Quantum arrival and dwell times via idealized clocks

J. M. Yearsley,* D. A. Downs,† J. J. Halliwell,‡ and A. K. Hashagen§
Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom
(Received 27 June 2011; published 16 August 2011)

A number of approaches to the problem of defining arrival- and dwell-time probabilities in quantum theory makes use of idealized models of clocks. An interesting question is the extent to which the probabilities obtained in this way are related to standard semiclassical results. In this paper, we explore this question using a reasonably general clock model, solved using path-integral methods. We find that, in the weak-coupling regime, where the energy of the clock is much less than the energy of the particle it is measuring, the probability for the clock pointer can be expressed in terms of the probability current in the case of arrival times, and the dwell-time operator in the case of dwell times, the expected semiclassical results. In the regime of strong system-clock coupling, we find that the arrival-time probability is proportional to the kinetic-energy density, consistent with an earlier model involving a complex potential. We argue that, properly normalized, this may be the generically expected result in this regime. We show that these conclusions are largely independent of the form of the clock Hamiltonian.

DOI: 10.1103/PhysRevA.84.022109 PACS number(s): 03.65.Xp, 03.65.Yz, 03.65.Ta

I. INTRODUCTION

A. Opening remarks

Questions involving time in quantum theory have a rich and controversial history, and there is still much debate about their status [1–4]. While historically most attention has been focused on tunneling times, because of their relevance to atomic processes, more recently, there has been considerable interest in the problem of defining arrival and dwell times for free particles. This shift in focus reflects the gradual acceptance that the study of time observables in quantum theory is as much a foundational issue as a technical one [3]. Arrival and dwell times for a free particle are in some ways the simplest time observables one could hope to define, and studying these quantities allows one to see the difficulties common to all time observables with the minimum of extra technical complication. There are many different approaches to defining arrival- and dwell-time probability distributions. In this paper, we use a clock model to define arrival and dwell times and to compare the results with standard semiclassical expressions.

B. Arrival and dwell times

We begin by reviewing some of the standard, mainly semiclassical, formulas for arrival and dwell times. We consider a free particle described by an initial wave packet with entirely negative momenta concentrated in $x > 0$. The arrival-time probability is the probability $\Pi(t)dt$ that the particle crosses the origin in a time interval $[t,t+dt]$. A widely discussed candidate for the distribution $\Pi(t)$ is the current density [3,5,6],

$$\Pi(t) = J(t) = \frac{(-1)}{2m}\langle\psi_t|\,[\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}]\,|\psi_t\rangle = \frac{i}{2m}\left(\psi^*(0,t)\frac{\partial\psi(0,t)}{\partial x} - \frac{\partial\psi^*(0,t)}{\partial x}\psi(0,t)\right). \quad (1.1)$$

(We use units in which $\hbar = 1$ throughout.) The distribution $\Pi(t)$ is normalized to 1 when integrated over all time, but it is not necessarily positive. (This is the backflow effect [7].) Nevertheless, Eq. (1.1) has the correct semiclassical limit [6].

For arrival-time probabilities defined by measurements considered in this paper, one might expect a very different result in the regime of strong measurements, since most of the incoming wave packet will be reflected at $x = 0$. Essentially, this is the Zeno effect [8]. In a complex potential model, it was found that the arrival-time distribution in this regime is the kinetic-energy density,

$$\Pi(t) = C\,\langle\psi_t|\,\hat{p}\delta(\hat{x})\hat{p}\,|\psi_t\rangle\,, \quad (1.2)$$

where C is a constant that depends strongly on the underlying measurement model [9,10]. (See Ref. [11] for a discussion of kinetic-energy density.) However, because the majority of the incoming wave packet is reflected, it is natural to normalize this distribution by dividing by the probability that the particle is ever detected, that is,

$$\Pi_N(t) = \frac{\Pi(t)}{\int_0^\infty ds\,\Pi(s)} = \frac{1}{m|\langle p\rangle|}\langle\psi_t|\hat{p}\delta(\hat{x})\hat{p}|\psi_t\rangle, \quad (1.3)$$

where $\langle p\rangle$ is the average momentum of the initial state. This normalized probability distribution does not depend on the details of the detector. This suggests that the form, Eq. (1.3), may be the generic result in this regime, although a general argument for this is yet to be found.

The dwell-time distribution is the probability $\Pi(t)dt$ that the particle spends a time between $[t,t+dt]$ in the interval $[-L,L]$. One approach to defining this is to use the dwell-time operator,

$$\hat{T}_D = \int_{-\infty}^{\infty} dt\,\chi(\hat{x}_t), \quad (1.4)$$

where $\chi(x)$ is the characteristic function of the region $[-L,L]$ [12]. Then, the distribution $\Pi(t)$ is

$$\Pi(t) = \langle\psi_0|\,\delta(t-\hat{T}_D)\,|\psi_0\rangle\,. \quad (1.5)$$

*james.yearsley@imperial.ac.uk
†delius.downs07@imperial.ac.uk
‡j.halliwell@imperial.ac.uk
§anna-lena.hashagen07@imperial.ac.uk

In the limit $|p|L \gg 1$, where p is the momentum of the incoming state, the dwell-time operator takes the approximate form $\hat{T}_D \approx 2mL/|\hat{p}|$ so that the expected semiclassical form for the dwell-time distribution is

$$\Pi(t) = \langle\psi_0|\,\delta\left(t - \frac{2mL}{|\hat{p}|}\right)|\psi_0\rangle. \tag{1.6}$$

In practice, it is found that measurement models for both arrival and dwell times lead to distributions depending on both the initial state of the particle and the details of the clock, typically of the form

$$\Pi_C(t) = \int_{-\infty}^{\infty} ds\, R(t,s)\Pi(s), \tag{1.7}$$

where $\Pi(t)$ is one of the ideal distributions discussed above and the response function $R(t,s)$ is some function of the clock variables. (In some cases, this expression will be a convolution.) However, it is of interest to coarse grain by considering probabilities $p(t_1,t_2)$ for arrival or dwell times lying in some interval $[t_1,t_2]$. The resolution function R will have some resolution time scale associated with it, and if the interval t_2-t_1 is much larger than this time scale, we expect the dependence on R to drop out so that

$$p(t_1,t_2) = \int_{t_1}^{t_2} dt\, \Pi_C(t) \approx \int_{t_1}^{t_2} dt\, \Pi(t). \tag{1.8}$$

This is the sense in which many different models are in agreement with semiclassical formulas at coarse-grained scales.

