# Hartman effect and weak measurements that are not really weak

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We show that in wave packet tunneling, localization of the transmitted particle amounts to a quantum measurement of the delay it experiences in the barrier. With no external degree of freedom involved, the envelope of the wave packet plays the role of the initial pointer state. Under tunneling conditions such "self-measurement" is necessarily weak, and the Hartman effect just reflects the general tendency of weak values to diverge, as postselection in the final state becomes improbable. We also demonstrate that it is a good precision, or a 'not really weak' quantum measurement: no matter how wide the barrier $d$, it is possible to transmit a wave packet with a width $\sigma$ small compared to the observed advancement. As is the case with all weak measurements, the probability of transmission rapidly decreases with the ratio $\sigma/d$.

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## I. INTRODUCTION

One often reflects on the controversial nature of the tunneling time issue. A common feature of many approaches (for a review, see [1–3]) is that proposed tunneling times appear as mere parameters, endowed, unlike most other quantities, with neither probability amplitudes nor probability distributions. Inclusion of such time parameters into the framework of standard quantum theory is clearly desirable. One such parameter is the phase (Wigner) time used to characterize transmission of a wave packet [1–3]. In the phase time analysis, one typically proceeds by expanding the logarithm of the transmission amplitude $T(p)$ in a Taylor series around the particle's mean momentum $p_0$. Retaining only linear terms, one finds the transmitted part of a Gaussian pulse to be given by

$$\Psi^T(x,t) \approx T(p_0)\exp(ip_0\bar{y})\Psi^0(x-\bar{y},t), \qquad (1)$$

where $\Psi^0(x,t)$ is the freely propagating state, and

$$\bar{y}(p_0) \equiv iT(p_0)^{-1}\frac{\partial T(p_0)}{\partial p} \qquad (2)$$

is a complex valued quantity. If spreading of the wave packet can be neglected, Re$\bar{y}$ gives the shift of the transmitted pulse $\Psi^T(x,t)$ relative to a freely propagating one, while $T(p_0)$ and Im$\bar{y}$ describe overall reduction of its size. The term *Hartman effect* (HE) [4] refers to the fact that as the width of a (e.g., rectangular) barrier $d$ tends to infinity, the tunneled pulse is advanced roughly by the width of the classically forbidden region Re$\bar{y} \sim d$. This creates an impression that the barrier has been crossed almost infinitely fast, and using the shift to evaluate the duration spent in the region, one arrives at the phase time $\tau_{\text{phase}} = (d - \text{Re}\bar{y})m/p_0 \ll md/p_0$, which is independent of the barrier width. There is a large volume of literature on the HE (see [2–8] and references therein) as well as current interest in its experimental observations [9]. One problem in defining the HE for wave packets is that one cannot simply fix the shape of the incident pulse and increase the width of the barrier [4,5], since eventually the transmission will become dominated by the momenta passing over the barrier, for which Eq. (1) no longer holds. One can make the pulse ever narrower in the momentum space, but then there is no guarantee that the spatial width $\sigma$ of the pulse need not become much greater than the advancement one wants to detect. In this vein, the authors of Ref. [8] suggested that the HE is an artifact of the stationary formulation of the scattering theory and cannot be realized once localization of the tunneling particle is taken into account. Their conclusions appear to agree with those of Winful [2], who pointed out that in a tunneling experiment the width of the incident wave packet must exceed the size of the barrier. One can, however, imagine an optimal case, where $\sigma$ would be large enough to justify the approximation (1), yet always smaller than the expected advancement $d$. If so, one would be able to observe the advancement associated with the HE in a single tunneling event.

A similar problem arises in the seemingly different context of the so-called weak measurements [10–15]. There one measures the value of an operator $\hat{A}$ using a pointer whose initial position is uncertain, so as not to perturb the measured system. If the uncertainty is large, the mean of the meter's readings coincides with the real part of the the weak value of $\hat{A}$, $\langle A\rangle_W$. This may lie well outside the spectrum of $\hat{A}$ or even tend to infinity, and one's wish is to observe such an "unusual" value. Often the spread of the readings exceeds Re$\langle A\rangle_W$, thus requiring a large number of trials before the value can be established. If, on the other hand, the large spread can be made significantly smaller than Re$\langle A\rangle_W$, a single measurement would yield information about $\langle A\rangle_W$. The authors of [11,12] gave a possible recipe for constructing such measurements, which they described as "weak" but not "really weak." Analogy between tunneling times and weak values has been studied in [16–19]. Discussion of causality in barrier penetration can be found in [20].

