $\nu_- = \sqrt{(V_{11} - V_{13})/m} = \omega_z\sqrt{\alpha^2 - 2}$, is decoupled from the remaining two and thus embodies the first of the normal modes of the system. The remaining two are found upon explicit diagonalization of the Hamiltonian matrix

$$\boldsymbol{V}'' = \begin{pmatrix} \frac{V_{11}+V_{33}}{2} & \frac{\sqrt{2}V_{12}\mu}{2} & 0 & 0 \\ \frac{\sqrt{2}V_{12}\mu}{2} & \frac{V_{22}\mu^2}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2m} & 0 \\ 0 & 0 & 0 & \frac{1}{2m}\end{pmatrix}, \tag{6}$$

which delivers the eigenenergies $E_j (j = 1,2)$ (their expressions are too lengthy to be reported) and thus the normal frequencies $\nu_j = \sqrt{2E_j/m}$. The corresponding normal modes $(\hat{\mathcal{X}}_j,\hat{\mathcal{P}}_j)$ are related to $(\hat{x}_2',\hat{p}_2')$ and $(\hat{x}_+,\hat{p}_+)$ by simple linear relations, which can be inverted so as to give the position operators

$$\begin{aligned} \hat{x}_1 &= (\hat{x}_- + a_{11}\hat{\mathcal{X}}_1 + a_{12}\hat{\mathcal{X}}_2)/\sqrt{2}, \\ \hat{x}_2 &= \mu(a_{21}\hat{\mathcal{X}}_1 + a_{22}\hat{\mathcal{X}}_2), \\ \hat{x}_3 &= (-\hat{x}_- + a_{11}\hat{\mathcal{X}}_1 + a_{12}\hat{\mathcal{X}}_2)/\sqrt{2}, \end{aligned} \tag{7}$$

and the momenta

$$\begin{aligned} \hat{p}_1 &= (\hat{p}_- + a_{11}\hat{\mathcal{P}}_1 + a_{12}\hat{\mathcal{P}}_2)/\sqrt{2}, \\ \hat{p}_2 &= (a_{21}\hat{\mathcal{P}}_1 + a_{22}\hat{\mathcal{P}}_2)/\mu, \\ \hat{p}_3 &= (-\hat{p}_- + a_{11}\hat{\mathcal{P}}_1 + a_{12}\hat{\mathcal{P}}_2)/\sqrt{2}, \end{aligned} \tag{8}$$

where the coefficients a_{jk} are the elements of $\boldsymbol{B}^{-1}$, with $\boldsymbol{B}$ being the orthogonal matrix that diagonalizes $\boldsymbol{V}''$.

The last step towards the solution of our model is the introduction of dimensionless quadratures ($\hat{X}_j = \sqrt{m_j\omega_j/\hbar}\hat{x}_j$, $\hat{P}_j = \hat{p}_j/\sqrt{\hbar m_j\omega_j}$) ($j = 1,2,3$) with $\omega_{1,3} = \sqrt{V_{11}/m} = \omega_z\sqrt{\alpha^2 - 9/5}$ and $\omega_2 = \sqrt{V_{22}/M} = \mu\omega_z\sqrt{\alpha^2\mu^2 - 16/5}$. By placing these into Eqs. (7) and (8), we finally obtain

$$\hat{X}_j = (\lambda_j\sqrt{\omega_j/\nu_-}\hat{X}_- + a_{11}\sqrt{\omega_j/\nu_1}\hat{\mathcal{X}}_1' + a_{12}\sqrt{\omega_j/\nu_2}\hat{\mathcal{X}}_2')/\sqrt{2},$$
$$\hat{P}_j = (\lambda_j\sqrt{\nu_-/\omega_j}\hat{P}_- + a_{11}\sqrt{\nu_1/\omega_j}\hat{\mathcal{P}}_1' + a_{12}\sqrt{\nu_2/\omega_j}\hat{\mathcal{P}}_2')/\sqrt{2}, \tag{9}$$

for $j = 1,3$, where $\lambda_1 = 1$, $\lambda_3 = -1$. Similarly, we have

$$\hat{X}_2 = a_{21}\sqrt{\omega_2/\nu_1}\hat{\mathcal{X}}_1' + a_{22}\sqrt{\omega_2/\nu_2}\hat{\mathcal{X}}_2',$$
$$\hat{P}_2 = a_{21}\sqrt{\nu_1/\omega_2}\hat{\mathcal{P}}_1' + a_{22}\sqrt{\nu_2/\omega_2}\hat{\mathcal{P}}_2', \tag{10}$$