Formulas, such as Eqs. (1.1) and (1.6) and their coarse-grained version Eq. (1.8) are not fundamental quantum mechanical expressions but postulated semiclassical formulas. However, they have the correct semiclassical limit, and any approach to defining arrival and dwell times must reduce to these forms in the appropriate regime.

C. Clock model

In this paper, we will derive arrival- and dwell-time distributions by coupling the particle to a model clock. We denote the particle variables by (x,p) and those of the clock by (y,p_y). We denote the initial states of the particle and clock by $|\psi\rangle$, $|\phi\rangle$, respectively, and the total system state by $|\Psi\rangle$. We couple this clock to the particle via the interaction $H_I = \lambda\chi(\hat{x})H_C$. Therefore, the total Hamiltonian of the system plus clock is given by

$$H = H_0 + \lambda\chi(\hat{x})H_C, \tag{1.9}$$

where H_0 is the Hamiltonian of the particle. Here, χ is the characteristic function of the region where we want our clock to run so that $\chi(x) = \theta(x)$ for the arrival-time problem and $\chi(x) = \theta(x+L)\theta(L-x)$ for the dwell-time problem. The operator,

$$H_c = H_c(\hat{y},\hat{p}_y) \tag{1.10}$$

describes the details of the dynamics of the clock, and we assume that it is such that the clock position y is the measured time. The physical situation is depicted in Fig. 1. For the moment, we will assume only that the clock Hamiltonian is self-adjoint so that it may be written in the following form:

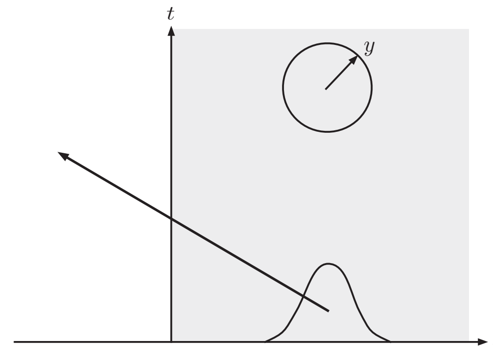

FIG. 1. The arrival-time problem defined using a model clock. The clock runs while the particle is in $x > 0$.

$$H_c = \int d\epsilon\, \epsilon\, |\epsilon\rangle\langle\epsilon|, \tag{1.11}$$

where the $|\epsilon\rangle$ form an orthonormal basis for the Hilbert space of the clock. Later on, we will restrict H_c further by considering the accuracy of the clock. We will also quote some results for the special choice of $H_c = \hat{p}_y$, whose action is to simply shift the pointer position y of the clock in proportion to time. This is the simplest and most frequently used choice for the clock Hamiltonian. The physical relevance of this and other clock models is discussed in Refs. [13,14].

Our aim, for both arrival and dwell times, is to first solve for the evolution of the combined system of particle and clock. We write this as

$$\begin{aligned}\Psi(x,y,\tau) &= \langle x,y|\,e^{-iH\tau}\,|\Psi_0\rangle \\ &= \langle x,y|\exp[-iH_0\tau - i\lambda\chi(\hat{x})H_c\tau]\,|\psi_0\rangle\,|\phi_0\rangle \\ &= \int d\epsilon\,\langle y|\epsilon\rangle\,\langle\epsilon|\phi_0\rangle \\ &\quad\times\langle x|\exp[-iH_0\tau - i\lambda\chi(\hat{x})\epsilon\tau]|\psi_0\rangle,\end{aligned} \tag{1.12}$$

and we then solve for the propagator in the integrand using path-integral methods. We will take the total time τ to be sufficiently large that the wave packet has left the region defined by $\chi(x)$. We then compute the final distribution of the pointer variable y, which is

$$\Pi(y) = \int_{-\infty}^{\infty} dx|\Psi(x,y,\tau)|^2. \tag{1.13}$$

Our main aim is to show that the predictions of the clock model, Eq. (1.13), reduce, in certain limits, to the standard forms described above.

D. Connections to earlier work

Clock models of the type, Eq. (1.9), for arrival and dwell times have been studied by numerous authors, including Peres [13], Aharanov *et al.* [15], Hartle [16], and Mayato *et al.* [14]. These studies are largely focused on the characteristics of clocks. References [13,15] are the works perhaps most closely related to the present paper. They concentrate on the case of a clock Hamiltonian linear in momentum with some elaborations on this basic model in the case of Ref. [15]. Here, we focus on a different issue not addressed by these works, namely, the dependence of the distribution, Eq. (1.13), on the initial state of the particle, for reasonably general clock Hamiltonians. In particular, we determine the extent to which the standard semiclassical forms derived above are obtained for general initial states of the particle. We also use path-integral methods to perform the calculations, in contrast to the scattering methods used in most of the previous works. Path-integral methods similar to those employed here have previously been used in Refs. [17,18] to explore the time taken to tunnel under a potential barrier, although these authors sought to define the tunneling time in terms of subsets of paths in the path integral, rather than by considering the behavior of a physical clock.

E. This paper

The rest of this paper is arranged as follows. In Sec. II, we review some path-integral techniques and, in particular, the path decomposition expansion (PDX), which we use to compute Eq. (1.12). We also introduce a useful semiclassical approximation. In Sec. III, we compute Eqs. (1.12) and (1.13) for the arrival-time problem and similarly, in Sec. IV, for the dwell-time problem. We conclude in Sec. V.

