Optimal unitary dilation for bosonic Gaussian channels

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A general quantum channel can be represented in terms of a unitary interaction between the information-carrying system and a noisy environment. In this paper the minimal number of quantum Gaussian environmental modes required to provide a unitary dilation of a multimode bosonic Gaussian channel is analyzed for both pure and mixed environments. We compute this quantity in the case of pure environment corresponding to the Stinespring representation and give an improved estimate in the case of mixed environment. The computations rely, on one hand, on the properties of the generalized Choi-Jamiolkowski state and, on the other hand, on an explicit construction of the minimal dilation for arbitrary bosonic Gaussian channel. These results introduce a new quantity reflecting "noisiness" of bosonic Gaussian channels and can be applied to address some issues concerning transmission of information in continuous variables systems.

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I. INTRODUCTION

Bosonic Gaussian channels (BGCs) constitute an important special class of transformations that act on a collection of bosonic modes preserving the Gaussian character of transmitted quantum states. In quantum information science the class of BGCs is singled out from a theoretical viewpoint [1], but most significantly also from the perspective of practical implementations, since it emerges naturally as the fundamental noise model in several experimental contexts. In the majority of physical implementations of quantum transmission lines quantum information is sent using photons—through optical fibers [2], in free space [3], or via superconducting transmission lines [4]—the physical situations for which BGCs provide a very satisfactory model. Moreover, BGCs play a major role in characterizing the open quantum system dynamics of various setups which use collective degrees of freedom to store and manipulate quantum information, including systems from cavity quantum electrodynamics, nanomechanical harmonic oscillators [5], or clouds of cold atomic gases [6].

The study of Gaussian quantum channels (or quasifree maps) has a long tradition [7]. Intense recent research was mostly focused on properties of BGCs with respect to their ability to preserve and transmit quantum information (for a review, see, e.g., Ref. [8] and references therein). Recent contributions include the computation of the quantum capacity [9] of a large class of single-mode BGCs [10], a characterization in terms of degradability [11] that allows one to identify zero-quantum capacity BGCs, and a criterion for BGCs being entanglement breaking [12]. A general unitary dilation theorem for BGCs was proven in Ref. [13]: It shows that each BGC Φ acting on a system A formed by n input bosonic modes admits a unitary dilation in terms of a bosonic environment E composed of $\ell \leqslant 2n$ modes, the initial state $\hat{\rho}_E$ of which is Gaussian, with a Gaussian unitary coupling $\hat{U}_{A,E}$ corresponding to Hamiltonians that are quadratic in the canonical observables (see Sec. II for more precise description of the Gaussian unitaries),

$$\Phi(\hat{\rho}) = \mathrm{Tr}_E[\hat{U}_{A,E}(\hat{\rho} \otimes \hat{\rho}_E)\hat{U}_{A,E}^{\dagger}]. \tag{1}$$

Here $\hat{\rho}$ is the input quantum state of the system A and Tr_E denotes the partial trace over the degrees of freedom associated with E. In the case of pure environment state $\hat{\rho}_E$ the representation (1) is closely related to the Stinespring dilation [14] and with some abuse of terminology, we then speak of Eq. (1) as of Gaussian Stinespring dilation.

Let us stress that here, as anywhere in quantum information science, "environment" means those degrees of freedom of the actual physical environment of the open quantum system which are essentially involved in the interaction and in the resulting information exchange with the system (cf. the term "faked continuum" in [15]). The fact that the number of Gaussian environmental modes ℓ entering the unitary dilation (1) can be bounded from above by $2n$ may be viewed as the continuous-variable counterpart of the upper bound on the minimal dilation set by the Stinespring theorem [16] for finite dimensional quantum channels: It indicates that any quantum channel can be described by using an environment which is no more than twice larger than the input system. Therefore, the question of such a *minimal environment* for a given channel arises quite naturally; correspondingly, for BGC the natural question to ask is in regard to the *minimal Gaussian environment*. Especially important is estimation of the minimal value $\ell^{(\Phi)}$ needed to represent a given BGC, specifically the minimal value $\ell^{(\Phi)}_{\rm pure}$ in the case of Gaussian Stinespring dilation. Along with the quantum capacity, this quantity may be used as a basic characteristic of a BGC since one can expect that the larger it is, the noisier and less efficient in preserving the initial state will be the corresponding channel. Furthermore, exact knowledge of such a number would allow one to simplify the degradability analysis of BGCs by minimizing the number of degrees of freedom of the corresponding complementary channel.

The main result of this work is explicit computation of the minimal value $\ell^{(\Phi)}_{\rm pure}$ and the construction of the corresponding dilation. This is accomplished by first determining a lower bound for $\ell^{(\Phi)}_{\rm pure}$ in terms of the minimum number $q^{(\Phi)}_{\min}$ of ancillary modes needed to construct a Gaussian purification of a (generalized) Choi-Jamiolkowski (CJ) state of Φ. This is motivated by the fact that any Gaussian Stinespring representation (1) naturally induces a Gaussian purification of the CJ states of the channel. Then we show that this lower bound can be achieved by explicitly constructing a Gaussian Stinespring dilation with $q_{\min}^{(\Phi)}$ modes.

Moreover, we address the case of unitary dilations (1) in which the environmental Gaussian state $\hat{\rho}_E$ is not necessarily pure. These representations turn out to be useful since, in some cases, they provide a mathematical description which is closer to the physical intuition (for instance, they allow for a natural description of channels induced by beam-splitter transformations of the input state with a thermal environmental state). In this context we provide a new estimate for the minimal number of modes ℓ required by the representation, which improves the one presented in Ref. [13].

The paper is organized as follows: In Sec. II we recall some basic definitions and formulate the results. The notion of a generalized CJ state of a BGC and the lower bound $q_{\min}^{(\Phi)}$ for $\ell_{\rm pure}^{(\Phi)}$ are presented in Sec. III. Then, in Sec. IV we give a recipe to construct such minimal dilation, while technical results and details of the construction are given in the appendices.

II. DEFINITIONS AND RESULTS

Consider a system A composed of n bosonic quantum mechanical modes described by the canonical observables $\hat{R} := (\hat{Q}_1,\ldots,\hat{Q}_n;\hat{P}_1,\ldots,\hat{P}_n)$ and by the *Weyl (or displacement) operators*

$$\hat{V}(z) := e^{i\hat{R}z}, \tag{2}$$

with $z := (x_1,\ldots,x_n,y_1,\ldots,y_n)^T \in \mathbb{R}^{2n}$ being a column vector [17]. To simplify notation, we choose units in which $\hbar = 1$. We denote $\mathbb{1}_n$ the $n\times n$ identity matrix and by

$$\sigma_{2n} := \begin{bmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{bmatrix} \tag{3}$$

the matrix of the *symplectic form* defining the canonical commutation relations for n modes. We denote $M(m,\mathbb{R})$ the algebra of all $m\times m$ real matrices and $\mathrm{Gl}(m,\mathbb{R})$ the group of $m\times m$ real invertible matrices.