for the remaining mode. Quite naturally, we have used here the dimensionless normal coordinates $\hat{X}_- = \sqrt{m\nu_-/\hbar}\hat{x}_-$, $\hat{P}_- = \hat{p}_-/\sqrt{\hbar m\nu_-}$, $\hat{\mathcal{X}}_i' = \sqrt{m\nu_i/\hbar}\hat{\mathcal{X}}_i$, and $\hat{\mathcal{P}}_i' = \hat{\mathcal{P}}_i/\sqrt{\hbar m\nu_i}$ ($i = 1,2$).

The quadratic nature of the interion interaction ensures that any input Gaussian states (i.e., a state having a Weyl characteristic function of the form of a multivariate Gaussian) preserves its Gaussian nature and is thus completely characterized by the first and second statistical moments of the quadrature operators. Because we are interested here in the correlation properties of the three-ion state, the first moment will not be relevant and we thus discard them. In the picture given by the local quadratures of the ionic system the second moments can be organized in the form of a covariance matrix $\boldsymbol{\sigma}$ defined as $\sigma_{ij} = \frac{1}{2}\langle\{\hat{v}_i,\hat{v}_j\}\rangle$, where the expectation value is evaluated over the state of the three-mode system, $\{\cdot,\cdot\}$ stands for an anticommutator and $\hat{\boldsymbol{v}} = (\hat{X}_1,\hat{P}_1,\hat{X}_2,\hat{P}_2,\hat{X}_3,\hat{P}_3)$.

On the other hand, in the picture given by the normal modes, $\boldsymbol{\sigma}$ is the direct sum of three independent blocks, each describing the local properties of one of the normal modes. That is,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_- \bigoplus_{k=1}^{2}\boldsymbol{\sigma}_k. \tag{11}$$

In what follows, we will assume that each normal mode is in a thermal state at temperature T with an average number of excitations given by $\overline{n}_j(\nu_j,T) = 1/\{\exp[\hbar\nu_j/(K_bT)] - 1\}$ ($j = 1,2,-$), where K_b is the Boltzmann constant. The corresponding covariance matrix is $\boldsymbol{\sigma}_j = [\overline{n}_j(\nu_j,T) + 1/2]\mathbb{1}_2$. With this at hand, the covariance matrix for the state in terms of the original coordinates corresponding to the individual ions can be easily obtained by using the relations in Eqs. (7) and (8). The explicit elements of the resulting covariance matrix are obtained straightforwardly, although they are too lengthy to be reported here. Nevertheless, a piece of information that can be gathered immediately from Eqs. (9) and (10) is that $\langle\hat{X}_j\hat{P}_k\rangle = 0$ for $j,k = 1,2,3$. Therefore, $\boldsymbol{\sigma}$ is automatically found to be in the so-called standard form [12].

In order for $\boldsymbol{\sigma}$ to describe a physically meaningful state, it has to satisfy the conditions that guarantee its *bona fide* nature. In detail, $\boldsymbol{\sigma}$ should be semipositive definite and satisfy the Heisenberg-Robertson uncertainty relation $\boldsymbol{\sigma} + (i/2)\boldsymbol{\Omega} \geqslant 0$, where $\boldsymbol{\Omega} = \bigoplus_{j=1}^{3} i\boldsymbol{\sigma}^y$ is the so-called symplectic matrix and $\boldsymbol{\sigma}^y$ is the y-Pauli matrix. This condition is equivalent to requiring that the eigenvalues of $|i\boldsymbol{\Omega}\boldsymbol{\sigma}|$, which are commonly referred to as symplectic eigenvalues, are in modulus larger than 1/2 [12]. This imposes some constraints to the values that α and μ could take. In the remainder of our calculations we work in a parameter regime such that physicality of $\boldsymbol{\sigma}$ is strictly guaranteed.

III. PHASE-SPACE NONLOCALITY TEST

Having determined the moments of the ionic vibrational modes, we are now in a position to straightforwardly investigate the quantum correlation properties of the system, which is also the main goal of our study.