II. THE PDX AND THE SEMICLASSICAL APPROXIMATION

In this section, we discuss the PDX, which we make use of in the rest of this paper to calculate Eqs. (1.12) and (1.13). We also introduce a useful semiclassical approximation, which significantly reduces the complexity of the calculations. Throughout this section, we will focus on the arrival-time case so that $\chi(x) = \theta(x)$.

To evaluate Eq. (1.12), we need to evaluate a propagator of the form

$$g(x_1,\tau|x_0,0) = \langle x_1|\exp\{-i[H_0 + V\theta(\hat{x})]\tau\}|x_0\rangle, \quad (2.1)$$

for $x_1 < 0$ and $x_0 > 0$ (more general situations are considered in Ref. [6]). Here, V is some real number. This may be calculated using a sum over paths,

$$g(x_1,\tau|x_0,0) = \int \mathcal{D}x \, \exp(iS), \quad (2.2)$$

where

$$S = \int_0^\tau dt \left(\frac{1}{2}m\dot{x}^2 - V\theta(x)\right), \quad (2.3)$$

and the sum is over all paths $x(t)$ from $x(0) = x_0$ to $x(\tau) = x_1$.

We can simplify the analysis by splitting off the sections of the paths lying entirely in $x > 0$ or $x < 0$. The way to do this is to use the PDX [19,20]. Every path from $x_0 > 0$ to $x_1 < 0$ will typically cross $x = 0$ many times, but all paths have a first crossing, at time t, say. As a consequence of this, it is possible

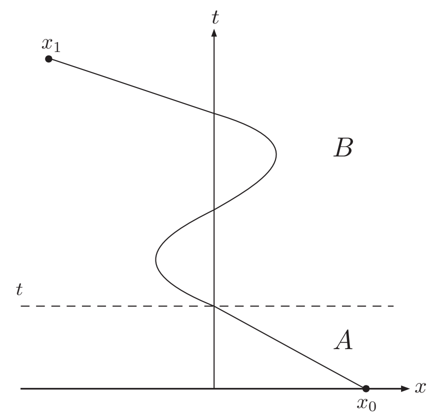

FIG. 2. The first crossing PDX: Paths from $x_0 > 0$ to $x_1 < 0$ consist of a restricted section of propagation to $x = 0$ (A), followed by unrestricted propagation along $x = 0$ and to $x_1 < 0$ (B).

to derive the formula,

$$g(x_1,\tau|x_0,0) = \frac{i}{2m}\int_0^\tau dt\, g(x_1,\tau|0,t)\frac{\partial g_r}{\partial x}(x,t|x_0,0)\bigg|_{x=0}. \quad (2.4)$$

Here, $g_r(x,t|x_0,0)$ is the restricted propagator obtained by a path integral of the form, Eq. (2.2), summed over paths lying entirely in $x > 0$. Its derivative at $x = 0$ is given by a sum over all paths in $x > 0$, which end on $x = 0$ [20]. Similar formulas may also be derived involving the last crossing time and both the first *and* last crossings times [19,20]. The PDX is depicted in Fig. 2.

Each element of these expressions can be calculated for a potential of the form $V\theta(x)$. The restricted propagator in $x > 0$ is given by the method of images expression,

$$g_r(y,\tau|x,0) = \theta(y)\theta(x)[g_f(y,\tau|x,0) - g_f(-y,\tau|x,0)]e^{iV\tau}, \quad (2.5)$$

where g_f denotes the free-particle propagator,

$$g_f(y,\tau|x,0) = \left(\frac{m}{2\pi i\tau}\right)^{1/2} \exp\left(\frac{im(y-x)^2}{2\tau}\right). \quad (2.6)$$

Note that this means that

$$\frac{\partial g_r}{\partial x}\bigg|_{x=0} = 2\,\frac{\partial g_f}{\partial x}\bigg|_{x=0}, \quad (2.7)$$

and, thus, Eq. (2.4) can be written as

$$\begin{aligned}\langle x_1|\exp\{-i[H_0 + V\theta(\hat{x})]\tau\}|x_0\rangle \\ = \frac{1}{m}\int_0^\tau dt\,\langle x_1|\exp\{-i[H_0 + V\theta(\hat{x})](\tau - t)\} \\ \times \delta(\hat{x})\hat{p}e^{-i(H_0+V)t}|x_0\rangle, \quad (2.8)\end{aligned}$$

where $\delta(\hat{x}) = |0\rangle\langle 0|$ and $|0\rangle$ denote a position eigenstate $|x\rangle$ at $x = 0$.

The propagator from $x = 0$ to $x_1 > 0$ is more difficult to calculate because it generally involves many recrossings of the origin. This propagator may be calculated exactly by using the last crossing version of the PDX [6], but it may also be approximated using a semiclassical expression, which we now describe.

The exact propagator from the origin to a point $x_1 < 0$ consists of propagation along the edge of the potential followed by restricted propagation from $x = 0$ to x_1. However, for sufficiently small V, we expect from the path-integral representation of the propagator that the dominant contribution will come from paths in the neighborhood of the straight line path from $x = 0$ to $x_1 < 0$. These paths lie almost entirely in $x < 0$, so we expect that the propagator may be approximated semiclassically by

$$\langle x_1|\exp\{-i[H_0 + V\theta(\hat{x})]t\}|0\rangle \approx \langle x_1|e^{-iH_0t}|0\rangle, \tag{2.9}$$

and, thus, Eq. (2.8) can be written as

$$\langle x_1|\exp\{-i[H_0 + V\theta(\hat{x})]\tau\}|x_0\rangle \approx \frac{1}{m}\int_0^\tau dt\, \langle x_1|e^{-iH_0(\tau-t)}\delta(\hat{x})\hat{p}e^{-i(H_0+V)t}|x_0\rangle. \tag{2.10}$$

In Ref. [6], it was shown that this semiclassical approximation holds for $E \gg V$, where E is the kinetic energy of the particle.

