

System of classical nonlinear oscillators as a coarse-grained quantum system

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Constrained Hamiltonian dynamics of a quantum system of nonlinear oscillators is used to provide the mathematical formulation of a coarse-grained description of the quantum system. It is seen that the evolution of the coarse-grained system preserves constant and minimal quantum fluctuations of the fundamental observables. This leads to the emergence of the corresponding classical system on a sufficiently large scale.

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I. INTRODUCTION

The relation between quantum and classical mechanics (QC relation) is a very complex one with many complementary facets. Over the years, since the discovery of quantum theory, many different and more or less related aspects of the QC relation have been investigated. The studied problems can be artificially divided into two main groups. The first group entails the problems of formal or mathematical relations between quantum and classical formalisms (an excellent review is [1]). Problems of the other group are related to the description of physical reasons or processes that effectuate the quantum to classical transition [2–4]. Our goal in this paper is to explore yet another formal QC relation and interpret it as the mathematical formulation of a coarse-graining that is necessary in the quantum to classical transition.

Comparison of typical formal features of classical and quantum mechanics is facilitated if the same mathematical framework is used in both theories. It is well known, since the work of Kibble [6–8], that the quantum evolution, determined by the linear Schrödinger equation, can be represented using the typical language of classical mechanics, that is, as a Hamiltonian dynamical system on an appropriate phase space, given by the Hilbert space geometry of the quantum system. This line of research was later developed into the full geometric Hamiltonian representation of quantum mechanics [9–16]. Such geometric formulation of quantum mechanics has inspired natural definitions of measures of entanglement [16] and has been used to model the spontaneous collapse of the state vector [17,18].

It has been realized recently that the geometric formulation of quantum mechanics provides a particularly suitable framework for discussions of nonlinear constraints that might be imposed on a quantum system [19–21]. In particular, it was shown in Ref. [19] that a quantum system of two qubits constrained to be always on the manifold of separable states shows the characteristic qualitative features of classical Hamiltonian dynamical systems, which cannot be realized by the unconstrained Schrödinger evolution. The idea is further explored in [26] and applied to a general spin system, i.e., to a quantum system with a finite Hilbert space. Study of the QC

relations for such systems is hampered by the fact that there is no classical mechanical model which after quantization gives the quantum system.

In this paper we consider systems based on the Heisenberg H_4 dynamical algebra, say, a collection of oscillators possibly nonlinear and interacting. Classical system of oscillators is quantized to give the quantum system of oscillators. Our main result is that the quantum system of oscillators constrained with a specific type of constraints is equivalent to a finite dimensional Hamiltonian system that preserves constant and minimal quantum fluctuations of the fundamental observables during the entire evolution. This Hamiltonian system is close to the classical one if some classically parameter is small. Finally we shall propose an interpretation of these formal results as the mathematical formulation of the emergence of classical systems from a coarse-grained description of quantum systems.

The paper is organized as follows. Geometric Hamiltonian formulation of quantum mechanics and in particular the quantum constrained evolution for a general quantum system is formulated in Sec. II. Our main result is given in Sec. III where the formalism is applied to study the evolution of a system of quantum oscillators with particular constraints. In Sec. IV a complete construction of the classical system based on the constrained quantum system is presented. Section V contains a discussion and an interpretation of the formal results from Secs. III and IV.

II. HAMILTONIAN FORMULATION OF CONSTRAINED QUANTUM DYNAMICS**A. Hamiltonian framework for quantum systems**

Consider a quantum system with separable and complete Hilbert space \mathcal{H} . The Schrödinger dynamical equation on \mathcal{H} generates a Hamiltonian dynamical system on an appropriate symplectic manifold. The symplectic structure, which is needed for the Hamiltonian formulation of the Schrödinger dynamics, is provided by the imaginary part of the unitary scalar product on \mathcal{H} . In fact the Hilbert space \mathcal{H} is viewed as a real manifold \mathcal{M} with a complex structure, given by a linear operator J such that $J^2 = -1$. If \mathcal{H} is finite n -dimensional then $\mathcal{M} \equiv \mathbb{R}^{2n}$, but in general \mathcal{M} is an infinite dimensional Euclidean manifold. Real coordinates $\{(x_i, y_i), i = 1, 2, \dots\}$ of a point $\psi \in \mathcal{H} \equiv \mathcal{M}$ are introduced using expansion