We start reminding that, given $\boldsymbol{\sigma}$, one can easily compute the Wigner function [13] associated with the state of the vibrational modes. For our three-mode continuous-variable (CV) state, the Wigner function is defined as the Fourier transform of the Weyl characteristic function $\chi(\boldsymbol{v},\boldsymbol{\sigma}) = \exp(-\boldsymbol{v}\boldsymbol{\sigma}\boldsymbol{v}^t)$ [14]. By working out explicitly the Fourier transform, one easily gets

$$W_{\boldsymbol{\sigma}}(\boldsymbol{v}) = \frac{\exp(-\boldsymbol{v}\boldsymbol{\sigma}^{-1}\boldsymbol{v}^t)}{\pi^3\sqrt{\det[\boldsymbol{\sigma}]}}. \tag{12}$$

In their seminal work [15], Banaszek and Wodkiewicz reprised an earlier observation connecting the value of the Wigner function at a given point in phase space to the expectation value of the displaced parity operator [16]. This was used in order to build up a phase-space version of the Bell-Clauser-Horne-Shimony-Holt inequality used to falsify local hidden-variable theories [15,17]. Very recently, this approach has been used to demonstrate nonlocal correlations in the state of two interacting ultracold neutral atoms [18]. Here, we go forward and extend the phase-space study of nonlocality to the multipartite scenario.

It is worth noticing that, in this context, one usually faces some important ambiguities in determining which type of inequality to utilize. In fact, in general, multiparticle entanglement should not be identified with multiparticle nonlocality [19]: the violation of an n-particle Bell-like inequality by an n-particle entangled state is not enough, *per se*, to prove genuine multipartite nonlocality. On the other hand, specific nonlocality inequalities exist to probe different types of correlation. Noticeably, Svetlichny has derived an inequality for the tripartite case that, while obeyed by models assuming two-particle nonlocality, is violated by quantum mechanical states that are genuinely three-particle entangled [20]. A Svetlichny inequality is a proper Bell inequality for the tripartite case and a valuable tool for the unambiguous assertion of the existence of genuine tripartite entanglement and nonlocality in any three-particle state. For this task, the use of a standard MK inequality [21] is not sufficient. We now proceed to ascertain the type of nonlocal state we have at hand when dealing with our system by considering the phase-space version of appropriate multipartite nonlocality inequalities.

Let us begin by observing that, similarly to what has been found in Ref. [10] for an array of identical ions, our system violates the MK inequality in an ample region of the (α,μ)

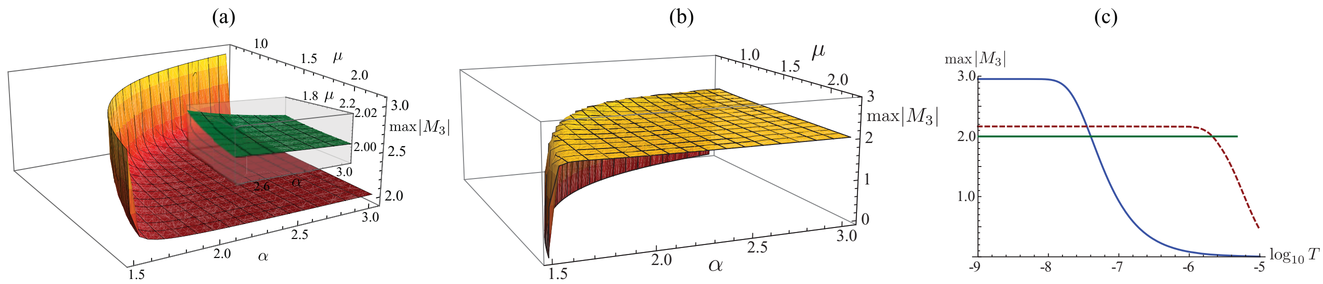

FIG. 2. (Color online) (a) The maximum value of $|M_3|$ as a function of α and μ at zero temperature. The inset shows that the quasiplateau seen in the main panel is always above the local-realistic bound. (b) The maximum value of $|M_3|$ as a function of α and μ at temperature $T = 10^{-6}$ K, with $\omega_z = 1$ MHz. (c) We show the maximum of $|M_3|$ against $\log_{10} T$ for two different working points. The solid line is for $(\mu,\alpha) = (1,2.1909)$ while the dashed one is for $(\mu,\alpha) = (1,2.3)$. The straight line indicates the boundary imposed by local hidden-variable theories. We have assumed the same value of ω_z as in panel (b).