III. ARRIVAL-TIME DISTRIBUTION FROM AN IDEALIZED CLOCK

We now turn to the calculation of the arrival-time distribution, Eq. (1.13), recorded by our model clock. Using the PDX in the form, Eq. (2.8), the state of the system, Eq. (1.12), can be written as

$$\begin{aligned}\Psi(x,y,\tau) &= \langle x,y|\exp\{-i[H_0 + \lambda\theta(\hat{x})H_c]\tau\}|\Psi_0\rangle \\ &= \frac{1}{m}\int d\epsilon\, \langle y|\epsilon\rangle\, \langle\epsilon|\phi_0\rangle \\ &\quad\times \int_0^\tau dt\, \langle x|\exp\{-i[H_0 + \lambda\epsilon\theta(\hat{x})](\tau - t)\} \\ &\quad\times \delta(\hat{x})\hat{p}\exp[-i(H_0 + \lambda\epsilon)t]|\psi_0\rangle.\end{aligned} \tag{3.1}$$

We can simplify this expression in two different regimes, the weak-coupling regime of $E \gg \lambda\epsilon$ and the strong-coupling regime of $E \ll \lambda\epsilon$.

A. Weak-coupling regime

In the limit $E \gg \lambda\epsilon$, we can make use of the semiclassical approximation to the PDX formula, Eq. (2.10). This yields

$$\begin{aligned}\Psi(x,y,\tau) &= \frac{1}{m}\int d\epsilon\, \langle y|\epsilon\rangle\, \langle\epsilon|\phi_0\rangle \\ &\quad\times \int_0^\tau dt\, \langle x|\exp[-iH_0(\tau - t)]\delta(\hat{x})\hat{p} \\ &\quad\times \exp[-i(H_0 + \lambda\epsilon)t]|\psi_0\rangle.\end{aligned} \tag{3.2}$$

This means that the arrival-time distribution is

$$\begin{aligned}\Pi(y) = \frac{1}{m^2}\int & d\epsilon\, d\epsilon'\langle\phi_0|\epsilon'\rangle\langle\epsilon'|y\rangle\langle y|\epsilon\rangle\langle\epsilon|\phi_0\rangle \\ &\times \int_0^\tau dt\, dt'\, \langle\psi_0|\exp[i(H_0 + \lambda\epsilon')t']\hat{p}\delta(\hat{x}) \\ &\times \exp[-iH_0(t' - t)]\delta(\hat{x})\hat{p}\, \exp[-i(H_0 + \lambda\epsilon)t]|\psi_0\rangle.\end{aligned} \tag{3.3}$$

[Recall that we are assuming τ is sufficiently large so that all the wave packet is in $x < 0$ at the final time, see the discussion below Eq. (1.12).] To proceed, we first note that, for any operator $\hat{A}$, we have

$$\delta(\hat{x})\hat{A}\delta(\hat{x}) = \delta(\hat{x})\, \langle 0|\,\hat{A}|0\rangle. \tag{3.4}$$

Using this in Eq. (3.3) gives

$$\begin{aligned}\Pi(y) = \frac{1}{m^2}\int & d\epsilon\, d\epsilon'\langle\phi_0|\epsilon'\rangle\langle\epsilon'|y\rangle\langle y|\epsilon\rangle\langle\epsilon|\phi_0\rangle \\ &\times \int_0^\tau dt\, dt'\, \langle\psi_0|\exp[i(H_0 + \lambda\epsilon')t']\hat{p}\delta(\hat{x})\hat{p} \\ &\times \exp[-i(H_0 + \lambda\epsilon)t]\,|\psi_0\rangle\, \langle 0|\exp[-iH_0(t' - t)]|0\rangle.\end{aligned} \tag{3.5}$$

Here, we see the appearance of the combination $\hat{p}\delta(\hat{x})\hat{p}$, and the main challenge is to show how this turns into the current operator $\delta(\hat{x})\hat{p} + \hat{p}\delta(\hat{x})$.

Next, we rewrite the integrals using

$$\int_0^\tau dt\, dt' = \int_0^\tau dt \int_t^\tau dt' + \int_0^\tau dt' \int_{t'}^\tau dt. \tag{3.6}$$

In the first term, we set $u = t$, $v = t' - t$, and in the second, we set $u = t'$, $v = t - t'$ to obtain

$$\begin{aligned}\Pi(y) = \frac{1}{m^2}\int & d\epsilon\, d\epsilon'\langle\phi_0|\epsilon'\rangle\langle\epsilon'|y\rangle\langle y|\epsilon\rangle\langle\epsilon|\phi_0\rangle \int_0^\tau du \\ &\times \int_0^{\tau-u} dv\{\langle\psi_0|\exp[i(H_0 + \lambda\epsilon')u]\hat{p}\delta(\hat{x})\hat{p} \\ &\times\, \exp[-i(H_0 + \lambda\epsilon)(u + v)]|\psi_0\rangle\langle 0|\exp(iH_0v)|0\rangle \\ &+ \langle\psi_0|\exp[i(H_0 + \lambda\epsilon')(u + v)]\hat{p}\delta(\hat{x})\hat{p} \\ &\times \exp[-i(H_0 + \lambda\epsilon)u]|\psi_0\rangle\langle 0|\exp(-iH_0v)|0\rangle\}.\end{aligned} \tag{3.7}$$

Since we take the time τ to be large, we can extend the upper limits of the integrals to infinity. The integral over v can then be carried out to give

$$\begin{aligned}\Pi(y) &= \int d\epsilon\, d\epsilon'\langle\phi_0|\epsilon'\rangle\langle\epsilon'|y\rangle\langle y|\epsilon\rangle\langle h|\phi_0\rangle\frac{(-1)}{2m} \\ &\quad\times \int_0^\infty du\langle\psi_u|e^{i\lambda\epsilon' u}[\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}]e^{-i\lambda\epsilon u}|\psi_u\rangle, \\ \Pi(y) &= \frac{(-1)}{2m}\int_0^\infty du|\Phi(y,u)|^2\langle\psi_u|[\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}]|\psi_u\rangle \\ &= \int_0^\infty dt|\Phi(y,t)|^2 J(t),\end{aligned} \tag{3.8}$$

where

$$\Phi(y,t) = \int d\epsilon\langle y|\epsilon\rangle\langle\epsilon|\phi_0\rangle e^{-i\lambda\epsilon t} = \langle y|e^{-i\lambda H_c t}|\phi_0\rangle \tag{3.9}$$

is the wave function of the clock and $J(t)$ is the current, Eq. (1.1).