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coefficients $\{c_i, i = 1, 2, \dots\}$ in some basis $\{|i\rangle, i = 1, 2, \dots\}$ of \mathcal{H} as follows:

$$|\psi\rangle = \sum_i c_i |i\rangle, \quad c_i = \frac{x_i + iy_i}{\sqrt{2}}, \quad (1a)$$

$$x_i = \sqrt{2} \operatorname{Re}(c_i), \quad y_i = \sqrt{2} \operatorname{Im}(c_i), \quad i = 1, 2, \dots \quad (1b)$$

Alternatively, if \mathcal{H} is identified with some space of functions $L_2(\mathbb{R}^N)$ with $q \in \mathbb{R}^N$ then the real and imaginary parts of $\psi(q) = [\phi(q) + i\pi(q)]/\sqrt{2}$ give two real fields $[\phi(q), \pi(q)]$ representing the coordinates of the real infinite manifold \mathcal{M} .

Besides the complex structure J , the real manifold \mathcal{M} has Riemannian and symplectic structure. Since \mathcal{M} is real, it is natural to decompose the unitary scalar product on \mathcal{H} into its real and imaginary parts,

$$\langle \psi_1 | \psi_2 \rangle = \frac{1}{2\hbar} G(\psi_1, \psi_2) + \frac{i}{2\hbar} \Omega(\psi_1, \psi_2). \quad (2)$$

It follows that G is Riemannian metric on \mathcal{M} and that Ω is a symplectic form on \mathcal{M} . Furthermore, J , G , and Ω satisfy $G(\psi_1, \psi_2) = \Omega(\psi_1, J\psi_2)$ so that the space \mathcal{M} is in fact a Kähler manifold. Thus the manifold \mathcal{M} associated with the Hilbert space \mathcal{H} can be viewed as a phase space of a Hamiltonian dynamical system. A vector $|\psi\rangle$ from \mathcal{H} , associated with a pure quantum state ψ , is represented by the corresponding point X_ψ in the phase space \mathcal{M} . It is convenient to add an abstract index $a = 1, 2, \dots$ to the points from \mathcal{M} such as X_ψ^a and to assume the standard summation convention over repeated abstract indices. On the other hand, summation over coordinate indices i, j as in (1) or integration over the argument q in $\phi(q), \pi(q)$ will always be written explicitly.

In the coordinates (x_i, y_i) the Riemannian and the symplectic structures of \mathcal{M} are given by

$$G = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad (3)$$

$$\Omega = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad (4)$$

where $\mathbf{0}$ and $\mathbf{1}$ are zero and unit matrices of dimension equal to the dimension of the Hilbert space. In the coordinates $(\phi(q), \pi(q))$ the analogous formulas are

$$G(\psi_1, \psi_2) = \int dq [\phi_1(q)\phi_2(q) + \pi_1(q)\pi_2(q)], \quad (5)$$

$$\Omega(\psi_1, \psi_2) = \int dq [\phi_1(q)\pi_2(q) - \pi_1(q)\phi_2(q)]. \quad (6)$$

Thus, coordinates $\{(x_i, y_i), i = 1, 2, \dots\}$ or $\{(\psi(q), \pi(q)), q \in \mathbb{R}^N\}$ represent canonical coordinates of a Hamiltonian dynamical system. Consequently, the Poisson bracket between two functions F_1 and F_2 on \mathcal{M} corresponding to the symplectic form Ω is in the canonical coordinate representation $(\phi(q), \pi(q))$ given by

$$\{F_1, F_2\} = \int dq \left[\frac{\delta F_1}{\delta \phi(q)} \frac{\delta F_2}{\delta \pi(q)} - \frac{\delta F_2}{\delta \phi(q)} \frac{\delta F_1}{\delta \pi(q)} \right]. \quad (7)$$

A one-parameter family of unitary transformations on \mathcal{H} generated by a self-adjointed operator \hat{A} is represented on