space. In terms of the Wigner function given in Eq. (12), local realistic theories impose the inequality $|M_3| \leqslant 2$ with

$$M_3 = \frac{\pi^3}{8}[W_{\sigma}(\boldsymbol{v}'_1,\boldsymbol{v}_2,\boldsymbol{v}_3)+W_{\sigma}(\boldsymbol{v}_1,\boldsymbol{v}'_2,\boldsymbol{v}_3) + W_{\sigma}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}'_3) - W_{\sigma}(\boldsymbol{v}'_1,\boldsymbol{v}'_2,\boldsymbol{v}'_3)], \tag{13}$$

where $\boldsymbol{v}_j = (X_j,P_j)$ and $\boldsymbol{v}'_j = (X'_j,P'_j)$ embody pairs of different values of the same quadrature operators of mode $j = 1,2,3$. Violation of the MK inequality signals nonlocality in the tripartite state although, as commented above, not necessarily of a genuine three-particle nature. In Fig. 2(a) we plot the maximum value of $|M_3|$ as a function of α and μ for $T = 0$. Evidently, there is a threshold in α at any given value of μ below which $|M_3|$ should not be considered, as $\boldsymbol{\sigma}$ is not *bona fide* any more. Close to such a boundary for physicality, the violation of the MK inequality is quite large (quantum mechanics imposes the bound $M_3 \leqslant 4$) and is abruptly reduced to values just above the local realistic bound of 2 away from it (and, in particular, in the region of unbalanced masses and large values of α). The state of the system then remains nonlocal for a wide range of values of (α,μ). Although the degree of violation of the inequality is very small, nonlocality persists at nonzero temperature, as shown in Fig. 2(b). By further studying the latter, an interesting effect emerges: differently from the region at large α and μ, close to the physicality bound max $|M_3|$ sharply decreases to zero for low temperatures, witnessing the extreme sensitivity of the corresponding ionic state to thermal excitations. This is more clearly seen from Fig. 2(c), where we compare the nonlocal behavior of two initial states of the system as the temperature increases. Clearly, the MK function corresponding to a state closer to the boundary fades rapidly with respect to the state violating the inequality by a lesser extent. Similar behaviors have been predicted for measures of nonclassicality of optical fields affected by environmental losses [22], as well as multipartite entanglement in interacting spin chains [23].

A remark is due here: Clearly, μ cannot be taken at will and is set by the atomic species used in an experiment. However, our aim here is to highlight the existing trade-off to be carefully sought among the physical parameters entering the current problem. We value such information as quite useful.

The natural step forward in this context is to address now the Svetlichny inequality [20] so as to ascertain the genuine tripartite nonlocality of our system. We refer to Refs. [19,23] for full (yet inessential for our tasks) details on the significance of such inequality and the details of its construction. Here, it is sufficient to state that it reads $|S_3| \leqslant 4$ with Greenberger-Horne-Zeilinger-like states achieving a maximum degree of violation equal to $\sqrt{2}$. The phase-space version of the Svetlichny function reads [24]

$$S_3 = \frac{\pi^3}{8}[W_{\sigma}(\boldsymbol{v}'_1,\boldsymbol{v}_2,\boldsymbol{v}_3) + W_{\sigma}(\boldsymbol{v}_1,\boldsymbol{v}'_2,\boldsymbol{v}_3) + W_{\sigma}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}'_3) + W_{\sigma}(\boldsymbol{v}_1,\boldsymbol{v}_1,\boldsymbol{v}_3) - W_{\sigma}(\boldsymbol{v}'_1,\boldsymbol{v}'_2,\boldsymbol{v}_3) - W_{\sigma}(\boldsymbol{v}'_1,\boldsymbol{v}_2,\boldsymbol{v}'_3) - W_{\sigma}(\boldsymbol{v}_1,\boldsymbol{v}'_2,\boldsymbol{v}'_3) - W_{\sigma}(\boldsymbol{v}'_1,\boldsymbol{v}'_2,\boldsymbol{v}'_3)]. \tag{14}$$