This form shows that, in the weak-coupling limit, our arrival-time probability distribution yields the current but smeared with a function depending on the clock state. We, thus, get agreement with the expected result, Eq. (1.7). Note that the physical quantity measured, the current, is not affected by the form of the clock Hamiltonian.

Although the form, Eq. (3.8), holds for a wide class of clock Hamiltonians, not all choices make for equally good clocks. To further restrict the coupling H_c, we require that different arrival times may be distinguished up to some accuracy δt. For this to be the case, we require that the clock wave functions, corresponding to different arrival times, are approximately orthogonal so that

$$\int dy\, \Phi^*(y,t')\Phi(y,t) \approx \begin{cases} 1, & \text{if } t \approx t', \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

We easily see that,

$$\begin{aligned} &\int dy\, \Phi^*(y,t')\Phi(y,t) \\ &\quad = \int d\epsilon\, d\epsilon' dy \langle\phi_0|\epsilon'\rangle\langle\epsilon'|y\rangle\langle y|\epsilon\rangle\langle\epsilon|\phi_0\rangle e^{i\lambda(t'\epsilon'-t\epsilon)} \\ &\quad = \int d\epsilon |\phi_0(\epsilon)|^2 e^{-i\lambda\epsilon\,\delta t}, \end{aligned} \tag{3.11}$$

where $\delta t = t - t'$. Clearly this expression is equal to 1 if $\delta t = 0$. Suppose now that $|\phi_0(\epsilon)|^2$ is peaked around some value ϵ_0 with width σ_ϵ. This integral will approximately vanish if

$$\lambda\sigma_\epsilon\delta t \gg 1, \tag{3.12}$$

and so the resolution of the clock is given by $1/\lambda\sigma_\epsilon$. The relationship between t and the pointer variable y will depend on the specific model. It is easily seen that a clock with good characteristics may be obtained using, for example, a free particle with a Gaussian initial state. But clocks with more general Hamiltonians can also be useful if they evolve an initial Gaussian along an approximately classical path (as many Hamiltonians do). See Refs. [13–16] for further discussion of clock characteristics.

For the special case $H_c = \hat{p}_y$, $|\epsilon\rangle = |p_y\rangle$, the expression for the arrival-time distribution simplifies, since

$$\Phi(y,t) = \int \frac{dp_y}{\sqrt{2\pi}} e^{ip_y(y-\lambda t)}\tilde{\phi}_0(p_y) = \phi_0(y-\lambda t). \tag{3.13}$$

The time is related to y by $t = y/\lambda$ and the expected form, Eq. (1.7), then becomes a simple convolution.

B. Strong-coupling regime

1. Special case: $H_c = \hat{p}_y$

We now turn to the limit of strong coupling between the particle and the clock. The analysis of the case of general clock Hamiltonian is rather subtle, so before we tackle this, we first examine the special case where the clock Hamiltonian is linear in the momentum. That is, we have,

$$H_c = \hat{p}_y = \int dp_y\, p_y |p_y\rangle\langle p_y|. \tag{3.14}$$

We start from Eq. (3.1) and insert a complete set of momentum states for the particle p to obtain

$$\begin{aligned} \Psi(x,y,\tau) = \frac{1}{m}\int \frac{dp_y dp}{\sqrt{2\pi}} &\langle y|p_y\rangle\tilde{\phi}_0(p_y) \\ &\times \exp[-i(E+\lambda p_y)\tau]p\langle p|\psi_0\rangle \int_0^\tau dt\langle x| \\ &\times \exp\{-i[H_0 - \lambda p_y\theta(-\hat{x})]t\}|0\rangle \exp(iEt), \end{aligned} \tag{3.15}$$

where $\tilde{\phi}_0(p_y)$ is the initial momentum space wave function of the clock and $E = p^2/2m$ is the kinetic energy of the particle. Note the appearance of the momentum p in the integrand. The expression involving the integral over t has been computed previously using the final crossing PDX [6,10]. In the limit $\tau \to \infty$, it is given by

$$\begin{aligned} &\int_0^\infty dt\langle x|\exp\{-i[H_0 - \lambda p_y\theta(-\hat{x})]t\}|0\rangle \exp(iEt) \\ &\quad = \sqrt{\frac{2m}{\lambda p_y}} \exp(-ix\sqrt{2m(E+\lambda p_y)}). \end{aligned}$$

We can now write our probability distribution for y. Carrying out the x integral, we obtain

$$\begin{aligned} \Pi(y) = \int &dp_y dp_y' dp\, dp' \tilde{\phi}_0^*(p_y')\tilde{\phi}_0(p_y)\langle p_y'|y\rangle\langle y|p_y\rangle \\ &\times \exp[-i\lambda(p_y - p_y')\tau]\frac{pp'}{m}\langle\psi_0|p'\rangle\langle p|\psi_0\rangle \\ &\times \exp[-i(E-E')\tau]\frac{2}{\sqrt{\lambda^2 p_y p_y'}} \\ &\times \delta(\sqrt{2m(E+\lambda p_y)} - \sqrt{2m(E'+\lambda p_y')}). \end{aligned} \tag{3.16}$$