\mathcal{M} by a flow generated by the Hamiltonian vector field $\Omega(-J\hat{A}\psi, \cdot) = (dA)(\cdot)$ with the Hamilton function

$$A(X_\psi) = \langle \psi | \hat{A} | \psi \rangle \equiv \langle \hat{A} \rangle_\psi. \quad (8)$$

Thus, quantum observables \hat{A} are represented by functions of the form $\langle \hat{A} \rangle_\psi$. Such and only such Hamiltonian flows with the Hamilton function of the form (8) also generate isometries of the Riemannian metric G . More general Hamiltonian flows on \mathcal{M} , corresponding to the Hamilton functions which are not of the form (8), do not generate isometries and do not have the physical interpretation of quantum observables. In what follows we shall often use the short-hand notation $A(X_\psi) \equiv A$ and $\langle \hat{A} \rangle_\psi \equiv \langle \hat{A} \rangle$, implicitly assuming relation to the state ψ . Not every such function has an interpretation as a classical variable. If it does then we shall denote it by the corresponding small letter $a \equiv A \equiv \langle \hat{A} \rangle$.

It can be seen easily that the Poisson bracket of two Hamilton functions relates to the commutator between corresponding observables,

$$\{A_1, A_2\} = \frac{1}{i\hbar} [\hat{A}_1, \hat{A}_2]. \quad (9)$$

The Schrödinger evolution generated by Hamiltonian \hat{H} ,

$$i\hbar |\dot{\psi}\rangle = \hat{H} |\psi\rangle, \quad (10)$$

is equivalent to the Hamilton equations on \mathcal{M} ,

$$\dot{X}_\psi^a = \Omega^{ab} \nabla_b H(X_\psi). \quad (11)$$

In the canonical coordinates (x_i, y_i) the Schrödinger evolution is given by

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, 2, \dots \quad (12)$$

or in $(\phi(q), \pi(q))$ coordinates by

$$\dot{\phi}(q) = \frac{\delta H}{\delta \pi(q)}, \quad \dot{\pi}(q) = -\frac{\delta H}{\delta \phi(q)}, \quad q \in \mathbb{R}^N. \quad (13)$$

We have constructed the Hamiltonian dynamical system corresponding to the Schrödinger evolution equation on \mathcal{H} . In fact, phase invariance and arbitrary normalization of the quantum states imply that the proper space of pure quantum states is not the Hilbert space used to formulate the Schrödinger equation, but the projective Hilbert space. This also is a Kähler manifold and can be used as a phase space of a completely geometrical Hamiltonian formulation of quantum mechanics. Nevertheless, we shall continue to use the formulation in which points of the quantum phase space are identified with the vectors from \mathcal{H} since it is sufficient for our main purpose.

B. Constrained quantum systems

The Hamiltonian framework for quantum dynamics enables one to describe the evolution of a dynamical system generated by the Schrödinger equation with quite general additional constraints [19–21]. Suppose that the evolution given by the Hamiltonian H is further constrained onto a submanifold Γ of \mathcal{M} given by a set of k independent functional equations,

$$f_l(X) = 0, \quad l = 1, 2, \dots, k. \quad (14)$$

Equations of motion of the constrained system are obtained using the method of Lagrange multipliers. In the Hamiltonian

form, the method assumes that the dynamics on Γ is determined by the following set of differential equations:

$$\dot{X} = \Omega(\nabla X, \nabla H_{tot}), \quad H_{tot} = H + \sum_{l=1}^k \lambda_l f_l, \quad (15)$$

which should be solved together with the equations of the constraints (14). Other approaches to realize the constraints are possible [21], but the resulting system is not explicitly of Hamiltonian form. Notice that the total Hamilton function H_{tot} need not be given as the quantum expectation of a linear operator on \mathcal{H} . The Lagrange multipliers λ_l are functions on \mathcal{M} that are to be determined from the following so-called compatibility conditions,

$$\begin{aligned} 0 = \dot{f}_l &= \Omega(\nabla f_l, \nabla H_{tot}) \\ &= \Omega(\nabla f_l, \nabla H) + \sum_{m=1}^k \lambda_m \Omega(\nabla f_l, \nabla f_m), \end{aligned} \quad (16)$$