The purpose of this paper is to introduce an amplitude distribution for the phase time $\tau_{\text{phase}}$ or, rather for the spacial delay associated with $\tau_{\text{phase}}$, to demonstrate that locating the transmitted particle amounts to a weak measurement of this delay, and prove that this measurement is of the "not really weak" kind mentioned above. The rest of the paper is organized as follows. In Secs. II and III, we establish formal equivalence between wave packet transmission and quantum measurements. In Sec. IV, we use the analogy to analyze a weak measurement in the limit where the weak value tends to infinity. In Sec. V, we consider a special case where such a measurement is not "really weak." In Sec. VI, we apply the analysis to the Hartman effect in tunneling. Section VII contains our conclusions.

## II. WAVE-PACKET TRANSMISSION AS A QUANTUM MEASUREMENT

Consider a one-dimensional wave packet with a mean momentum $p_0$ incident from the left on a short-range potential $W(x)$. Its transmitted part is given by (we put $\hbar = 1$)

$$\Psi^T(x,t) = \int T(p)C(p)\exp[ipx - i\epsilon(p)t]dp, \quad (3)$$

where $p$ is the momentum, $T(p)$ is the transmission amplitude, $C(p)$ is the momentum distribution of the initial pulse, and the energy $\epsilon(p)$ is $p^2/2m$ for massive nonrelativistic particles or $cp$ ($c$ is the speed of light) for the photons. The freely propagating $[W(x) = 0, T(p) = 1]$ state is given by

$$\Psi^0(x,t) = \int C(p)\exp[ipx - i\epsilon(p)t]dp. \quad (4)$$

Writing $T(p)$ as a Fourier integral,

$$T(p) = \int \xi(y)\exp(-iyp)dy, \quad (5)$$

we note that upon transmission, an incident plane wave $\exp(ip_0x)$ becomes a superposition of plane waves with various spacial shifts, $T(p_0)\exp(ip_0x) = \int dy\xi(y)\exp[ip_0(x - y)]dy$. Thus, the value of the spacial shift (delay relative to free propagation) with which a particle with a momentum $p_0$ emerges from a barrier is indeterminate unless the superposition is destroyed. Such a destruction can be achieved by employing a wave packet with a finite spacial width. Inserting Eq. (5) into (3), we write down the transmitted pulse as a superposition of freely propagating states with all possible spacial shifts $y$:

$$\Psi^T(x,t) = \int \xi(y)\Psi^0(x - y,t)dy. \quad (6)$$

If the potential $W(x)$ is a barrier, and, therefore, does not support bound states, the causality principle requires $\xi(y)$ must vanish for all $y > 0$ [20]. Thus, the Fourier spectrum of $T(p)$ (5) contains no positive shifts, and in Eq. (6), there are no terms advanced with respect to free propagation.

In the following, we will consider an incident pulse with a Gaussian envelope of a width $\sigma$ and a mean momentum $p_0$, $2/\sigma << p_0$, located at $t = 0$ far to the left of the barrier at some $x_0 < 0$. Thus, in Eq. (3), we have

$$C(p) = \sigma^{1/2}/(2\pi)^{3/4}\exp[-(p - p_0)^2\sigma^2/4 - i(p - p_0)x_0], \quad (7)$$

and the free state in Eq. (4) takes the form

$$\Psi^0(x,t) = \exp[ip_0x - i\epsilon(p_0)t]G^0(x,t), \quad (8)$$

where $G^0(x,t)$ is a time dependent envelope, whose explicit form is given in the Appendix A. Finally, with the help of (8) and (6), we rewrite Eq. (3) as

$$\Psi^T(x,t) = \exp[ip_0x - i\epsilon(p_0)t]\int \eta(y,p_0)G^0(x - y,t)dy,$$
$$\eta(y,p_0) = \exp(-ip_0y)\xi(y). \quad (9)$$

Next, we demonstrate that by finding the transmitted particle at a point $x$, we do, in fact, perform a quantum measurement of the shift $y$ for a particle with the momentum $p_0$. In order to do so, we compare Eq. (9) with the one describing a von Neumann quantum measurement on a pre- and post-selected system.