Using the formula $\delta(f(x)) = \delta(x)/f'(0)$, we can carry out the p_y' integral to give

$$\begin{aligned} \Pi(y) \approx \int &dp_y dp\, dp' |\tilde{\phi}_0(p_y)|^2 \frac{pp'}{m^2}\langle\psi_0|p'\rangle\langle p|\psi_0\rangle \\ &\times \exp\left(-i\frac{(E-E')}{\lambda}y\right)\frac{2}{\lambda}\sqrt{\frac{2m}{\lambda p_y}} \end{aligned} \tag{3.17}$$

$$\begin{aligned} &= \int dp_y |\tilde{\phi}_0(p_y)|^2 \frac{2}{m^2}\sqrt{\frac{2m}{\lambda p_y}}\langle\psi_0| \\ &\quad \times \exp\left(iH_0\frac{y}{\lambda}\right)\hat{p}\delta(\hat{x})\hat{p}\exp\left(-iH_0\frac{y}{\lambda}\right)|\psi_0\rangle \\ &= A\langle\psi_0|\exp\left(iH_0\frac{y}{\lambda}\right)\hat{p}\delta(\hat{x})\hat{p}\exp\left(-iH_0\frac{y}{\lambda}\right)|\psi_0\rangle, \end{aligned} \tag{3.18}$$

where A is some constant whose explicit form is not required and we have used the fact that $E \ll \lambda p_y$ to approximate

$$\phi\left(p_y + \frac{E-E'}{\lambda}\right) \approx \phi(p_y). \tag{3.19}$$

Therefore, we see that, in this limit, the probability of finding the clock at a position y is given by the kinetic-energy density of the system at the time $t = y/\lambda$, in agreement with Eqs. (1.2) and (1.3).

Note that there is no response function involved in this case, as one might have expected from the general form, Eq. (1.7). (A similar feature was noted in the complex potential model of Ref. [10]). It seems likely that this is because the strong measurement prevents the particle from leaving $x > 0$ until the last moment so that the response function $R(t,s)$ is effectively a δ function concentrated around the latest time.

2. General case

As well as the approximations valid for $E \ll \lambda p_y$, the key to the analysis in the special case presented above is that the position space eigenfunction of the clock Hamiltonian with eigenvalue p_y takes the simple form

$$\langle y|p_y\rangle = \frac{1}{\sqrt{2\pi}}\exp(i y p_y). \tag{3.20}$$

This greatly simplifies the resulting calculation. For the case of a more general clock Hamiltonian, the eigenstates will not have this simple form. Instead, we perform a standard WKB approximation for the eigenstates of the clock,

$$\langle y|\epsilon\rangle = C(y,\epsilon)\exp[i S(y,\epsilon)], \tag{3.21}$$

where $S(y,\epsilon)$ is the Hamilton-Jacobi function of the clock at fixed energy. This means Eq. (3.17) becomes

$$\begin{aligned}\Pi(y) &\approx \int d\epsilon\, dp\, dp' |\langle\epsilon|\phi_0\rangle|^2 \frac{pp'}{m^2}\langle\psi_0|p'\rangle\langle p|\psi_0\rangle \\ &\quad\times\left\langle \epsilon + \frac{(E-E')}{\lambda}|y\right\rangle\langle y|\epsilon\rangle\frac{2}{\lambda}\sqrt{\frac{2m}{\lambda\epsilon}} \\ &\approx \int d\epsilon |\langle\epsilon|\phi_0\rangle|^2\frac{2}{m^2}\sqrt{\frac{2m}{\lambda\epsilon}}|C(y,\epsilon)|^2 \\ &\quad\times\langle\psi_0|\exp\left(iH_0\frac{1}{\lambda}\frac{\partial S(y,\epsilon)}{\partial\epsilon}\right)\hat{p}\delta(\hat{x})\hat{p} \\ &\quad\times\exp\left(-iH_0\frac{1}{\lambda}\frac{\partial S(y,\epsilon)}{\partial\epsilon}\right)|\psi_0\rangle, \end{aligned} \tag{3.22}$$

where we have used

$$\begin{aligned}&\left\langle \epsilon + \frac{(E-E')}{\lambda}|y\right\rangle\langle y|\epsilon\rangle \\ &\quad\approx |C(y,\epsilon)|^2\exp\left(-i\frac{(E-E')}{\lambda}\frac{\partial S(y,\epsilon)}{\partial\epsilon}\right), \end{aligned} \tag{3.23}$$

which is valid for $E - E' \ll \lambda\epsilon$.

We now suppose that the clock state is a simple Gaussian in y, or equivalently, in p_y. It follows that it will be peaked in ϵ about some value ϵ_0. This means that the integral over ϵ_0 in Eq. (3.22) may be carried out. The result for $\Pi(t)$ will again be proportional to the kinetic-energy density of the form, Eq. (1.2), where the relationship between t and the pointer variable y is defined by the equation,

$$t = \frac{1}{\lambda}\frac{\partial S(y,\epsilon_0)}{\partial\epsilon}, \tag{3.24}$$

as one might expect from Hamilton-Jacobi theory [21]. Hence, the arrival-time distribution has the expected general form, Eq. (1.2) [and, therefore, Eq. (1.3) also holds], but the precise definition of the time variable depends on the properties of the clock.

IV. DWELL TIMES

We now turn to the related issue of dwell times. Here, the aim is to measure the time spent by the particle in a given region of space, which, for simplicity, we take to be the region $[-L,L]$. This is portrayed in Fig. 3. In this section, we will work exclusively in the weak-coupling regime where $E \gg \lambda\epsilon$.

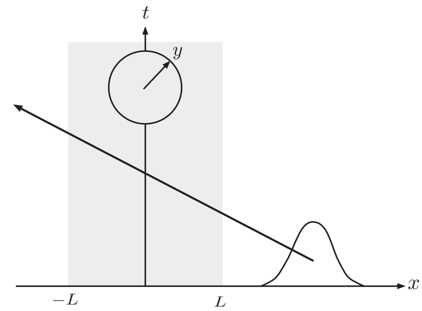

FIG. 3. The dwell-time problem, defined using a model clock. The clock runs while the particle is in the region $[-L,L]$.