A state $\hat{\rho}$ is called *Gaussian* if its symmetrically ordered *characteristic function*,

$$\phi(\hat{\rho};z) := \mathrm{Tr}[\,\hat{\rho}\,\hat{V}(z)\,], \tag{4}$$

is a Gaussian exponent with the covariance matrix $\gamma \in \mathrm{Gl}(2n,\mathbb{R})$ satisfying the matrix uncertainty relation $\gamma \geqslant i\sigma_{2n}$. A *BGC* Φ is defined as the linear map which induces the following transformation of states:

$$\phi(\hat{\rho};z) \mapsto \phi[\Phi(\hat{\rho});z] := \phi(\hat{\rho};Xz)e^{-\frac{1}{4}z^TYz+iv^Tz}.$$

Here $v\in\mathbb{R}^{2n}$ is a real vector, X is a $2n\times 2n$ real matrix, and Y,X is a $2n\times 2n$ real symmetric matrix satisfying the inequality

$$Y \geqslant i\Sigma, \qquad \text{with} \quad \Sigma := \sigma_{2n} - X^T\sigma_{2n}X, \tag{5}$$

which ensures complete positivity of the map Φ. A BGC Φ maps Gaussian states into Gaussian states; in the case $Y = 0$, $X \in \mathrm{Sp}(2n,\mathbb{R})$ [the group of symplectic transformations preserving the form (3)] we speak of *Gaussian unitary* channel. By the Stone–von Neumann uniqueness theorem, it is given by conjugation with a unitary operator U, which we also call Gaussian unitary. Those are the operators implementing linear Bogoliubov transformations through the unitary metaplectic representation; any such unitary operator is a product of at most two exponentials of quadratic polynomials in the canonical variables [18].

In the construction of the Gaussian unitary dilation (1) of Φ, the vector v plays a marginal role since it can be eliminated via a unitary rotation acting on the output state (see, e.g., Ref. [13]). In contrast, the matrices in Eq. (5) are of fundamental importance; in particular, we see that the values of $\ell_{\rm pure}^{(\Phi)}$ and of our estimate for $\ell_{\rm mix}^{(\Phi)}$, depend upon the ranks of Y, Σ, and $Y - i\Sigma$. It is thus worth anticipating some relevant facts that concern these matrices. First of all, we notice that the inequality (5) implies the following relations:

$$\ker[\Sigma]\cap\ker[Y-i\Sigma] \subseteq \ker[Y] \subseteq \ker[Y-i\Sigma], \tag{6}$$

$$\ker[Y] \subseteq \ker[\Sigma], \tag{7}$$

where, throughout the paper, given a generic (possibly real) $d\times d$ matrix M, we denote its kernel (null subspace) by $\ker[M] := \{w\in\mathbb{C}^d : Mw = 0\}$ [19]. The first inclusion in Eq. (6) follows from the definition, the remaining one and the inclusion of Eq. (7) are derived from the observation that $w^\dagger Yw = 0 \Rightarrow w^\dagger(i\Sigma)w = 0 \Rightarrow w^\dagger(Y-i\Sigma)w = 0 \Rightarrow (Y-i\Sigma)w = 0 \Rightarrow \Sigma w = 0$. Putting these identities together we also find that

$$\ker[Y] = \ker[\Sigma]\cap\ker[Y-i\Sigma]. \tag{8}$$

Other useful properties are the identities

$$2\,\mathrm{rank}[Y-i\Sigma] = \mathrm{rank}[Y] + \mathrm{rank}[Y-\Sigma Y^{\ominus 1}\Sigma^T] \tag{9}$$

and the inequalities

$$\mathrm{rank}[Y] \geqslant \mathrm{rank}[\Sigma] \geqslant \mathrm{rank}[Y] - \mathrm{rank}[Y-\Sigma Y^{\ominus 1}\Sigma^T] \geqslant 0, \tag{10}$$

where $\mathrm{rank}[M]$ stands for the rank of the matrix M (i.e., the dimension over the complex field of the complement to $\mathbb{C}^d$ of the matrix $\ker[M]$), and $Y^{\ominus 1}$ is the Moore-Penrose (MP) inverse of Y [20]. The explicit proof of these relations is rather technical and thus we postpone it to Appendix A. Here we point out that the first inequality of Eq. (10) is a consequence of the fact that $\ker[Y]$ is included in $\ker[\Sigma]$, while the last inequality is an immediate consequence of the fact that $\Sigma Y^{\ominus 1}\Sigma^T$ is positive semidefinite.

In Ref. [13], an upper bound for $\ell_{\rm pure}^{(\Phi)}$ was set by showing that one can construct a Gaussian Stinespring dilation of Φ that involves $\ell = 2n - r'/2$ environmental modes with

$$r' := \mathrm{rank}[Y] - \mathrm{rank}[Y-\Sigma Y^{\ominus 1}\Sigma^T]. \tag{11}$$

In this paper we strengthen this result by showing that *the minimum number of modes necessary to build a Gaussian Stinespring dilation for Φ is given by*

$$\ell_{\rm pure}^{(\Phi)} = \mathrm{rank}[Y-i\Sigma] = \mathrm{rank}[Y] - r'/2, \tag{12}$$

where we used Eq. (9) when formulating the second identity. Since Y is a $2n\times 2n$ matrix, we have $2n - \mathrm{rank}[Y] \geqslant 0$, and so the optimal bound we prove here leads to an improvement compared to the results of Ref. [13]. In particular, for Gaussian unitary channels the optimal bound (12) yields $\ell^{(\Phi)}_{\rm pure} = 0$—no environment is required to construct the dilation—while Ref. [13] had this value equal to $2n$. To prove Eq. (12) we shall first show that the quantity rank$[Y - i\Sigma]$ provides a lower bound for $\ell^{(\Phi)}_{\rm pure}$ (see Sec. III) and then construct an explicit Stinespring dilation (1) for Φ, which attains such bound (see Sec. IV).

In Ref. [13] it was also shown that for arbitrary (not necessarily Stinespring) dilations one can reduce the number of environmental modes from $2n - r'/2$ to $2n - \text{rank}[\Sigma]/2$ [this is smaller than the former because of Eq. (10)]. In the present paper we improve such estimate, showing that one needs at most

$$\ell^{(\Phi)}_{\rm mix} = \text{rank}[Y] - \text{rank}[\Sigma]/2, \tag{13}$$

environmental modes [which is less than the quantity in (12), again due to inequality Eq. (10)]. This bound is derived in the last part of Appendix C. A reasonable conjecture is that such a bound is also optimal, but CJ argument used in Sec. III to derive a lower bound in the pure dilation case no longer works for the mixed case.