This is obtained by following exactly the same line of thoughts leading to the phase-space formulation of a Bell-Clauser-Horne-Shimony-Holt inequality [15], by exploiting the formal identity between the expectation value of the displaced parity operator $\langle\hat{D}(\zeta)(-1)^{\hat{n}}\hat{D}^{\dagger}(\zeta)\rangle$ (with $\hat{n}$ being the operator counting the number of excitations in a boson and $\hat{D}^{\dagger}(\zeta)$ being the operator performing the displacement by an amplitude $\zeta \in \mathbb{C}$ in phase space) and the value of the Wigner function in ζ. An explicit derivation will be given somewhere else [24]. The maximum of $|S_3|$ exhibits a behavior similar to that of max $|M_3|$, reaching a top value of about 3 close to the origin of the phase space, even at $T = 0$. Clearly, this is not enough to falsify the Svetlichny inequality and thus demonstrate genuine multipartite nonlocality. The reason behind this probably lies in the linear arrangement of the ions that enforces the interaction between the end-chain particles to be the weakest. In turn, this unbalances the distribution of quantum correlations within the system in a way unfavorable to the violation of the Svetlichny inequality. It should be remarked, indeed, that the violation of the constraints imposed on S_3 by local realistic theories is rather demanding, with genuinely multipartite entangled states of the W form achieving values barely larger than 4 [19].

IV. TRIPARTITE CV ENTANGLEMENT

Notwithstanding the failure to violate the Svetlichny inequality, we claim that genuine tripartite entanglement is indeed shared among the ions of our system. In order to prove this statement, we refer to the multimode inseparability classification put forward by Giedke *et al.* [25]. Let us

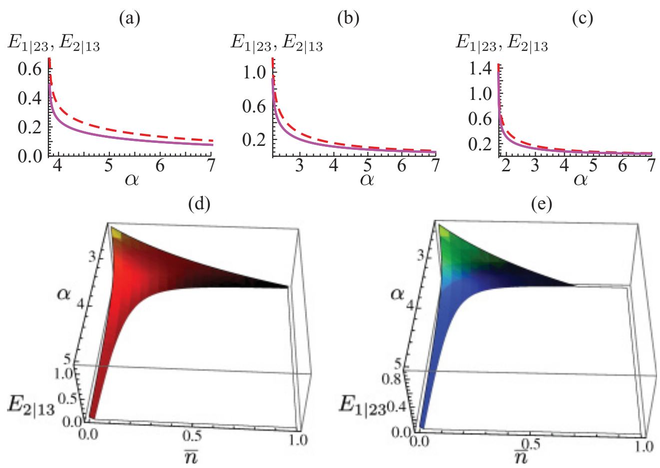

FIG. 3. (Color online) Panels (a), (b), and (c): We show the behavior of $E_{1|23}$ (solid lines) and $E_{2|13}$ (dashed ones) against α for $T = 0$ and $\mu = 0.5$, 1, and 1.5, respectively. Panels (d) and (e): The same quantities are plotted against α and $\bar{n}$ for $\mu = 1$. Qualitatively similar behaviors hold for other values of μ.

call $E_{a|b}$ an entanglement measure providing a quantitative estimate of the entanglement between party a and b of a multipartite system. According to Ref. [25], a state having $E_{i|jk} \neq 0$ ($\forall\, i,j,k = 1,2,3$) is genuinely tripartite entangled (or three-way entangled). If one of the multimode bipartitions $i|jk$ is separable, the state is said to be two-way entangled. We have thus checked the conditions for three-way inseparability by calculating $E_{1|23}$ and $E_{2|13}$ as functions of the relevant parameters in our model (by symmetry $E_{1|23} = E_{3|12}$). For this task, we have employed the logarithmic negativity as our entanglement measure [26], which is calculated as

$$E_{i|jk} = \max\left[0, -\sum_k \ln(2s_k^-)\right], \qquad (15)$$

with s_k^- being the k symplectic eigenvalue whose modulus is smaller than 1/2 of matrix $\boldsymbol{F\sigma F}$ and $\boldsymbol{F}$ being the matrix performing the sign-flip operation $\hat{P}_j \to -\hat{P}_j$ to mode j. The results, which are given in Fig. 3, show that, depending on the masses of the particles being used, quite a large range of values of α ensure genuine tripartite entanglement in the system. Panels 3(a)–3(c) reveal that $E_{2|13}$ and $E_{1|23}$ die off with increasing α and more quickly as μ grows (each plot is given in a range of parameters such that both stability and the validity of the second-order approach are guaranteed). As for the effects of a thermal character of the ionic modes, Figs. 3(d) and 3(e) show that $E_{2|13}$ is slightly more robust to a nonzero value of T than $E_{1|23}$ (and therefore $E_{3|12}$).