## III. QUANTUM MEASUREMENT AS TRANSMISSION

Consider next a freely moving pointer with a position $x$ and an energy $\epsilon(p) = p^2/2m$. The pointer is prepared in a Gaussian state (7) so that its free evolution is described by Eq. (8). At a time $t = t_i$, the pointer is briefly coupled to a quantum system which is, at that time, in some state $|\psi_I\rangle$. Our aim is to measure the system's variable represented by an operator $\hat{A}$, so that (neglecting for simplicity the system's Hamiltonian) we write the total Hamiltonian as

$$\mathcal{H}(t) = -i\partial_x\hat{A}\delta(t - t_i) - (2m)^{-1}\partial_x^2, \quad (10)$$

where $\delta(z)$ is the Dirac delta. After a brief interaction at $t \approx t_i$, the pointer becomes entangled with the system [21], and at some $t > t_i$, the meter is read, i.e., the pointer position is accurately determined. Taking into account the pointer's free evolution, for the state of the composite system $|\Psi(t)\rangle$, we have

$$\langle x|\Phi(t)\rangle = \sum_n \langle n|\psi_I\rangle\Psi^0(x - A_n,t_i)|n\rangle, \quad (11)$$

where $A_n$ and $|n\rangle$ are the eigenvalues and eigenstates of the operator $\hat{A}$, $\hat{A}|n\rangle = A_n|n\rangle$. Postselecting the system in some final state $|\psi_F\rangle = \sum_n \langle n|\psi_F\rangle|n\rangle$ purifies the state of the meter, which then becomes

$$\Psi^{F\leftarrow I}(x,t) = \exp[ip_0x - i\epsilon(p_0)t] \times \int \eta^{F\leftarrow I}(y,p_0)G^0(x - y,t)dy, \quad (12)$$

with

$$\eta^{F\leftarrow I}(y,p_0) \equiv \exp(-ip_0y) \times \sum_n \langle\psi_F|n\rangle\langle n|\psi_I\rangle\delta(y - A_n), \quad (13)$$

which has the same form as (9)

Defining a state-dependent "transmission amplitude"

$$T^{F\leftarrow I}(p) \equiv \int \eta^{F\leftarrow I}(y,p_0)\exp[-i(p - p_0)y]\,dy = \sum_n \langle\psi_F|n\rangle\langle n|\psi_I\rangle\exp(-iA_np) \quad (14)$$

allows us rewrite Eq. (12) also in a form similar to Eq. (3):

$$\Psi^{F\leftarrow I}(x,t) = \int T^{F\leftarrow I}(p)C(p)\exp[ipx - i\epsilon(p)t]\,dp, \quad (15)$$

where $C(p)$ is given in Eq. (7). The Fourier series of $T^{F\leftarrow I}(p)$ (14) only contains frequencies $A_n$ from the spectrum of $\hat{A}$ and, since $T^{F\leftarrow I}(p)$ is a transition amplitude for the Hamiltonian (10), we have

$$|T^{F\leftarrow I}(p)| \leqslant 1. \tag{16}$$

Both representations, [(3),(15)] and [(9),(12)], are useful. Equations (12) and (9) highlight the nature of the measured quantity and the accuracy of the measurement. In particular, using a von Neumann measurement, the meter determines the value of $\hat{A}$ to accuracy $\sigma$. If the system is postselected in $|\psi_F\rangle$ and no meter is employed, possible values of $A$, $A_n$, are distributed with probability amplitudes $\langle\psi_F|n\rangle\langle n|\psi_I\rangle$, and the exact value of $A$ remains indeterminate. With the meter switched on, only those values of $A$ which fit under the Gaussian $G^0$ centered at the observed value $x$ contribute to the amplitude $\Psi^{F\leftarrow I}(x,t)$. Thus, finding the pointer at $x$ guarantees that $A$ has the value roughly in the interval $[x-\sigma,x+\sigma]$. Similarly, in Eq. (9), finding the tunneling particle at a location $x$ determines to accuracy $\sigma$ the delay $y$. Again, for a plane wave with a momentum $p_0$, the value of $y$ is indeterminate, its amplitude distribution is $\eta(y,p_0)$, and only those values of $y$ which fit under $G^0$ centered at $x$ contribute to $\Psi^T(x,t)$ in Eq. (9).

For their part, Eqs. (3) and (15) show that both wave-packet transmission and a quantum measurement explore local behavior of the correspondening transmission amplitude, $T(p)$ or $T^{F\leftarrow I}(p)$, in a region of the width $\sigma_p = 2/\sigma$ around $p_0$. They are, therefore, convenient for studying the limit in which the momentum width of the initial Gaussian $\sigma_p$ becomes small, i.e., the case of a nearly monochromatic initial pulse or an initial meter state broad in coordinate space. More discussion of the attributes of the measurement formalism can be found in the Appendix B, and in the next section, we consider such inaccurate or weak quantum measurements.