III. LOWER BOUND ON $\ell^{(\Phi)}_{\rm pure}$ VIA GENERALIZED CJ STATES OF BGCS

In this section we review the notion of generalized Choi-Jamiolkowski (CJ) state for a multimode BGC (see Ref. [22]) and use it to show that the term on the right-hand side of Eq. (12) provides a lower bound for $\ell^{(\Phi)}_{\rm pure}$. Consider a state vector $|\Psi_{\hat\Lambda}\rangle_{A,B}$ providing a purification of a quantum state $\hat\Lambda = \sum_{j=1}^{\infty} \lambda_j (|j\rangle\langle j|)_A$ of the system labeled A which has full rank (e.g., a Gibbs state of n modes),

$$|\Psi_{\hat\Lambda}\rangle_{A,B} = \sum_{j=0}^{\infty} \sqrt{\lambda_j}\, |j\rangle_A \otimes |j\rangle_B$$
$$= (\hat\Lambda^{1/2} \otimes \mathbb{1}) \sum_{j=0}^{\infty} |j\rangle_A \otimes |j\rangle_B,$$

with A indicating the input space of the channel Φ, B being an ancillary system isomorphic to A, and $\{|j\rangle : j = 0,1,\ldots\}$ denoting an orthonormal basis. A generalized CJ state of the channel Φ is now obtained as

$$\hat\rho_{A,B}(\Phi) = (\Phi \otimes \mathcal{I})(|\Psi_{\hat\Lambda}\rangle\langle\Psi_{\hat\Lambda}|)_{A,B}, \tag{14}$$

with $\mathcal{I}$ being the identity map. The state $\hat\rho_{A,B}(\Phi)$ provides a complete description of the channel via the inversion formula,

$$\Phi(\hat\rho) = \text{Tr}_B\big[\big(\mathbb{1}_A \otimes \hat\Lambda_B^{-1/2} \hat\rho_B^T \hat\Lambda_B^{-1/2}\big)\hat\rho_{A,B}(\Phi)\big], \tag{15}$$

where $\hat\rho_B$ and $\hat\Lambda_B$ are copies of the states $\hat\rho$ and $\hat\Lambda$ on B, respectively, while $\hat\rho_B^T$ is its transpose with respect to the orthonormal basis introduced above.

For a finite-dimensional system $\hat\rho_{A,B}(\Phi)$ provides a standard CJ state representation when $\hat\Lambda$ is taken to be the maximally mixed state. In the infinite-dimensional case such a limit in general is well defined only in the context of positive forms (see Ref. [22]). However, Eq. (15) shows that we do not need to approach such a limit in order to build a proper representation of the channel: It is defined for any state diagonal in the distinguished basis of full rank. Furthermore, it is easy to verify that it is always possible to work with CJ states $\hat\rho_{A,B}(\Phi)$ which are Gaussian: To do so, simply take $(|\Psi_{\hat\Lambda}\rangle\langle\Psi_{\hat\Lambda}|)_{A,B}$ to be Gaussian and use the fact that the Gaussian map $\Phi \otimes \mathcal{I}$ maps Gaussian states into Gaussian states. In the following we choose to take such Gaussian reference states. In particular, we assume $(|\Psi_{\hat\Lambda}\rangle\langle\Psi_{\hat\Lambda}|)_{A,B}$ to be a Gaussian purification of a multimode Gibbs (thermal) state of quantum mechanical oscillators.

An important observation concerning the generalized CJ representation is that, given a Stinespring representation of Φ involving an environmental system E, one can construct a purification of $\hat\rho_{A,B}(\Phi)$ that uses E as an ancillary system. Indeed, assuming that $\hat U_{A,E}$ and $(|0\rangle\langle 0|)_E$ give rise to a Stinespring representation for Φ, we have that the pure state

$$|\chi\rangle_{A,B,E} = \hat U_{A,E} |\Psi_{\hat\Lambda}\rangle_{A,B} \otimes |0\rangle_E \tag{16}$$

is a purification of $\hat\rho_{A,B}(\Phi)$. Furthermore, if $\hat\rho_{A,B}(\Phi)$ is Gaussian and E represents a collection of ℓ environmental bosonic modes with $|0\rangle_E$ being a Gaussian state vector and $\hat U_{A,E}$ being a Gaussian unitary, it follows that also $|\chi\rangle_{A,B,E}$ will define a Gaussian purification of $\hat\rho_{A,B}(\Phi)$. Putting these facts together it follows that a lower bound for the minimal number $\ell^{(\Phi)}_{\rm pure}$ of environmental modes that are needed to build a Gaussian Stinespring representation of Φ is provided by the minimal number $q^{(\Phi)}_{\rm min}$ of Gaussian ancillary modes that are required to purify a generalized Gaussian CJ state $\hat\rho_{A,B}(\Phi)$ of Φ, that is,

$$\ell^{(\Phi)}_{\rm pure} \geqslant q^{(\Phi)}_{\rm min}. \tag{17}$$

To compute $q^{(\Phi)}_{\rm min}$ we first make a specific choice for $|\Psi_{\hat\Lambda}\rangle_{A,B}$. In particular, since A is composed of n bosonic modes, we can take $|\Psi_{\hat\Lambda}\rangle_{A,B}$ to be a product of n identical two-mode state vectors of the form [23]

$$|\Psi_{\hat\Lambda}\rangle_{A,B} = \bigotimes_{i=1}^{n} |\psi\rangle_{A_i,B_i}, \tag{18}$$

where $|\psi\rangle_{A_iB_i}$ denotes a purification of a Gibbs state of the ith mode A_i of A built by coupling it with the corresponding ancillary system B_i: This is nothing but what is usually referred to as a *two-mode squeezed state* [24]. The resulting state is Gaussian and it is fully characterized by its covariance matrix. To express it in a compact form note that the kernel of the natural symplectic form for the $2n$ modes of A,B is given by

$$\sigma_{A,B} := \begin{bmatrix} \sigma_{2n} & 0 \\ 0 & \sigma_{2n} \end{bmatrix}, \tag{19}$$

where the upper-left and lower-right block matrices represent the symplectic forms of the n modes of A and B, respectively, defined as in Eq. (3).

With this choice the covariance matrix γ of $(|\Psi_{\hat\Lambda}\rangle\langle\Psi_{\hat\Lambda}|)_{A,B}$ is given by the following $M(4n,\mathbb{R})$ matrix:

$$\gamma = \begin{bmatrix} \alpha & \delta \\ \delta^T & \beta \end{bmatrix}, \tag{20}$$

where α, $\beta \in M(2n,\mathbb{R})$ are the covariance matrices of the A and B modes, respectively, with δ, $\delta^T \in M(2n,\mathbb{R})$ being the cross-correlation terms. Explicitly, they are given by

$$\alpha = \begin{bmatrix} \theta \mathbb{1}_n & 0 \\ 0 & \theta \mathbb{1}_n \end{bmatrix} = \beta, \quad \delta = \begin{bmatrix} 0 & f(\theta)\mathbb{1}_n \\ f(\theta)\mathbb{1}_n & 0 \end{bmatrix} = \delta^T,$$

with $\theta > 1$ and

$$f(\theta) := -(\theta^2 - 1)^{1/2}. \tag{21}$$

The parameter θ determines the *temperature* of the Gibbs states we used to build the vector $|\Psi_{\hat{\Lambda}}\rangle_{A,B}$, or equivalently, the *two-mode squeezing parameter* of the purification. In particular, the case $\theta = 1$ corresponds to the limit in which all the modes of A and B are prepared in the vacuum state: In this case the state $\hat{\Lambda}$ no longer has maximum support and thus does not provide a proper starting point to build a CJ state. For $\theta \to \infty$, in contrast, the state $|\Psi_{\hat{\Lambda}}\rangle_{A,B}$ approaches a purification of a maximally mixed state for the modes (for details, see Ref. [22]). Equivalently, it corresponds to the limit of large squeezing in the two-mode squeezed state of the purification. Notice also that by construction, for all values of $\theta \geqslant 1$, γ satisfies the condition $\gamma \geqslant i\sigma_{A,B}$, as it indeed represents a physical pure state.