on the constrained manifold Γ . There is a standard Dirac approach to the constrained classical Hamiltonian dynamics [22,23]. We shall not go into its details that stress the distinction between the first- and the second-class constraints. In order to apply the standard procedure, the constraints have to be regular. A set of constraints is irregular if there is at least one such that the derivative of the constraint with respect to at least one of the coordinates is zero in at least one point on the constrained manifold. Otherwise the constraints are regular. In our case the constraints are regular if for all l

$$\frac{\delta f_l}{\delta \phi(q)} \neq 0, \quad \frac{\delta f_l}{\delta \pi(q)} \neq 0 \quad (17)$$

for all $q \in \mathbb{R}^N$ and everywhere on the constrained manifold. If this is not satisfied the Dirac classification is blurred and the straightforward application of Dirac's recipe is not possible. It will turn out that the case of interest here involves precisely the irregular constraints that cannot be easily replaced by an equivalent set of regular constraints.

We shall now briefly recapitulate the main steps of the general analysis of the constrained dynamics. The Eq. (16) can be satisfied in two fundamentally different ways. First, if the matrix of Poisson brackets $\{f_l, f_m\} = \Omega(\nabla f_l, \nabla f_m) \equiv (\Omega_f)_{l,m}$ computed on Γ is nonsingular, then the multipliers are uniquely determined from

$$\lambda_l = \sum_{m=1}^k (\Omega_f^{-1})_{l,m} \Omega(\nabla f_m, \nabla H). \quad (18)$$

The equations of motion (15) assume the form

$$\dot{X} = \Omega(\nabla X, \nabla H) + \sum_{l,m=1}^k f_l (\Omega_f^{-1})_{l,m} \Omega(\nabla f_m, \nabla H) \quad (19)$$

and should be solved together with the constraints (14). In this case all the constraints (14) are called primary and of the second class. In this case Γ is symplectic manifold with the symplectic structure determined by the so-called Dirac-Poisson brackets,

$$\{F_1, F_2\}_D = \{F_1, F_2\} + \sum_{l,m=1}^k \{f_l, F_1\} (\Omega_f^{-1})_{l,m} \{f_m, F_2\}. \quad (20)$$

A very different situation occurs if all of the Poisson brackets $\{f_l, f_m\}$ and $\{f_m, H\}$ are zero on the constrained manifold Γ and the regularity condition (17) is trivially satisfied. In this case the constraints are said to be of the first class. The compatibility conditions do not specify the multipliers and the constrained dynamics is not uniquely determined. Nevertheless, once a system with regular first-class constraints and the Hamiltonian H_{tot} is put onto the constrained manifold, the system remains on that manifold whatever choice is made for the Lagrange multipliers. Different choices of the multipliers must be considered as leading to the same physical situation.

Let us stress that the described scheme can be applied and leads to the above conclusions only if the constraints are regular. If this is not the case then it might be necessary to fix some or all of the multipliers even if the constraints appear to be of the first class. The system analyzed in the next section is precisely of this type.

If some of the compatibility equations do not contain multipliers, then for that condition $\dot{f}_l = \{f_l, H\} = 0$ represents an additional constraint. These are called secondary constraints, and they must be added to the system of original constraints (14). They could be of the first or of the second class. If this enlarged set of constraints is functionally independent one can repeat the procedure. At the end one either obtains a contradiction, in which case the original problem has no solution, or one obtains appropriate multipliers λ_l that need not be uniquely determined.