## IV. WEAK QUANTUM MEASUREMENT

If the momentum width of the initial meter's state $\sigma_p$ is small, expanding $\ln T^{F\leftarrow I}(p)$ around $p = p_0$, we arrive at an analog of Eq. (1):

$$\Psi^{F\leftarrow I}(x,t) \approx T^{F\leftarrow I}(p_0)\exp[i\bar{A}p_0]\Psi^0(x-\bar{A}(p_0),t), \tag{17}$$

where

$$\bar{A}(0) = \bar{A}_1 + i\bar{A}_2 \equiv \frac{i}{T^{F\leftarrow I}(p_0)}\frac{\partial T^{F\leftarrow I}(p_0)}{\partial p} = \int y\eta^{F\leftarrow I}(y,p_0)dy\Big/\int \eta^{F\leftarrow I}(y,p_0)dy. \tag{18}$$

The second equality in (18) defines $\bar{A}(p_0)$ as an "improper" average [22] calculated with the amplitude distribution $\eta^{F\leftarrow I}(y,p_0)$. For $p_0 = 0$, the complex valued $\bar{A}(p_0)$ coincides with the "weak value" of $A$, $\langle A\rangle_W$, introduced in [11]:

$$\bar{A}(0) = \sum_n A_n\langle\psi_F|n\rangle\langle n|\psi_I\rangle/\langle\psi_F|\psi_I\rangle = \langle A\rangle_W. \tag{19}$$

Thus, if approximation (17) holds, the final state of the meter is a reduced copy of its freely propagating state $\Psi^0$ translated into the complex coordinate plane by $\bar{A}(p_0)$. A von Neumann measurement typically employs a heavy pointer at rest, prepared in a the Gaussian state (8) centered at the origin:

$$m \to \infty, \quad p_0 = 0, \quad x_0 = 0, \tag{20}$$

which will be assumed throughout the rest of this section. With (20), the Gaussian pointer state (17) becomes

$$\Psi^{F\leftarrow I}(x,t) \approx K\exp(2i\bar{A}_2x/\sigma^2)\exp[-(x-\bar{A}_1)^2/\sigma^2],$$
$$K \equiv T^{F\leftarrow I}(0)(2/\pi\sigma^2)^{1/4}\exp[(\bar{A}_2^2 - 2i\bar{A}_1\bar{A}_2)/\sigma^2], \tag{21}$$

so that the complex translation results in a real coordinate shift $\mathrm{Re}\bar{A}$ and a momentum "kick" of $2\bar{A}_2/\sigma^2$. This is a known result (see, for example, Ref. [13]), and from it we proceed to the main question of this section.

It is well known [10–12] that weak values can exhibit unusual properties. For example, $\langle A\rangle_W$ could be arbitrarily large for initial and final states that are nearly orthogonal, $\langle\psi_F|\psi_I\rangle \approx 0$, even though the spectrum of $\hat{A}$ is bounded. We ask next whether such large shifts can, in principle, be observed with a pointer state whose width $\sigma$ is less than $\mathrm{Re}\langle A\rangle_W$, so that the uncertainty in the final pointer position is smaller than the mean measured value? We note that one can always justify the approximation (17) by making the pointer state narrow in the momentum space, i.e., by sending $\sigma \to \infty$, but there is no guarantee that the spread in the meter reading will not exceed $\mathrm{Re}\langle A\rangle_W$, however large it may be.

As a simple example, consider the case where one measures the $z$ component of a spin 1/2, $\hat{A} = \hat{\sigma}_z$, pre- and post-selected in the states $|\psi_I\rangle = (|\uparrow\rangle + |\downarrow\rangle)/2^{1/2}$ and $|\psi_F\rangle = [|\uparrow\rangle - (1-d^{-1}|\downarrow\rangle)]/N^{1/2}$, $N(d) \equiv 1 + (1-d^{-1})$. Here the parameter $d$ controls the overlap $\langle\psi_F|\psi_I\rangle$, so that as $d \to \infty$, we have $\langle\psi_F|\psi_I\rangle \to 0$. With the help of Eqs. (14) and (18), we easily find $T^{F\leftarrow I}(p) = [2i\sin(p) - \exp(ip)/d]/(2N)^{1/2}$ and $\bar{A} = 2d - 1$. Expanding the logarithm of the transmission amplitude in a Taylor series around $p = 0$, $\ln T^{F\leftarrow I}(p) = \sum_{n=0}(n!)^{-1}\partial^n \ln T^{F\leftarrow I}(0)/\partial p^n p^n$, we note that as $d \to \infty$, $\partial^n \ln T^{F\leftarrow I}(0)/\partial p^n \to d^n$. The range of $p$'s contributing to the integral (15) is determined by the momentum width of the initial state, $\sigma_p$, so that we have $p \lesssim 1/\sigma$. Thus, we can truncate the above Taylor series and, therefore, satisfy the approximation (17), only if $d/\sigma << 1$. Consequently, no matter how large the weak value, the coordinate width of the initial pointer state must be even larger. This weak measurement is, in terms of Ref. [12], really weak.