The starting point is the final state of the particle plus the clock, Eq. (1.12), which we write as

$$\Psi(x,y,\tau) = \langle x,y|\exp\{-i[H_0 + \lambda H_c\chi(\hat{x})]\tau\}|\Psi_0\rangle, \tag{4.1}$$

where $\chi(\hat{x}) = \theta(\hat{x}+L)\theta(L-\hat{x})$. We wish to reexpress this using the path decomposition in a similar way to Eq. (3.1). For this case, we need a PDX, which is more general than the one used for the arrival time, since there are now crossings for two surfaces. One way to proceed is to use the path-integral expression for the first crossing of $x = L$ and $x = -L$, which is

$$\begin{aligned}\Psi(x,y,\tau) = \frac{1}{m^2}\int d\epsilon\langle\epsilon|\phi_0\rangle\langle y|\epsilon\rangle\int_0^\infty ds\int_{-\infty}^{\tau-s}dt\langle x| \\ \times\exp\{-i[H_0+\lambda\chi(\hat{x})\epsilon](\tau-s-t)\}|-L\rangle \\ \times\langle -L|\hat{p}\exp\{-i[H_0+\lambda\chi(\hat{x})\epsilon]s\}|L\rangle\langle L|\hat{p} \\ \times\exp(-iH_0t)|\psi_0\rangle, \end{aligned} \tag{4.2}$$

This is shown in Fig. 4. However, there are other choices we could make. We could consider the first crossing of $x = L$ and the last crossing of $x = -L$, for example. In the semiclassical limit, these choices lead to equivalent expressions for the dwell time. It would be interesting to explore what differences do arise in other regimes. This will be addressed elsewhere.

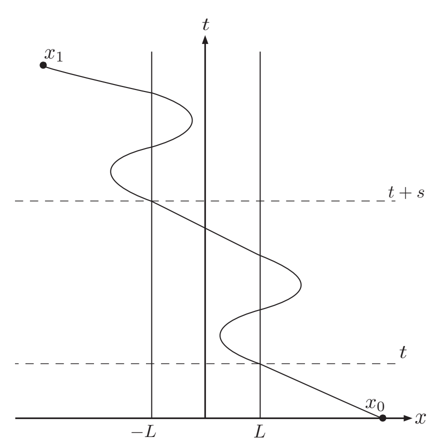

FIG. 4. The PDX used for the dwell-time problem: Paths from $x_0 > 0$ to $x_1 < 0$ have a first crossing of $x = L$ at a time t and a first crossing of $x = -L$ at a time $t + s$.

It will prove more useful to work with the wave function in position space for the clock and momentum space for the particle. Changing to this representation and making use of the semiclassical approximation, Eq. (2.10), we obtain

$$\begin{aligned}\langle p,y|\Psi_\tau\rangle \approx \frac{1}{m^2}\int d\epsilon \langle y|\epsilon\rangle\langle\epsilon|\phi_0\rangle \int_0^\infty ds \int_{-\infty}^{\tau-s} dt \langle p| \\ \times \exp[-iH_0(\tau-s-t)]|-L\rangle \\ \times \langle -L|\hat{p}\exp[-i(H_0+\lambda\epsilon)s]|L\rangle\langle L|\hat{p} \\ \times \exp(-iH_0 t)|\psi_0\rangle. \qquad (4.3)\end{aligned}$$

This is a semiclassical version of the PDX for the first crossing of $x = L$ and $x = -L$. Now, we make the standard scattering approximation of letting the upper limit of the integral over t go to infinity. This means we can carry out the t and s integrals to obtain

$$\begin{aligned}\langle p,y|\Psi_\tau\rangle \approx \int d\epsilon\, \langle y|\epsilon\rangle\, \langle\epsilon|\phi_0\rangle \exp(-iE\tau) \\ \times\, \exp(-i2Lm\lambda\epsilon/|p|)\langle p|\psi_0\rangle, \qquad (4.4)\end{aligned}$$

where we have used the standard integral [6],

$$\int_0^\infty ds\, \langle x|\exp(-iH_0 s)\hat{p}\,|0\rangle\, e^{iEs} = m\, \exp(i|x|\sqrt{2mE}), \qquad (4.5)$$

and

$$\begin{aligned}\int_{-\infty}^\infty \frac{dt}{2\pi}\, \langle x|\, \hat{p}\, \exp[-i(H_0-E)t]\,|\psi_0\rangle \\ = \langle x|\, \hat{p}\delta(H_0-E)\,|\psi_0\rangle = m\,\langle x|\,\delta(\hat{p}-p)|\psi_0\rangle. \qquad (4.6)\end{aligned}$$

Here, we have neglected the term involving $\delta(\hat{p}+p)$ since this corresponds to reflection, which is negligible in this semiclassical limit. We have also used the fact that $E \gg \lambda\epsilon$ to approximate

$$\exp(i2L\sqrt{2m(E-\lambda\epsilon)}) \approx \exp\left(i2L|p| - i2Lm\lambda\epsilon/|p|\right). \qquad (4.7)$$