III. DYNAMICS OF A QUANTUM SYSTEM OF OSCILLATORS WITH CONSTRAINTS

The Hilbert space $\mathcal{H} = L_2(\mathbb{R}^n)$ is the unique irreducible representation space of the canonical commutation relations given by the n -term direct sum of Heisenberg H_4 algebras. Up to the normalization and the global phase invariance, this Hilbert space is the state space of a collection of n quantum oscillators. The fundamental observables of such a system are represented by $2n$ operators (\hat{Q}_i, \hat{P}_i) , $i = 1, 2, \dots, n$, satisfying $[\hat{Q}_i, \hat{P}_j] = i\delta_{i,j}$ on a dense domain in \mathcal{H} . The symplectic phase space \mathcal{M} of the Hamiltonian formulation of the quantum oscillators system is given as the product of n infinite dimensional symplectic spaces. The canonical coordinates of this infinite dimensional symplectic space can be written using the continuous index as $\phi(q_1, \dots, q_n), \pi(q_1, \dots, q_n)$ ($q_i \in \mathbb{R}$) or using discrete indices as (x_i^l, y_i^l) ($l = 1, 2, \dots, n$, $i = 1, 2, \dots$). A Hermitian operator \hat{A} is in the Hamiltonian formulation represented as function $A(X_\psi) = \langle \psi | \hat{A} | \psi \rangle$ on \mathcal{M} . In particular, fundamental observables \hat{Q}_i, \hat{P}_j give $2n$ fundamental variables as functions on the infinite quantum phase space \mathcal{M} , which we shall denote as $q_i = \langle \hat{Q}_i \rangle$, $p_j = \langle \hat{P}_j \rangle$. The Poisson brackets of the infinite phase space \mathcal{M} between the fundamental variables q_i, p_j are given by the general formula (6) as

$$\{q_i, p_j\}_{\mathcal{M}} = \delta_{i,j}, \quad i, j = 1, 2, \dots, n, \quad (21)$$

where we stress by the subscript \mathcal{M} that the Poisson bracket is computed on the infinite manifold \mathcal{M} , for example as in (6). Notice that the quantum variables of the oscillator system are represented as functions of the fundamental variables of

the infinite phase space \mathcal{M} [x_i^l, y_i^l or the canonical fields $\phi(q_1, \dots, q_n), \pi(q_1, \dots, q_n)$] but most of them cannot be represented as functions only of the fundamental variables q_i, p_j . A nonlinear operator expression in terms of \hat{Q}_i, \hat{P}_j is represented as a function of x_i^l, y_i^l ($l = 1, 2, \dots, n, i = 1, 2, \dots$) or functional of $\phi(q_1, \dots, q_n), \pi(q_1, \dots, q_n)$, but cannot be written as function only of q_i, p_j ($i, j = 1, 2, \dots, n$). Such expressions involve terms that represent quantum fluctuations of the fundamental observables, i.e., contain the second or higher order moments, for example fluctuations $(\Delta \hat{Q}_i)^2 = \langle \hat{Q}_i^2 \rangle - q_i^2$, $(\Delta \hat{P}_i)^2 = \langle \hat{P}_i^2 \rangle - p_i^2$ and correlations $\langle \hat{P}_i \hat{Q}_i + \hat{Q}_i \hat{P}_i \rangle - 2p_i q_i$. Of course, these are functions on \mathcal{M} but cannot be presented as functions only of q_i, p_j ($i, j = 1, 2, \dots, n$). A polynomial expression of \hat{Q}_i, \hat{P}_j thus involves a function of q_i, p_j plus additional terms involving the correlations. The important observation is that the correlations can become arbitrarily large during typical Schrödinger evolution. However, there is an important exception. Namely, when the system is in the coherent state, all moments of \hat{Q}_i (\hat{P}_i) of order higher than two are expressible solely in terms of q_i and $\Delta \hat{Q}_i$ (p_i and $\Delta \hat{P}_i$), while the correlations $\langle \hat{Q}_i^m \hat{P}_i^n + \hat{P}_i^n \hat{Q}_i^m \rangle - 2q_i^m p_i^n$ ($m, n \in \mathbb{N}$) vanish.

Previous discussion suggests that a closed dynamical system expressed solely in terms of the fundamental variables q_i, p_j could be obtained from the quantum system if the Schrödinger evolution is additionally constrained to the appropriate coherent state manifold, i.e., to preserve constant and minimal values of the fluctuations of the fundamental observables \hat{Q}_i, \hat{P}_j . The formalism of the constrained quantum Hamiltonian system sketched in the previous section is ideally suited for the analysis of such systems. However, as we shall see, the construction of the most appropriate set of constraints and the analysis thereof is not straightforward. In this section we deal with the construction of the constrained system. Physical interpretation of the constrained system will be discussed in the following sections.