## V. WEAK QUANTUM MEASUREMENT WHICH IS "NOT REALLY WEAK"

A "not really weak" measurement can be realized for a system whose Hilbert space has sufficiently many dimensions as follows. One can choose initial $|I\rangle$ and final $|F\rangle$ states of the system in such a way that in some vicinity of $p = 0$, the transmission amplitude can be approximated as [23]

$$T^{F\leftarrow I}(p) \approx B\exp[-iF(p)d], \quad B = \mathrm{const},$$
$$F(p) = F_1(p) + iF_2(p), \quad F_2(0) < 0, \tag{22}$$

where $d$ is, as before, a large parameter and $F^{(n)} \equiv \partial^n F/\partial p^n$, $n = 0,1,2,\ldots$, are all of order of unity. As in the first example, the weak value of $\hat{A}$ tends to infinity as $d \to \infty$:

$$\langle A\rangle_W = d[F_1'(0) + iF_2'(0)]. \quad (23)$$

In Eq. (15), we have $p \lesssim 1/\sigma$, and so may choose

$$\sigma = \gamma d^{1/2+\epsilon/2}, \quad \gamma < 1 = \text{const.}, \quad 0 < \epsilon \leqslant 1, \quad (24)$$

so that the width $\sigma$, although large for a large $d$, is always smaller than $\text{Re}\langle A\rangle_W = F_1'(0)d$. (For $\epsilon = 1$, we have $\sigma/\text{Re}\bar{A} = \gamma$; otherwise, $\lim_{d\to\infty} \sigma/\text{Re}\bar{A} = 0$.) Returning to the Taylor expansion of $\ln T^{F\leftarrow I}(p)$, $-iF(p)d = -id\sum_{n=0}(n!)^{-1}F^{(n)}(0)p^n$, we note that while the first two terms are proportional to $d$ and $d^{1/2-\epsilon/2}$, respectively, the higher order terms behave as $d^{1-n(\epsilon+1)/2}$, $n = 2,3,\ldots$, and can, therefore, be neglected as $d \to \infty$. With this, the Gaussian meter state (21) becomes

$$\begin{aligned}
&\lim_{d\to\infty}\Psi^{F\leftarrow I}(x,t)\\
&\quad\approx K\exp[2ixF_2'(0)/(\gamma^2 d^\epsilon)]\\
&\qquad\times\exp\{-[x - F_1'(0)d]^2/\gamma^2 d^{1+\epsilon}\}, \quad (25)\\
&K = B[2/(\pi\gamma^2 d^{1+\epsilon})]^{1/4}\exp\{-iF(0)d\\
&\qquad + d^{1-\epsilon}[2iF_1'(0)F_2'(0) + \bar{F}_2'(0)^2]/\gamma^2\}.
\end{aligned}$$

In Eq. (25), we have, apart from a constant and a phase factor, a reduced copy of the original Gaussian shifted by a distance exceeding its width. This weak measurement is, therefore, not really weak.

Finally, to demonstrate that as $d \to \infty$, $\Psi^{F\leftarrow I}$ does indeed build up from the momenta in an ever narrower vicinity of $p = 0$, it is helpful to evaluate the contribution to the integral (15) from the tail of the momentum distribution $A(p)$, i.e., from $p$'s greater then some fixed $p_{\min}$:

$$\begin{aligned}
I &\equiv \left|\int_{p_{\min}}^{\infty}\exp(-p^2\sigma^2/4)T^{F\leftarrow I}(p)\exp(ipx)dp\right|\\
&\leqslant \int_{p_{\min}}^{\infty}\exp(-p^2\sigma^2/4)|T^{F\leftarrow I}(p)|dp \quad (26)\\
&\leqslant \int_{p_{\min}}^{\infty}\exp(-p^2\sigma^2/4)dp = \pi^{1/2}\sigma^{-1}\text{erfc}(\sigma p_{\min}/2),
\end{aligned}$$

where erfc$(z)$ is the complementary error function, and we have used Eq. (16) in going from the second inequality to the third. Using the large argument asymptotic of erfc$(z)$ shows that as $d \to \infty$,

$$I \sim (\gamma^2 p_{\min} d^{1+\epsilon})^{-1}\exp(-d^{1+\epsilon}\gamma^2 p_{\min}^2/4). \quad (27)$$

For any $0 < \epsilon \leqslant 1$, and $d \to \infty$, $I$ can, therefore, be neglected in comparison with $\Psi^{F\leftarrow I}(x) \sim d^{-(1+\epsilon)/4}\exp[-|F_2(0)|d]$, which proves the above point.