Therefore, we obtain the distribution for y as

$$\begin{aligned}\Pi(y) &= \int dp\, |\langle p,y|\Psi_\tau\rangle|^2 \\ &= \int d\epsilon\, d\epsilon' \langle\phi_0|\epsilon'\rangle\langle\epsilon'|y\rangle\langle y|\epsilon\rangle\langle\epsilon|\phi_0\rangle \\ &\quad\times \int dp|\psi_0(p)|^2 \exp\left(\frac{i2Lm\lambda}{|p|}(\epsilon'-\epsilon)\right) \\ &= \int dp|\psi_0(p)|^2|\Phi(y,2Lm/|p|)|^2, \qquad (4.8)\end{aligned}$$

where

$$\begin{aligned}\Phi(y,2Lm/|p|) &= \int d\epsilon\, \langle y|\epsilon\rangle\, \langle\epsilon|\phi_0\rangle \exp\left(\frac{-i2Lm\lambda}{|p|}\epsilon\right) \\ &= \langle y|\exp\left(\frac{-i2Lm\lambda}{|p|}H_c\right)|\phi_0\rangle \qquad (4.9)\end{aligned}$$

is the clock wave function. We may rewrite this as

$$\Pi(y) = \int dt\, |\Phi(y,t)|^2\langle\psi_0|\delta\left(t-\frac{2mL}{|\hat{p}|}\right)|\psi_0\rangle. \qquad (4.10)$$

Therefore, it is of precisely the desired form, Eqs. (1.6) and (1.7), with $|\Phi(y,t)|^2$ playing the role of the response function. The discussion of clock characteristics is then exactly the same as the arrival-time case discussed in Sec. III.

V. CONCLUSION

We have studied the arrival- and dwell-time problems defined using a model clock with a reasonably general Hamiltonian. We found that, in the limit of weak-particle-clock coupling, the time of arrival probability distribution is given by the probability current density, Eq. (1.1), smeared with some function depending on the initial clock wave function, Eq. (1.7). This is expected semiclassically, agrees with previous studies, and is independent of the precise form of the clock Hamiltonian.

In the regime of strong coupling, we found that the arrival-time distribution is proportional to the kinetic-energy density of the particle, in agreement with earlier approaches using a complex potential. The fact that two very different models give the same result in this regime suggests that the form, Eq. (1.3), is the generic result in this regime, independent of the method of measurement. It would be of interest to develop a general argument to prove this. (See Ref. [9] for further discussion of this regime.)

For the case of dwell time, we have shown that the dwell-time distribution measured by our model clock may be written in terms of the dwell-time operator in semiclassical form, smeared with some convolution function, Eqs. (1.6) and (1.7).

In all of these cases, the precise form of the clock Hamiltonian and clock initial state determine the relationship between the time t and the pointer variable y, and they determine the form of the response function R in the general form, Eq. (1.7). These are particularly simple for the special case $H_C = \hat{p}_y$ explored previously. However, what is important is that, once the definition of the time variable is fixed, the clock characteristics do not effect the form of the underlying distributions—the $\Pi(s)$ in Eq. (1.7). The $\Pi(s)$ are always one of the general forms, Eqs. (1.1), (1.2), and (1.6), no matter what the clock characteristics are. This means that these general forms will always play a central role, irrespective of how they are measured.

[1] *Time in Quantum Mechanics*, edited by J. G. Muga, R. Sala Mayato, and I. L. Egusquiza (Springer, Berlin, 2002).

[2] *Time in Quantum Mechanics*, edited by J. G. Muga, A. Ruschhaupt, and A. del Campo, Vol. 2 (Springer, Berlin, 2010).

[3] J. G. Muga and C. R. Leavens, Phys. Rep. **338**, 353 (2000).

[4] G. R. Allcock, Ann. Phys. **53**, 253 (1969); **53**, 286 (1969); **53**, 311 (1969).

[5] J. G. Muga, J. P. Palao, and C. R. Leavens, Phys. Lett. A **253**, 21 (1999).

[6] J. J. Halliwell and J. M. Yearsley, Phys. Rev. A **79**, 062101 (2009); Phys. Lett. A **374**, 154 (2009).
[7] A. J. Bracken and G. F. Melloy, J. Phys. A **27**, 2197 (1994); S. P. Eveson, C. J. Fewster, and R. Verch, Ann. I. H. P. Phys. Theor. **6**, 1 (2005); M. Penz, G. Grübl, S. Kreidl, and P. Wagner, J. Phys. A **39**, 423 (2006); M. V. Berry, *ibid.* **43**, 1 (2010).
[8] B. Misra and E. C. G. Sudarshan, J. Math. Phys. **18**, 756 (1977); A. Peres, Am. J. Phys. **48**, 931 (1980).
[9] J. Echanobe, A. del Campo, and J. G. Muga, Phys. Rev. A **77**, 032112 (2008).
[10] J. J. Halliwell, Phys. Rev. A **77**, 062103 (2008).
[11] J. G. Muga, D. Seidel, and G. C. Hegerfeldt, J. Chem. Phys. **122**, 154106 (2005).
[12] J. Munoz, I. L. Egusquiza, A. del Campo, D. Seidel, and J. Gonzalo Muga, in *Time in Quantum Mechanics*, Vol. 2, Chap. 5 (Ref. [2]).
[13] A. Peres, Am. J. Phys. **48**, 552 (1980); *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, 1993).
[14] R. S. Mayato, D. Alonso, and I. L. Egusquiza, in *Time in Quantum Mechanics*, Chap. 8 (Ref. [1]).
[15] Y. Aharonov, J. Oppenheim, S. Popescu, B. Reznik, and W. G. Unruh, Phys. Rev. A **57**, 4130 (1998).
[16] J. B. Hartle, Phys. Rev. D **38**, 2985 (1988).
[17] D. Sokolovski and J. N. L. Connor, Solid State Commun. **89**, 475 (1994); Phys. Rev. A **44**, 1500 (1991).
[18] H. A. Fertig, Phys. Rev. Lett. **65**, 2321 (1990).
[19] A. Auerbach and S. Kivelson, Nucl. Phys. B **257**, 799 (1985); P. van Baal, in *Lectures on Path Integration*, edited by H. A. Cerdeira *et al.* (World Scientific, Singapore, 1993); J. J. Halliwell, Phys. Lett. A **207**, 237 (1995).
[20] J. J. Halliwell and M. E. Ortiz, Phys. Rev. D **48**, 748 (1993).
[21] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1980).