A system of quantum nonlinear oscillators is given by the following Hamiltonian:

$$\begin{aligned} \hat{H} &= \sum_{i=1}^n \frac{1}{2m_i} \hat{P}_i^2 + V(\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_n) \\ &= \sum_{i=1}^n \frac{1}{2m_i} \hat{p}_i^2 + \frac{m_i \omega_i^2}{2} \hat{Q}_i^2 + \dots, \end{aligned} \quad (22)$$

where V is some function of $(\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_n)$ having the properties $\partial^2 V / \partial Q_i^2|_{Q_i=0} = m_i \omega_i^2$ ($i = 1, 2, \dots, n$).

In the general case when the Hamiltonian is not only quadratic in $\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_n$, the dispersions $\Delta \hat{Q}_i, \Delta \hat{P}_i$ ($i = 1, 2, \dots, n$) will assume different arbitrarily high values in the states along an orbit generated by \hat{H} . However, the constrained system defined by the Hamiltonian (22) and the following set of $2n$ constraints,

$$f_q^i(X) = (\Delta \hat{Q}_i)^2 - \frac{\hbar}{2m_i \omega_i} = 0, \quad (23a)$$

$$f_p^i(X) = (\Delta \hat{P}_i)^2 - \frac{m_i \omega_i \hbar}{2} = 0, \quad (23b)$$

should preserve the dispersions of all fundamental quantum observables. The values of the dispersions in (23) are the

minimal values that can be achieved simultaneously by the coordinates and momenta, and are obtained if and only if the state of the i th oscillator is a coherent state. However, the constraints (23) are irregular and we shall see that the conservation of minimal dispersions is achieved by a more suitable set of constraints.

Let us consider in detail a single nonlinear oscillator. This example is in fact sufficient to indicate the typical features of the general case. In this case there are only two constraints of the form (23):

$$f_q(X) = (\Delta \hat{Q})^2 - \frac{\hbar}{2m\omega} = 0, \quad (24a)$$

$$f_p(X) = (\Delta \hat{P})^2 - \frac{m\omega\hbar}{2} = 0. \quad (24b)$$

The constrained manifold Γ defined by (24) coincides with the set of coherent states, which is a finite-dimensional submanifold of all quantum states \mathcal{M} . This shows that there exists an infinite set of constraints on \mathcal{M} with the same constrained manifold as the one given by the two constraints (24). Furthermore, this indicates that it might not be possible to treat the two constraints (24) within the standard Dirac scheme for regular constraints. Nevertheless, in order to illustrate the problems that occur, we shall proceed with the analysis of the constrained Hamiltonian equations (15) with the two constraints (24).

The general dynamical equations for the fundamental variables $q = \langle \hat{Q} \rangle, p = \langle \hat{P} \rangle$ of the constrained quantum Hamiltonian system with the constraints (24) assume the form

$$\dot{q} = \{q, H + \lambda_q f_q + \lambda_p f_p\}_{\mathcal{M}}, \quad (25)$$

$$\dot{p} = \{p, H + \lambda_q f_q + \lambda_p f_p\}_{\mathcal{M}},$$

and should be solved together with the constraints equations (24). Notice that in (25) the Poisson brackets are those of the full quantum phase space \mathcal{M} , and H is a function on \mathcal{M} and not on the constrained manifold.

The general procedure requires first to compute the Poisson brackets between the constraints and between the constraints and the Hamiltonian H , and then to check the values they assume on the constrained manifold Γ . Computations are facilitated using the relations

$$\frac{\delta}{\delta \psi(q)} \langle \psi | \hat{A} | \psi \rangle = \langle \psi | \hat{A} | q \rangle, \quad (26a)$$

$$\frac{\delta}{\delta \psi^*(q)} \langle \psi | \hat{A} | \psi \rangle = \langle q | \hat{A} | \psi \rangle \quad (26b)$$

and equality

$$\begin{aligned} &\frac{\delta A_1}{\delta \phi(q)} \frac{\delta A_2}{\delta \pi(q)} - \frac{\delta A_2}{\delta \phi(q)} \frac{\delta A_1}{\delta \pi(q)} \\ &= -i \left[\frac{\delta A_1}{\delta \psi(q)} \frac{\delta A_2}{\delta \psi^*(q)} - \frac{\delta A_2}{\delta \psi(q)} \frac{\delta A_1}{\delta \psi^*(q)} \right], \end{aligned} \quad (27)$$

where $\psi^*(q) = [\phi(q) - i\pi(q)]/\sqrt{2}$.