The generalized CJ state $\hat{\rho}_{A,B}(\Phi)$ for a Gaussian channel characterized by matrices Y and X as in Eq. (5) is now computed as in Eq. (14). The resulting state is still Gaussian and has the covariance matrix $\gamma' \in M(4n,\mathbb{R})$ given by

$$\gamma' = \begin{bmatrix} X^T\alpha X + Y & X^T\delta \\ \delta^T X & \beta \end{bmatrix} = \begin{bmatrix} \theta X^T X + Y & f(\theta) X^T \sigma_x \\ f(\theta)\sigma_x X & \theta \mathbb{1}_{2n} \end{bmatrix},$$

where

$$\sigma_x := \begin{bmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{bmatrix}. \tag{22}$$

In general, it will be a mixed state and we are interested in the minimum number $q_{\min}^{(\Phi)}$ of ancillary modes q that is needed to construct a Gaussian purification of it. As discussed in Appendix B, this is given by the quantity

$$\begin{aligned} q_{\min}^{(\Phi)} &= \operatorname{rank}[\gamma' - i\sigma_{A,B}] - 2n \\ &= 2n - \dim\ker[\gamma' - i\sigma_{A,B}] \end{aligned} \tag{23}$$

[note that in this case $\gamma', \sigma_{A,B} \in \mathrm{Gl}(4n \times 4n,\mathbb{R})$]. In what follows we will compute this quantity, showing that it coincides with the right-hand side of Eq. (12). To do so, we first notice that the dimension of the kernel of $\gamma' - i\sigma_{A,B}$ can be expressed as

$$\begin{aligned} &\dim\ker[\gamma' - i\sigma_{A,B}] \\ &\quad = \dim\ker \begin{bmatrix} \theta X^T X + Y - i\sigma_{2n} & f(\theta) X^T \\ f(\theta) X & \theta \mathbb{1}_{2n} + i\sigma_{2n} \end{bmatrix}, \end{aligned} \tag{24}$$

where the equality was obtained by rotating $\gamma' - i\sigma_{A,B}$ with the transformation

$$T := \begin{bmatrix} \mathbb{1}_{2n} & 0 \\ 0 & \sigma_x \end{bmatrix}. \tag{25}$$

As for any positive semidefinite matrix M, the kernel in Eq. (24) can be computed as the set of vectors $w \in \mathbb{C}^d$ which satisfy the condition $w^\dagger M w = 0$ [19]. Writing $w = (w_A, w_B)$, we arrive at the condition

$$\begin{aligned} &\theta(w_A^* X^T X w_A - w_A^* X^T w_B - w_B^* X w_A + w_B^* w_B) \\ &\quad + w_A^*(Y - i\sigma) w_A + w_B^* i\sigma w_B \\ &\quad + O(1/\theta)(w_A^* X^T w_B + w_B^* X w_A) = 0, \end{aligned} \tag{26}$$

where in the first line we have collected all terms which are linear in θ. For θ sufficiently large this implies the following conditions:

$$w_A^* X^T X w_A - w_A^* X^T w_B - w_B^* X w_A + w_B^* w_B = 0, \tag{27}$$

$$w_A^*(Y - i\sigma) w_A + w_B^* i\sigma w_B = 0. \tag{28}$$

The first equation means $X w_A = w_B$, whereas the second reads $w_A^*(Y - i\sigma) w_A + w_A^* i X^T \sigma X w_A = 0$; that is,

$$w_A^*[Y - i\Sigma] w_A = 0. \tag{29}$$

There is one-to-one correspondence between solutions w_A of Eq. (29) and $w = (w_A, X w_A)$ of Eq. (26); hence,

$$\dim\ker[\gamma' - i\sigma_{A,B}] = \dim\ker[Y - i\Sigma].$$

Substituting this into Eq. (23) we finally get

$$q_{\min}^{(\Phi)} = 2n - \dim\ker[Y - i\Sigma] = \operatorname{rank}[Y - i\Sigma], \tag{30}$$

which, thanks to Eq. (17), yields the bound

$$\ell_{\text{pure}}^{(\Phi)} \geqslant \operatorname{rank}[Y - i\Sigma]. \tag{31}$$

IV. OPTIMALITY OF THE BOUND

In this section we describe how to construct a Gaussian unitary dilation with $q_{\min}^{(\Phi)} = \operatorname{rank}[Y - i\Sigma]$ environmental modes. In this way, we demonstrate that the lower bound derived in the previous section is tight, concluding the derivation of Eq. (12). To do so, let us assume that the number of modes which define the state $\hat{\rho}_E$ in Eq. (1) is $q_{\min}^{(\Phi)}$. Without loss of generality, we write the form corresponding to the commutation relations of our $n + q_{\min}^{(\Phi)}$ modes in the block structure

$$\sigma := \sigma_{2n} \oplus \sigma^E_{2q_{\min}^{(\Phi)}} = \begin{bmatrix} \sigma_{2n} & 0 \\ 0 & \sigma^E_{2q_{\min}^{(\Phi)}} \end{bmatrix}, \tag{32}$$

where σ_{2n} and $\sigma^E_{2q_{\min}^{(\Phi)}}$ are $2n \times 2n$ and $2q_{\min}^{(\Phi)} \times 2q_{\min}^{(\Phi)}$ matrices associated with the system and environment, respectively. While σ_{2n} is defined as in Eq. (5), for $\sigma^E_{2q_{\min}^{(\Phi)}}$ we do not make any assumption at this point, leaving open the possibility of defining it later on. Accordingly, the Gaussian unitary $\hat{U}_{A,E}$ of Eq. (1) will be determined by a symplectic matrix $S \in \mathrm{Sp}[2(n + q_{\min}^{(\Phi)}),\mathbb{R}]$ of block form

$$S := \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}, \tag{33}$$

satisfying the condition $S\sigma S^T = \sigma$. In the above expressions, s_1 and s_4 are $2n \times 2n$ and $2q_{\min}^{(\Phi)} \times 2q_{\min}^{(\Phi)}$ real square matrices, while s_2 and s_3^T are $2n \times 2q_{\min}^{(\Phi)}$ real rectangular matrices. As noticed in Ref. [13], the possibility of realizing the unitary dilation (1) can now be proven by simply taking

$$s_1 = X^T, \tag{34}$$

and finding s_2 and a $q_{\min}^{(\Phi)}$-mode covariance matrix $\gamma_E \geqslant i\sigma^E_{2q_{\min}^{(\Phi)}}$ satisfying the conditions

$$s_2\, \sigma^E_{2q_{\min}^{(\Phi)}}\, s_2^T = \Sigma, \qquad s_2\, \gamma_E\, s_2^T = Y, \tag{35}$$

with γ_E being the covariance matrix of the Gaussian state $\hat{\rho}_E$ of Eq. (1). The explicit construction of the matrices s_2,γ_E is given in Appendix C.