Rather than proceed with the construction of a weak von Neumann measurement corresponding to (22), we continue with the analysis of tunneling across the potential barrier. Equivalence between the two cases has been demonstrated in Secs. II and III.

## VI. HARTMAN EFFECT WITH WAVE PACKETS

Consider the tunneling of a Gaussian wave packet (8) representing a nonrelativistic particle of unit mass $m = 1$ across a rectangular potential barrier of a height $W$ and width $d$. The frequently quoted transmission amplitude is given by

$$T(p) = \frac{4ip\kappa\exp(-ipd)}{(p+i\kappa)^2\exp(\kappa d) - (p-i\kappa)^2\exp(-\kappa d)}, \quad (28)$$

where $\kappa = (2W - p^2)^{1/2}$. For a fixed initial momentum $p_0$ and a height $W$, $p_0^2/2m < W$, we increase the barrier width $d$ in order observe the advancement of the transmitted pulse relative to free propagation. As in the case of a not really weak quantum measurement, we wish to find a condition where the advancement exceeds the wave-packet width by as much as possible. As $d \to \infty$, and for $p \approx p_0$, we have

$$\begin{aligned}
T(p) &\approx \frac{4ip_0\kappa_0}{(p_0+i\kappa_0)^2}\exp[-i(p-i\kappa)d],\\
\kappa_0 &= \left(2W - p_0^2\right)^{1/2},
\end{aligned} \quad (29)$$

which clearly has the form (22) with $F(p) = [p - i\kappa(p)]$, and we can use Eq. (24) to define the wave-packet width $\sigma$ in such a way that it increases with the barrier width $d$ while always remaining smaller than $d$. The only difference with the case analyzed in Sec. V is that now the tunneling particle plays the role of a pointer whose mass $m = 1$ and mean momentum $p_0$ are both finite. In particular, in the language of measurement theory, Eq. (2) which now reads,

$$\bar{y}(p_0) \sim d + ip_0 d/\kappa_0, \quad (30)$$

gives the weak value of the shift relative to free propagation, experienced by a tunneling particle with momentum $p_0$.

The weak value is of an unusual kind mentioned above: $\text{Re}\bar{y}(p_0) \sim d$ lies far beyond the range $-\infty < y \leqslant 0$ allowed by causality [20].

We note further that in Eq. (17), spreading of the wave packet $\Psi^0(x,t)$ can be neglected, since spreading results in replacing initial width $\sigma$ by a complex time dependent width $\sigma_t^2 \equiv (\sigma^2 + 2it)$ [cf. Eq. (A2)]. We wish to compare positions of the freely propagating and tunneling pulse roughly at the time it takes the free particle to cross the barrier region, i.e., at $t \sim d/p_0$. Thus, as $d \to \infty$, we have $\sigma_t^2 \approx \gamma^2 d^{1+\epsilon}[1 + 2i/(d^\epsilon p_0)] \sim \gamma^2 d^{1+\epsilon} = \sigma^2$. Equation (17) for the transmitted Gaussian wave packet now reads

$$\begin{aligned}
\Psi^T(x,t) &\approx K\exp[ip_0x - i\epsilon(p_0)t]\exp[2ip_0X/(\kappa_0 d^\epsilon\gamma^2)]\\
&\quad\times\exp[-(X-d)^2/(\gamma^2 d^{1+\epsilon})], \quad (31)\\
K &= T(p_0)[2/(\pi\gamma^2 d^{1+\epsilon})]^{1/4}\\
&\quad\times\exp\left[\left(p_0^2/\kappa_0^2 - 2ip_0/\kappa_0\right)d^{1-\epsilon}/\gamma^2\right],
\end{aligned}$$

where

$$X(x,p_0,x_0,t) \equiv x - p_0t - x_0 \quad (32)$$

is the particle's position relative to the center of the freely propagating pulse. It is readily seen that the peak of the transmitted density, $\rho^T(x,t) \equiv |\psi^T(x,t)|^2 \approx |K|^2\exp[-2(X-d)^2/(\gamma^2 d^{1+\epsilon})]$ is advanced, as desired, by a distance $\sim d$ exceeding its width $\sim\gamma d^{(1+\epsilon)/2}/\sqrt{2}$. Note that