Having qualitatively characterized the existence of multipartite entanglement in the system, we now aim at quantifying it. This can be done by relying on monogamy relations [27] for multipartite entanglement in (generally mixed) CV states. Pure instances of such states are known to satisfy the set of inequalities

$$\Delta_{i|jk} \equiv C_{i|jk} - C_{i|j} - C_{i|k} \geqslant 0 \quad (i,j,k = 1,2,3), \qquad (16)$$

with $C_{a|b}$ the so-called CV tangle (or contangle) of systems a and b, which is a proper entanglement monotone defined

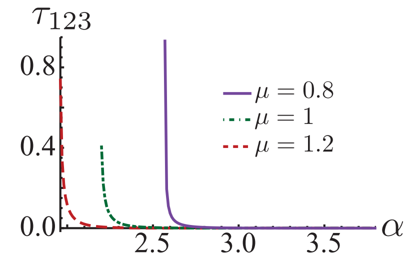

FIG. 4. (Color online) Residual tripartite entanglement in the vibrational state of the transverse modes of our system. We study τ_{123} against α at $T = 0$ and for three values of μ. Each curve is given in a range of parameters such that both stability and the validity of the second-order approach are guaranteed. The residual tripartite entanglement is always finite, with a maximum value that depends on μ.

as the squared logarithmic negativity [12]. Equation (16) exemplifies the constraints on the distribution of entanglement in a multimode state and can be extended to mixed multipartite states by taking the convex-roof extension of contangle [12]. Equipped with such tools, we quantify the residual tripartite entanglement within our system as

$$\tau_{123} = \min[\Delta_{1|23}, \Delta_{2|13}, \Delta_{3|12}]. \qquad (17)$$

We defer the details of the calculations required in order to quantitatively determine τ_{123} to the appendix, while here we report the results of our analysis. Figure 4 shows the behavior of τ_{123} at $T = 0$ and for various choices of the mass ratio. The degree of genuine tripartite entanglement is a function of μ that quickly decays for mismatched vibrational frequencies. As the ratio between the masses of the ions grows, $\max[\tau_{123}]$ has a nonmonotonic behavior, as it increases first and then decreases for large values of μ. Moreover, at $\mu = 1$, it achieves a minimum due to the fact that, for equal masses, the entanglement in any i-j bipartition ($i \neq j = 1,2,3$) is very large. An example of this is given in Fig. 4 where the maximum for $\mu = 1$ is smaller than the corresponding values achieved at $\mu = 0.8$ and $\mu = 1.2$.

The analysis at nonzero temperature presents features that are close to those addressed before, with a range of values of T where residual tripartite entanglement can be found to be nonzero that is consistent with the estimates given for the degree of nonlocality.

V. CONCLUSIONS

We have studied a system of three trapped ions interacting via the mutual Coulomb force in a bidimensional trap. We have focused on the degree of freedom embodied by the transversal vibrational modes of each individual particle. Under the harmonic approximation, a quadratic Hamiltonian was achieved, which has allowed us to make use of the powerful covariance-matrix formalism for Gaussian states. In turn, this allowed us to straightforwardly investigate multipartite nonlocality in phase space in the form of both an MK and a Svetlichny inequality. The existence of genuine tripartite entanglement in the vibrational state of the system has then been proven and characterized both qualitatively and quantitatively, showing resilience against the chosen working conditions and, to some extent, the thermal nature of such transversal modes. An important point to address is embodied by the identification of the techniques to follow for the experimental inference of the information required in order to perform the nonlocality tests studied here and the quantitative estimate of the multipartite entanglement shared by the three ions. We now briefly assess this issue by pointing out that both the Weyl characteristic function and the Wigner function of the motional state of trapped ions can be reconstructed exploiting the interplay between vibrational and spin degrees of freedom, along the lines of the schemes put forward in Refs. [18,28,29]. While such methods work, in principle, for arbitrary vibrational states, one could also take advantage of the Gaussian-preserving nature of the mechanism for multipartite entanglement studied in this work. By restricting our attention to the class of Gaussian states, in fact, one can explore the results presented here by means of a limited number of measurements directed at the reconstruction of the elements of the covariance matrix of the system, which can be performed by means of homodyne-like detection, which can be done in trapped-ion settings using various experimental techniques [30]. We will report on a practical scheme for such purposes elsewhere.