V. CONCLUSIONS

We have analytically derived the minimum number of environmental modes necessary for a Gaussian unitary dilation of a general multimode BGC. Moreover, we have also explicitly demonstrated how to construct such a dilation in terms of the covariance matrix of the Gaussian noisy environment and the symplectic transformation associated to the unitary system-environment interaction. These results allow one to compare BGCs in terms of the corresponding noise, by using the minimum number of environmental modes to represent such channels. Moreover, constructing a dilation with a minimal number of auxiliary modes may be used to reduce the size of the corresponding complementary channel and hence to simplify the degradability analysis, which is extremely helpful in the calculation of the quantum capacity of these continuous-variable quantum maps.

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APPENDIX A: A MATRIX IDENTITY

In this Appendix, we prove the important identity (9) and the inequality (10), by the following more general lemma.

Lemma 1. Let A,B be $m\times m$ real matrices with A symmetric, B skew-symmetric, which satisfy the inequality

$$A \geqslant iB. \tag{A1}$$

Then given $A^{\ominus 1}$ the MP inverse [20] of A, the following identity holds:

$$2\,\mathrm{rank}[A - iB] = \mathrm{rank}[A] + \mathrm{rank}[A - BA^{\ominus 1}B^T]. \tag{A2}$$

Furthermore, the following inequality applies:

$$\mathrm{rank}[B] \geqslant \mathrm{rank}[A] - \mathrm{rank}[A - BA^{\ominus 1}B^T]. \tag{A3}$$

Proof. Let us start by reviewing some general properties of A and B. Because of Eq. (A1) the matrix A must be positive semidefinite, and its support must contain the support of B. Consequently, indicating with $a = \mathrm{rank}[A]$ and $b = \mathrm{rank}[B]$ the ranks of the two matrices, we must have $a \geqslant b$ with b even. Furthermore, defining $\Pi \in M(m,\mathbb{R})$ to be the projector on the support of A, it will commute with A and B and hence satisfy the following identity:

$$\Pi\, A = A\, \Pi = A, \qquad \Pi\, B = B\, \Pi = B. \tag{A4}$$

Consider then the invertible matrix

$$\bar{A} := A + (\mathbb{1}_m - \Pi) \tag{A5}$$

The MP inverse [21] of A is defined by

$$A^{\ominus 1} := \Pi \bar{A}^{-1}\Pi. \tag{A6}$$

To prove the validity of Eq. (A2) we note that it is possible to identify a congruent transformation $A \mapsto A' = CAC^T$, $B \mapsto B' = CBC^T$, with $C \in Gl(m,\mathbb{R})$ such that

$$A' = \left[\begin{array}{c|c} \mathbb{1}_a & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}\,a \\ \}\,m-a\,, \end{array} \tag{A7}$$

and

$$B' = \left[\begin{array}{c|c} \begin{array}{c|c} \begin{array}{c|c} 0 & \mu \\ \hline -\mu & 0 \end{array} & 0 \\ \hline 0 & 0 \end{array} & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}\,b/2 \\ \}\,b/2 \\ \}\,a-b \\ \}\,m-a\,, \end{array} \tag{A8}$$

with $\mu = \mathrm{diag}(\mu_1,\mu_2,\ldots,\mu_{b/2})$ being the $b/2\times b/2$ diagonal matrix formed by the strictly positive eigenvalues of $|B'|$ (by construction, they satisfy $1 \geqslant \mu_j \geqslant 0$). The matrix C can be explicitly constructed as follows. First we identify the orthogonal matrix $O \in \mathrm{Gl}(m,\mathbb{R})$ which diagonalizes A and Π puts them in the following block forms:

$$OAO^T = \left[\begin{array}{c|c} A'' & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}\,a \\ \}\,m-a\,, \end{array} \tag{A9}$$

$$O\,\Pi\,O^T = \left[\begin{array}{c|c} \mathbb{1}_a & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}\,a \\ \}\,m-a\,, \end{array} \tag{A10}$$

with $A'' \in \mathrm{Gl}(a,\mathbb{R})$ being a $a\times a$ positive definite diagonal matrix. Then we construct the invertible matrix $K \in \mathrm{Gl}(m,\mathbb{R})$ defined as

$$K = \left[\begin{array}{c|c} A''^{-1/2} & 0 \\ \hline 0 & \mathbb{1}_{m-a} \end{array}\right] \begin{array}{l} \}\,a \\ \}\,m-a\,, \end{array} \tag{A11}$$

(notice that the matrix $A''^{-1/2} \in \mathrm{Gl}(a,\mathbb{R})$ is well defined since A'' is invertible). Finally, we take $O' \in \mathrm{Gl}(a,\mathbb{R})$ to be an orthogonal $a\times a$ matrix and define C as follows:

$$C = \left[\begin{array}{c|c} O' & 0 \\ \hline 0 & \mathbb{1}_{m-a} \end{array}\right] KO = \left[\begin{array}{c|c} O'A''^{-1/2} & 0 \\ \hline 0 & \mathbb{1}_{m-a} \end{array}\right] O \tag{A12}$$

By construction we have that for all the choices of O' the resulting matrix is invertible and Eq. (A7) is satisfied. Vice versa, Eq. (A8) can be satisfied by noticing that, since the support of B is included into the support of A, we must have

$$KO\,B\,O^TK^T = \left[\begin{array}{c|c} B'' & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}\,a \\ \}\,m-a \end{array}\,, \tag{A13}$$

with $B'' \in \mathrm{Gl}(a,\mathbb{R})$ skew symmetric. By using a theorem from linear algebra one can then find an orthhogonal $O' \in \mathrm{Gl}(a,\mathbb{R})$ such that

$$O'\,B''\,O'^T = \left[\begin{array}{cc|c} 0 & \mu & \multirow{2}{*}{0} \\ -\mu & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right], \tag{A14}$$

with μ being a positive diagonal matrix of dimension equal to the rank of B'' (the elements $\pm i\mu_j$ are its not null eigenvalues). Using such an O' in order to build C as in Eq. (A12) we can then satisfy Eq. (A8).