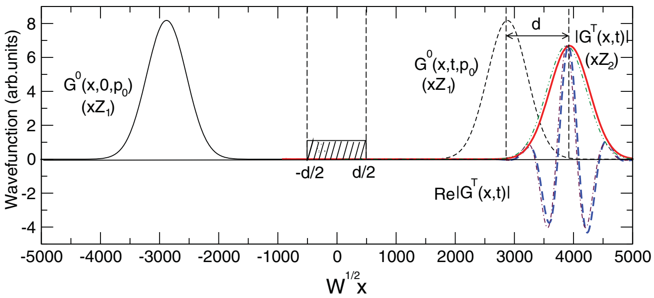


FIG. 1. (Color online) Modulus (thick solid) and real part (thick dashed) of the transmitted pulse, $G^T(x,t) \equiv \exp[-ip_0x - i\epsilon(p_0)t]\psi^T(x,t)$, multiplied by a (large) factor $Z_2 = \exp[i(p_0 - ik_0)d]$. The same quantities evaluated using Eq. (31) are shown by the dotted-dotted- and dotted-dashed lines, respectively. Also shown are the initial (solid) and final (dashed) free envelopes (A2) multiplied by a factor $Z_1 = 200$. Other parameters are $W^{1/2}d = 10^3$, $\epsilon = 0.85$, $W^{1/2-\epsilon}\gamma = 0.8$, $W^{-1/2}p_0 = 1$, and $Wt = W(d + 10\sigma)/p_0 = 5765$.

Eqs. (26) and (27) guarantee that the contributions from the momenta passing above the barrier, $p > p_{\min} = \sqrt{2W}$, are negligible and the transmission is always dominated by tunneling, rather than by the momenta passing above the barrier.

The advancement mechanism relies on the wave packet exploring local behavior of the transmission amplitude $T(p)$ which oscillates around $p = p_0$ with the period $\tau_p = 2\pi/\mathrm{Re}\bar{y}(p_0) = 2\pi/d$. The number of times $T(p)$ must oscillate within the momentum width of the pulse $\sigma_p = 2/\sigma$ in order to ensure a given ratio of the spacial width of the pulse to the observed advancement, $\nu = \sigma/d$, is given by

$$n_{\mathrm{osc}} \equiv \sigma_p/\tau_p = 1/\pi\nu \approx d^{(1-\epsilon)/2}/\pi\gamma. \tag{33}$$

The cost of advancement (31) in terms of the tunneling probability $P^T$ for a particle with an energy equal to half of the barrier height, $P^T \equiv \int |\psi^T(x,t)|^2 dx \sim |T(p_0)|^2 \sim \exp(-2W^{1/2}d)$, can be estimated by recalling that the approximation (1) requires $|2d\sigma^{-2}F''(p_0)| = 2d\sigma^{-2}\partial^2\kappa(p_0)/\partial p^2 = 4(d/\sigma)^2/(W^{1/2}d) \ll 1$. Thus, for a bound on $P^T$, given the value of the ratio $\nu$, we have

$$P^T(\nu) \ll \exp(-8/\nu^2). \tag{34}$$

We conclude by giving in Fig. 1 a comparison between the exact (3) and approximate (31) forms of the tunneled wave function for a broad, $W^{1/2}d \gg 1$, rectangular potential barrier.

## VII. CONCLUSIONS

In summary, spacial delay in transmission is conveniently analyzed in terms of quantum measurement theory. Classically, a particle with a momentum $p_0$ passing over a potential barrier experiences a unique delay $y$ (an advancement if $y > 0$) relative to the free propagation. This delay can, if one wishes, be used to determine the duration $\tau$ the particle has spent in the barrier region. Quantally, there is no unique spacial delay for tunneling with a momentum $p_0$, but rather a continuous spectrum of possible delays, which for a barrier extends from $-\infty$ to 0, $-\infty < y \leqslant 0$. Finding the particle with a mean momentum $p_0$ at a location $x$, one effectively performs a quantum measurement of the delay $y$. This is evident from comparing the state for the transmitted pulse (9) and the final state of the pointer in a von Neumann measurement with postselection (12). The physical conditions are clearly different. A von Neumann measurement requires coupling to an additional (pointer) degree of freedom, while a tunneling particle "measures itself," with the role of the initial meter state played by the envelope of the wave packet, superimposed on a plane wave with momentum $p_0$. Yet a further analogy is valid. Just as the width of the initial pointer state determines the accuracy of a von Neumann measurement, the spacial width of the incident wave packet $\sigma$ determines the uncertainty with which one can know the delay. For a large $\sigma$, the measurement is weak and is, therefore, capable of producing unusual results outside the spectrum of available delays. Spacial advancement of the transmitted pulse, while causality limits the spectrum of delays to $y \leqslant 0$, is just another example of such an unusual value. Quantally, there is no reason for converting it into an estimate for the duration spent in the barrier or subbarrier velocity. Yet even if such a conversion is made, a tunneling electron can be said to "travel at a speed greater than $c$" no more than a spin 1/2 for which a weak measurement of the $z$ component finds a value of 100 [10] can be said "to be a spin 100."