It remains an open question as to whether a configuration exists such that genuine tripartite nonlocality can be achieved and indeed observed. We are currently investigating along these lines by considering a genuine bidimensional arrangement of the ions that will mix longitudinal and transversal modes providing, at the same time, the necessary degree of symmetry necessary to violate a Svetlichny inequality.

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APPENDIX

In this appendix we briefly discuss the details of the approach needed for the calculation of the residual tripartite entanglement in the ion system.

In the pure-state case corresponding to taking $T=0$ in our model, each $C_{i|jk}$ is easily calculated as the squared logarithmic negativity. First, by means of local symplectic transformations, consisting in our case of a series of simple beam-splitter operations, the covariance matrix $\boldsymbol{\sigma}$ of the system can be brought into the form

$$\boldsymbol{\sigma}''=\begin{pmatrix} a_1\mathbb{1} & \boldsymbol{C}_{12} & \boldsymbol{C}_{13} \\ \boldsymbol{C}_{12}^T & a_2\mathbb{1} & \boldsymbol{C}_{12} \\ \boldsymbol{C}_{13}^T & \boldsymbol{C}_{12}^T & a_1\mathbb{1} \end{pmatrix} (a_{1,2}\geqslant 1), \tag{A1}$$

where, for ease of calculation, we have rescaled our variables in such a way that the single-mode vacuum state has a covariance matrix equal to $\mathbb{1}$. Here, each diagonal matrix $\boldsymbol{C}_{jk}$ embodies the correlations between modes j and k and the bisymmetric nature of $\boldsymbol{\sigma}''$ is evident: the covariance matrix is invariant under the exchange of modes 1 and 3 in virtue of the identity of the end-chain ions. The contangle between mode i and modes (j,k) is then simply evaluated as $C_{i|jk}=\text{arcsinh}^2\sqrt{a_i^2-1}$ (with, quite obviously, $a_3=a_1$). As for the $C_{i|j}$ terms appearing in Eq. (16), they are readily calculated by exploiting the fact that the reduced two-mode state of any pure three-mode state embodies a Gaussian least-entangled mixed state (or GLEMS) [31], as can be easily checked in our case by inspection.

In the mixed-state case of $T>0$, on the other hand, the monogamy inequalities have to be rephrased in terms of the convex-roof extension of the contangle [32]. As any multimode Gaussian state admits a decomposition in terms of pure Gaussian states only, we can restrict our attention to a convex-roof extension performed over the set of Gaussian states, which provides the Gaussian contangle $\mathcal{G}(\boldsymbol{\sigma})=\inf_{\boldsymbol{\sigma}^p\leqslant\boldsymbol{\sigma}}C(\boldsymbol{\sigma})$ with $\boldsymbol{\sigma}^p$ being the covariance matrices of pure Gaussian states such that $\boldsymbol{\sigma}^p-\boldsymbol{\sigma}\geqslant 0$. Equation (16) can then be restated by replacing the contangle with such a Gaussian convex-roof extension. The calculation of $\mathcal{G}_{i|jk}$ may appear difficult due to the minimization in $\mathcal{G}$. However, the bisymmetric nature of our covariance matrix allows for important simplifications. First, the minimization inherent in the definition of τ_{123} is achieved by the splitting, where mode i is the one with smallest mixedness [12]. In our case, such a mode is almost always 3 (i.e., equivalently, 1) and, without losing generality, we will assume this situation from now on. By means of simple beam splitting, it is possible to reduce the tripartite covariance matrix $\boldsymbol{\sigma}$ into one having the form

$$\boldsymbol{\sigma}_{loc}=\begin{pmatrix} \boldsymbol{\sigma}_{l,1'} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_{l,2'} & \boldsymbol{\sigma}_{l,2'3'} \\ \mathbf{0} & \boldsymbol{\sigma}_{l,2'3'} & \boldsymbol{\sigma}_{l,3'} \end{pmatrix}, \tag{A2}$$

where the new mode $1'$ is now uncorrelated from the set of new modes $(2',3')$ and, clearly, $\mathcal{G}_{3'|2'}=\mathcal{G}_{3|21}$. The Gaussian contangle $\mathcal{G}_{3'|2'}$ as well as the two-mode reductions $\mathcal{G}_{3|1}$ and $\mathcal{G}_{3|2}$ can then be calculated by making use of the formalism put forward in Ref. [33] and the explicit formulas provided in Ref. [12].

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