Now we notice that, since any congruent transformation preserves the rank of a matrix, the following identity holds:

$$\begin{aligned}\mathrm{rank}[A - iB] &= \mathrm{rank}[C(A - iB)C^T] \\ &= \mathrm{rank}\left[\begin{array}{ccc|c} \mathbb{1}_{b/2} & -i\mu & 0 & \multirow{3}{*}{0} \\ i\mu & \mathbb{1}_{b/2} & & \\ \multicolumn{2}{c}{0} & \mathbb{1}_{a-b} & \\ \hline \multicolumn{3}{c|}{0} & 0 \end{array}\right] = a - \#_1(\mu)\,,\end{aligned}$$

where $\#_1(\mu)$ counts the number of eigenvalues of the matrix μ which are equal to 1. The last identity follows from counting the nonzero eigenvalue of the matrix on the left-hand side of the second line. This can be easily done by observing that its spectrum contains $m - a$ explicit zeros (these are the terms in the zero block diagonal term), $a - b$ ones (these are the ones on the diagonals of the first block), and $1 \pm \mu_j$ with $\mu_j \in [1,0]$ being the eigenvalues of μ. Consequently, the nonzero eigenvalues are obtained by subtracting from k (rank of the first block) the number $\#_1(\mu)$ of eigenvalues of μ which are equal to 1. To compute the latter quantity we note that

$$B'B'^T = \left[\begin{array}{cc|c|c} \mu^2 & 0 & \multirow{2}{*}{0} & \multirow{3}{*}{0} \\ 0 & \mu^2 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & \\ \hline \multicolumn{3}{c|}{0} & 0 \end{array}\right] \begin{array}{l} \}\,b/2 \\ \}\,b/2 \\ \}\,a-b \\ \}\,m-a\,, \end{array} \tag{A15}$$

which yields

$$\mathrm{rank}[\mathbb{1}_m - B'B'^T] = m - 2\,\#_1(\mu). \tag{A16}$$

Using the fact that C is invertible, one has

$$\begin{aligned}\mathrm{rank}[\mathbb{1}_m - B'B'^T] &= \mathrm{rank}[\mathbb{1}_m - CBC^TCB^TC^T] \\ &= \mathrm{rank}[C^{-1}C^{-T} - BC^TCB^T].\end{aligned}$$

Since O' and O are orthogonal, we notice that $C^{-1}C^{-T}$ is composed of two terms that span orthogonal supports. Specifically, we can rewrite it as

$$\begin{aligned}C^{-1}C^{-T} &= O^TKO = O^T\left[\begin{array}{c|c} A'' & 0 \\ \hline 0 & \mathbb{1}_{m-a} \end{array}\right] O \\ &= A + O^T\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_{m-a} \end{array}\right] O = A + (\mathbb{1}_m - \Pi) = \bar{A}\,,\end{aligned}$$

where Eqs. (A9) and (A11) have been used. Similarly, BC^TCB^T is only supported on the support of A. Indeed, we have

$$\begin{aligned}BC^TCB^T &= (\Pi\ B\ \Pi)\,[C^{-1}C^{-T}]^{-1}\,(\Pi\ B^T\ \Pi) \\ &= (\Pi\ B\ \Pi)\ \bar{A}^{-1}\ (\Pi\ B^T\ \Pi) \\ &= (\Pi\ B\ \Pi^2)\ \bar{A}^{-1}\ (\Pi^2\ B^T\ \Pi) \qquad\qquad \text{(A17)} \\ &= (\Pi\ B\ \Pi)(\Pi\ \bar{A}^{-1}\ \Pi)(\Pi\ B^T\ \Pi) \\ &= (\Pi\ B\ \Pi)A^{\ominus 1}\,(\Pi\ B^T\ \Pi) = B\ A^{\ominus 1}\ B^T.\end{aligned}$$

Using these identities, we can then rewrite Eq. ((A17)) as

$$\begin{aligned}\mathrm{rank}[\mathbb{1}_m - B'B'^T] &= \mathrm{rank}[\bar{A} - BA^{\ominus 1}\,B^T] \\ &= \mathrm{rank}[\mathbb{1}_m - \Pi] + \mathrm{rank}[A - BA^{\ominus 1}\,B^T] \\ &= m - a + \mathrm{rank}[A - BA^{\ominus 1}\,B^T].\end{aligned} \tag{A18}$$

Thanks to Eq. (A16), the above identity finally yields

$$\#_1(\mu) = \frac{\mathrm{rank}[A] - \mathrm{rank}[A - BA^{\ominus 1}\,B^T]}{2}, \tag{A19}$$

which gives Eq. (A2) when inserted into Eq. (A16). The inequality (A3) can finally be proven by noticing that because of the invertibility of C, one has $\mathrm{rank}[B] = \mathrm{rank}[B'] = b$ which, by construction, is larger than $2\#_1(\mu)$. The result then follows simply by applying Eq. (A19).

APPENDIX B: MINIMAL GAUSSIAN PURIFICATIONS OF GAUSSIAN STATES

Here we consider the minimal number of ancillary modes which are necessary to construct a Gaussian purification of a generic multimode Gaussian state $\hat{\rho}$. Of course, the requirement of Gaussianity of the purification is fundamental for our purposes: Without it, any finite number of modes can always be embedded into a single one due to infinite dimensionality of the underlying Hilbert spaces.

Let $\gamma \in \mathrm{Gl}(2n,\mathbb{R})$ be the covariance matrix of a Gaussian state $\hat{\rho}$ of a system A formed by n bosonic modes. We know that it must satisfy the following inequality:

$$\gamma \geqslant i\sigma_{2n}, \tag{B1}$$

with σ_{2n} being the skew-symmetric matrix in Eq. (3) representing the symplectic form of the modes. Due to the Williamson's theorem [25] there exists a symplectic transformation $S \in \mathrm{Sp}(2n,\mathbb{R})$ which allows us to diagonalize γ in the following form:

$$\gamma \mapsto S\gamma S^T = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & D \end{array}\right], \tag{B2}$$

with $D \in \mathrm{Gl}(n,\mathbb{R})$ being the diagonal matrix formed by the symplectic eigenvalues D_j of γ which satisfy the condition $D_j \geqslant 1$ as follows from Eq. (B1). The values $\{D_j\}$ are the *symplectic eigenvalues* of γ [17,24], equal to the positive square roots of the eigenvalues of the matrix $-\sigma_{2n}\gamma\sigma_{2n}\gamma \in \mathrm{Gl}(2n,\mathbb{R})$. The transformation $\gamma \mapsto S\gamma S^T$ corresponds to applying a Gaussian unitary to the state which transforms it into a product state of the n modes, in fact, a product of Gibbs states of unit harmonic oscillators. Hence, without loss of generality, we can assume that γ is of the form of the right-hand side of Eq. (B2).