Mathematically, the Hartman effect reflects a general property of the weak values which become infinite as the probability to reach the final (in this case, transmitted) state vanishes [cf. Eq. (19)]. In Section VI, we have demonstrated that the measurement of the delay as the barrier width $d \to \infty$ is not just weak, but also belongs to the class of good precision measurements which "are not really weak." Contrary to the suggestions of Refs. [2,8], if the barrier is broad, one can always find a wave packet with a width large but smaller than $d$, which would tunnel and exhibit an advancement by approximately the barrier width. This is a consequence of the exponential behavior of the transmission amplitude in Eq. (29) and some properties specific to a Gaussian wave packet. We note also that in our estimate the width $\sigma$ must increase with $d$ at a rate no slower than $\sim d^{1/2}$. This agrees with the findings of Ref. [5], whose authors analyzed the Hartman effect using flux-based arrival times. It also agrees qualitatively with the best relative uncertainty of above $1/N^{1/2}$, achievable in a weak measurement on a system consisting of a large number $N \gg 1$ of spins 1/2 [11,12]. Thus, just as in the case of a good precision weak measurement, a single tunneling event may suffice to observe the Hartman effect. One would also have to wait a long time for that single event, as the tunneling probability rapidly decreases with the decrease of the ratio $\sigma/d$. We will follow the authors of [12] in assuming that whoever might perform the experiment is sufficiently patient and has time on his/her hands.


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## APPENDIX A: SOME GAUSSIAN INTEGRALS

Evaluating Gaussian integral (4) with $C(p)$ given by Eq. (7) yields

$$G^0(x,t,p_0) = \left[2\sigma^2/\pi\sigma_t^4\right]^{1/4} \exp\left[-(x-p_0t-x_0)^2/\sigma_t^2\right], \tag{A1}$$

where for a particle of a unit mass $\sigma_t^2 \equiv (\sigma^2 + 2it)$ and $\epsilon(p) = p^2/2$. For a photon, $\epsilon(p) = cp$, we have

$$G^0(x,t,p_0) = [2/\pi\sigma^2]^{1/4} \exp[-(x-ct-x_0)^2/\sigma^2]. \tag{A2}$$

## APPENDIX B: CONNECTION WITH THE TWO-STATE FORMALISM

In [12] (see also references therein), the authors have formulated a time-symmetric description for a system pre- and post-selected in the states $|\psi_I\rangle$ and $|\psi_F\rangle$ at some times $t_1$ and $t_2$, respectively. Evolving $|\psi_I\rangle$ and $|\psi_F\rangle$ forward and backward in time to the moment $t_i$ when the system interacts with an external meter yields what the authors of [12] called a two-state vector, $\langle\hat{U}^{-1}(t_2,t_i)\psi_F||\hat{U}(t_i,t_1)\psi_I\rangle$, where $\hat{U}(t,t')$ is the system's evolution operator. The vector contains sufficient information to describe the statistical properties of the observed system at $t_i$ [12]. For example, for $\hat{U} \equiv 1$, the weak value (19) of an operator $\hat{A}$ takes the simple form $A_W = \langle\psi_F|\hat{A}|\psi_I\rangle/\langle\psi_F|\psi_I\rangle$. A similar description can also be applied to the case of barrier penetration. One recalls that $T(p_0)$ is a transmission amplitude for a particle preselected in the plane wave $|p_0\rangle$ traveling to the right, $\langle x|p_0\rangle = \exp(ip_0x)$, at some $t_1$ in the distant past, and then postselected in the same state at some $t_2$ in the distant future,

$$\exp[-i\epsilon(p_0)(t_2-t_1)]T(p_0) = \langle p_0|\hat{U}(t_2,t_1)|p_0\rangle, \tag{B1}$$

where $\hat{U}(t_2,t_1)$ may include effects of adiabatic switching of the barrier potential. The postselection excludes the possibility of reflection, i.e., the particle ending up in the state $|-p_0\rangle$. With $|\psi_I\rangle$ and $|\psi_F\rangle$ thus defined, one can introduce a two-state vector $\langle\hat{U}^{-1}(t_2,t)p_0||\hat{U}(t,t_1)p_0\rangle$ for any $t_1 \leqslant t \leqslant t_2$. However, an immediate advantage of such a description is not clear since transmission, unlike an impulsive von Neumann interaction, is a continuous process and no simple expression, e.g., for the weak shift (2), is obtained as a result.


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[23] This can be done [11,19], for example, by choosing the states $|\psi_I\rangle$ and $|\psi_F\rangle$ in such a way that $K$ first moments of $\eta^{F\to I}$ in Eq. (13) coincide with $T^{(n)}(p_0)/T(p_0)$, $T(p)$ given by Eq. (22).