Let Γ be the covariance matrix of the minimum purification of γ, viewed as being defined on a bipartite system labeled A—the original system—and B. Since the spectrum of the reduced state with respect to B is identical to the spectrum of the reduced state of A, also the symplectic spectra of the two reductions are the same. Hence, it does not restrict generality to take Γ of the form

$$\Gamma = \left[\begin{array}{c|c} \begin{array}{c|c} D & 0 \\ \hline 0 & D \end{array} & C \\ \hline C^T & \begin{array}{c|c} D & 0 \\ \hline 0 & D \end{array} \end{array}\right], \tag{B3}$$

with suitable $C \in M(2n,\mathbb{R})$ such that the symplectic spectrum of Γ consists of units only, with respect to the symplectic form (19). Now, by taking

$$C = \left[\begin{array}{c|c} 0 & \eta \\ \hline \eta & 0 \end{array}\right], \tag{B4}$$

with $\eta = \mathrm{diag}[f(D_1),\ldots,f(D_n)]$, one clearly arrives at the covariance matrix of required purification. This purification essentially involves as many modes as there are symplectic eigenvalues different from 1; those modes associated with unit symplectic eigenvalues correspond to pure Gaussian states already. Denoting the number of unit values in D by $\#_1(D)$, this purification hence involves $n - \#_1(D)$ modes. Invoking the definition of the symplectic spectrum, one finds that

$$\#_1(D) = n - \mathrm{rank}\big[\gamma - \sigma_{2n}\gamma^{-1}\sigma_{2n}^T\big]/2. \tag{B5}$$

It is also easy to see that no purification can involve fewer modes than that. Consequently,

$$q_{\min} = n - \#_1(D). \tag{B6}$$

The covariance matrix of the reduced Gaussian state of the purification with respect to B is necessarily given by the right-hand side of Eq. (B2), up to local symplectic transformations $S \in Sp(2n,\mathbb{R})$. Hence, any Gaussian purification must involve at least $n - \#_1(D)$ modes, as so many symplectic eigenvalues are different from 1. Needless to say, only Gaussian states are uniquely defined by their second moments (the covariance matrix) and the first moments (of no relevance here): If one does not require a Gaussian purification, one can always embed the purification into a single mode if the state is mixed, and with no additional mode if the state is already pure.

APPENDIX C: EXPLICIT CONSTRUCTION OF THE BGC UNITARY DILATION

A. Pure environment case

First, let us consider the case in which rank$[Y]$ is an even number. To identify valid s_2 and γ_E which solve Eq. (35), it is useful to transform Y and Σ as in Eqs. (A7) and (A8) of Appendix A (take $A = Y$, $B = \Sigma$, $m = 2n$, $a = k$, and $b = r = \mathrm{rank}[\Sigma]$). Actually, applying an extra orthogonal matrix, Y' is still like in (A7), while Σ' can be written as

$$\Sigma' := C\Sigma C^T = \left[\begin{array}{c|c|c} 0 & \begin{array}{c|c} \mu & 0 \\ \hline 0 & 0 \end{array} & 0 \\ \hline \begin{array}{c|c} -\mu & 0 \\ \hline 0 & 0 \end{array} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, (k-r)/2 \\ \}\, r/2 \\ \}\, (k-r)/2 \\ \}\, 2n-k, \end{array} \tag{C1}$$

where $C \in \mathrm{Gl}(2n,\mathbb{R})$ and $\mu = \mathrm{diag}(\mu_1,\ldots,\mu_{r/2})$ is the $r/2 \times r/2$ diagonal matrix formed by the strictly positive eigenvalues of $|\Sigma'|$ (satisfying $\mathbb{1}_{r/2} \geqslant \mu$ as in Appendix A). In the transformed frame the parameter r' of Eq. (11) acquires a clear meaning: It provides the number of eigenvalues having modulus 1 of the matrix Σ', that is,

$$r' = 2n - \mathrm{rank}[\mathbb{1}_{2n} - \Sigma'(\Sigma')^T], \tag{C2}$$

as can be easily shown by using Eq. (A18) with $A = Y$ and $B = \Sigma$ (accordingly, $r'/2$ counts also the number of the eigenvalues of μ which are equal to 1). Furthermore, introducing $s_2' := C\, s_2$ the conditions of Eqs. (35) can be equivalently written as

$$s_2'\, \sigma^E_{2q^{(\Phi)}_{\min}}\, (s_2')^T = \Sigma', \qquad s_2'\, \gamma_E\, (s_2')^T = Y'. \tag{C3}$$

The explicit expressions for γ_E and s_2' satisfying the identity (C3) are obtained in the following way. We assume the environmental symplectic form to be

$$\sigma^E_{2q^{(\Phi)}_{\min}} = \sigma_k \oplus \sigma_{k-r'}, \tag{C4}$$

where we have set $k := \mathrm{rank}[Y]$ and used the identities (9) and (30) to write $q^{(\Phi)}_{\min} = k - r'/2$. Then we can take the $2n \times 2q^{(\Phi)}_{\min}$ rectangular matrix s_2' as

$$s_2' = \left[\begin{array}{c|c} \tilde{K}^{-1} & A \\ \hline 0 & 0 \end{array}\right], \tag{C5}$$

with $\tilde{K}$ being the $k \times k$ symmetric matrix defined by

$$\tilde{K} := \left[\begin{array}{c|c} \begin{array}{c|c} \mu^{-1/2} & 0 \\ \hline 0 & \mathbb{1}_{(k-r)/2} \end{array} & 0 \\ \hline 0 & \begin{array}{c|c} \mu^{-1/2} & 0 \\ \hline 0 & \mathbb{1}_{(k-r)/2} \end{array} \end{array}\right], \tag{C6}$$

and A being a rectangular matrix $k \times (k - r')$ of the form

$$A := \left[\begin{array}{c|c} 0 & \begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & \mathbb{1}_{(k-r)/2} \end{array} \\ \hline \begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & \mathbb{1}_{(k-r)/2} \end{array} & 0 \end{array}\right] \begin{array}{l} \}\, r'/2 \\ \}\, (r-r')/2 \\ \}\, k/2 - r/2 \\ \}\, r'/2 \\ \}\, (r-r')/2 \\ \}\, k/2 - r/2 \end{array}. \tag{C7}$$

By direct substitution one can easily verify that the first condition of Eq. (C3) is indeed satisfied. Vice versa, the $q^{(\Phi)}_{\min} \times q^{(\Phi)}_{\min}$ covariance matrix γ_E can be defined as follows. First we notice that writing it in the block form

$$\gamma_E = \left[\begin{array}{c|c} \alpha & \delta \\ \hline \delta^T & \beta \end{array}\right], \tag{C8}$$

the second condition of Eq. (C3) rewrites as

$$\alpha + A\,\delta^T + \delta\,A^T + A\,\beta\,A^T = \tilde{K}^2 \tag{C9}$$

(in these expressions α, β, and δ are, respectively, $k \times k$, $(k - r') \times (k - r')$, and $k \times (k - r')$ real matrices). A solution can now be derived, for instance, by organizing the matrix elements of μ in decreasing order and then taking

$$\alpha := \left[\begin{array}{c|c} \begin{array}{c|c} \mu^{-1} & 0 \\ \hline 0 & \frac{5}{4}\mathbb{1}_{(k-r)/2} \end{array} & 0 \\ \hline 0 & \begin{array}{c|c} \mu^{-1} & 0 \\ \hline 0 & \frac{5}{4}\mathbb{1}_{(k-r)/2} \end{array} \end{array}\right],$$

$$\beta := \left[\begin{array}{c|c} \begin{array}{c|c} \mu_o^{-1} & 0 \\ \hline 0 & \frac{5}{4}\mathbb{1}_{(k-r)/2} \end{array} & 0 \\ \hline 0 & \begin{array}{c|c} \mu_o^{-1} & 0 \\ \hline 0 & \frac{5}{4}\mathbb{1}_{(k-r)/2} \end{array} \end{array}\right],$$

$$\delta := \left[\begin{array}{c|c} 0 & \begin{array}{c|c} 0 & 0 \\ \hline f(\mu_o^{-1}) & 0 \\ \hline 0 & -\frac{3}{4}\mathbb{1}_{(k-r)/2} \end{array} \\ \hline \begin{array}{c|c} 0 & 0 \\ \hline f(\mu_o^{-1}) & 0 \\ \hline 0 & -\frac{3}{4}\mathbb{1}_{(k-r)/2} \end{array} & 0 \end{array}\right],$$

with $f(\theta)$ being defined as in Eq. (21) and with μ_o being the $(r-r')/2 \times (r-r')/2$ diagonal matrix formed by the elements of μ which are strictly smaller than 1 (again, the eigenvalues being organized in decreasing order).

With the choice we made on $\sigma^E_{2q^{(\Phi)}_{\rm min}}$, the matrix α defined above is a $k \times k$ covariance matrix for a set of independent $k/2$ bosonic modes, the matrix β is a $(k-r') \times (k-r')$ covariance matrix for a set of independent $(k-r')/2$ modes, and the matrices δ and δ^T represent cross-correlation terms among such sets. Furthermore, for all diagonal matrices μ compatible with the constraint $\mathbb{1}_{r/2} \geqslant \mu$, the γ_E defined in Eq. (C8) satisfies the uncertainty relation

$$\gamma_E \geqslant i\sigma^E_{2q^{(\Phi)}_{\rm min}} \tag{C10}$$

(i.e., it is a proper covariance matrix of $q^{(\Phi)}_{\rm min}$ modes). Finally, since it has $\mathrm{Det}[\gamma_E] = 1$, it describes a minimal uncertainty state, that is, a pure Gaussian state of $q^{(\Phi)}_{\rm min}$ modes [17]. By a close inspection one realizes also that γ_E is composed of three independent pieces. The first one describes a collection of $r'/2$ vacuum states. The second one, in turn, describes $(r-r')/2$ thermal states characterized by the matrices μ_o^{-1} which have been purified by adding further $(r-r')/2$ modes. The third one, finally, reflects a collection of $k-r$ modes prepared in a pure state formed by $k/2 - r/2$ independent pairs of modes which are entangled. Let us point out again that this covariance matrix is indeed formed by $q^{(\Phi)}_{\rm min}$ modes.

The derivation can finally be extended for $k = \mathrm{rank}[Y]$ odd. In this case the solutions are provided by those defined above by replacing k with $k-1$ everywhere, while adding an extra mode initialized into the vacuum state into the definition of γ_E (i.e., replacing it with $\gamma_E \oplus \mathbb{1}_2$) and substituting s_2' of Eq. (C5) with

$$s_2' = \left[\begin{array}{c|c|c} \tilde{K}^{-1} & A & 0 \\ \hline 0 & 0 & \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \\ \hline 0 & 0 & 0 \end{array}\right] \begin{array}{l} \}\, k-1 \\ \}\, 1 \\ \}\, 1 \\ \}\, 2n-k-1\,. \end{array} \tag{C11}$$

It is also worth stressing that for the special case in which $k = 1$ one has $r = \mathrm{rank}[\Sigma] = 0$ (this is a consequence of the fact that we have $r \leqslant k$ with r even, the latter being the rank of a skew-symmetric matrix). Therefore, $\Sigma = 0$ and $q^{(\Phi)}_{\rm min} = \mathrm{rank}[Y] = 1$. In this case the above construction gives us $\gamma_E = \mathbb{1}_2$, with s_2' being a $2n \times 2$ rectangular matrix with all null entries but the one in position 1,1, which is set equal to 1.

B. Mixed environment case

For the sake of simplicity, again we will treat explicitly only the case of $k = \mathrm{rank}[Y]$ even (the analysis, however, can be easily extended to the odd case). Because of the structure of A given in Eq. (C7), the $(k-r)$ environmental modes prepared in a pure state enter explicitly in the identity in Eq. (C9): Consequently, if we wish to satisfy such relation, we cannot remove any of these modes without changing A. Vice versa, we can drop some of the auxiliary modes which were introduced only for purifying the environmental state. Since they are $(r-r')/2$, we can reduce the number of modes from $\ell^{(\Phi)}_{\rm pure}$ to

$$\ell^{(\Phi)}_{\rm mix} = \ell^{(\Phi)}_{\rm pure} - (r-r')/2 = k - r/2. \tag{C12}$$

To see this explicitly, take

$$\sigma^E_{2\ell^{(\Phi)}_{\rm mix}} = \sigma_k \oplus \sigma_{k-r}. \tag{C13}$$

The matrix s_2' can be still expressed as above but with A being a rectangular matrix $k \times (k-r)$ of the form

$$A := \left[\begin{array}{c|c} 0 & \begin{array}{c} 0 \\ \hline \mathbb{1}_{(k-r)/2} \end{array} \\ \hline \begin{array}{c} 0 \\ \hline \mathbb{1}_{(k-r)/2} \end{array} & 0 \end{array}\right]. \tag{C14}$$

Similarly, β and δ entering in the definition of γ_E become, respectively, the following $(k-r) \times (k-r)$ and $k \times (k-r)$ real matrices:

$$\beta := \left[\begin{array}{c|c} \frac{5}{4}\mathbb{1}_{(k-r)/2} & 0 \\ \hline 0 & \frac{5}{4}\mathbb{1}_{(k-r)/2} \end{array}\right], \tag{C15}$$

and

$$\delta := \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -\frac{3}{4}\mathbb{1}_{(k-r)/2} \\ \hline 0 & 0 \\ \hline -\frac{3}{4}\mathbb{1}_{(k-r)/2} & 0 \end{array}\right]. \tag{C16}$$

This covariance matrix now consists of two independent parts: The first one describes a collection of $r/2$ thermal states described by the matrices μ^{-1}. The second one reflects a collection of $k-r$ modes prepared in a pure state formed by $k/2 - r/2$ independent couples of modes which are entangled.

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