The Poisson brackets between the constraints are

$$\begin{aligned} \{f_q, f_p\} &= -i \int dq \left[\frac{\delta f_q}{\delta \psi(q)} \frac{\delta f_p}{\delta \psi^*(q)} - \frac{\delta f_p}{\delta \psi(q)} \frac{\delta f_q}{\delta \psi^*(q)} \right] \\ &= -i \int dq [(\langle \psi | \hat{Q}^2 | q \rangle - 2\langle \hat{Q} \rangle \langle \psi | \hat{Q} | q \rangle) (\langle q | \hat{P}^2 | \psi \rangle \end{aligned}$$

$$\begin{aligned}
 & -2\langle \hat{P} \rangle \langle q | \hat{P} | \psi \rangle - \leftrightarrow = -i \langle [\hat{Q}^2, \hat{P}^2] \rangle \\
 & -2\langle \hat{Q} \rangle \langle [\hat{Q}, \hat{P}^2] \rangle - 2\langle \hat{P} \rangle \langle [\hat{Q}^2, \hat{P}] \rangle \\
 & + 4\langle \hat{Q} \rangle \langle \hat{P} \rangle \langle [\hat{Q}, \hat{P}] \rangle \\
 & = 2\hbar \langle (\hat{Q}\hat{P} + \hat{P}\hat{Q}) - 2\langle \hat{Q} \rangle \langle \hat{P} \rangle \rangle \equiv 4\hbar \Delta(\hat{Q}, \hat{P}).
 \end{aligned} \tag{28}$$

The symbol \leftrightarrow means the term of the same form as the previous one but having \hat{Q} replaced by \hat{P} and vice versa.

Similar calculations give the brackets between the constraints and the Hamiltonian H :

$$\{f_q, H\} = \frac{2\hbar}{m} \Delta(\hat{Q}, \hat{P}), \tag{29}$$

$$\{f_p, H\} = -2\hbar \Delta(V'(\hat{Q}), \hat{P}), \tag{30}$$

where $V'(\hat{Q})$ denotes the derivative of the function $V(\hat{Q})$.

All three expressions (28), (29), and (30) are zero on the constrained manifold Γ of the coherent states. Thus the constraints (24) appear to be of the first class and there are no secondary constraints. According to the general theory for the regular first-class constraints the Lagrange multipliers in the total Hamiltonian H_{tot} should be left unspecified. However the constraints are not regular because, for example, the derivative with respect to the coordinate $\psi(q) = \psi(0)$ is

$$\frac{\delta f_q}{\delta \psi(q)} = [q^2 \psi^*(q) - 2q \langle \hat{Q} \rangle \psi^*(q)]|_{q=0} = 0, \tag{31}$$

indicating that the multipliers have to be specified from some other condition. In order to correctly fix the multipliers one might use the following reasoning. Consider the Poisson bracket $\{f_q, f_p\}$. If the constraints were regular, this bracket would be a first-class constraint and would be preserved by the constrained evolution. In fact, the bracket $\{f_q, f_p\}$ computed on Γ , as seen from (28), represents the correlation between \hat{Q} and \hat{P} in a coherent state. This must be preserved by the evolution on Γ generated by the total Hamiltonian H_{tot} . The dynamical equation with H_{tot} for the correlation $\Delta(\hat{Q}, \hat{P})$ reads

$$\begin{aligned}
 \frac{d}{dt} \Delta(\hat{Q}, \hat{P}) & = \{\Delta(\hat{Q}, \hat{P}), H_{tot}\} \\
 & = 2\hbar \left(\frac{1}{2m} (\Delta \hat{P})^2 - \frac{\langle V''(\hat{Q}) \rangle}{2} (\Delta \hat{Q})^2 \right. \\
 & \quad \left. + \lambda_p (\Delta \hat{P})^2 - \lambda_q (\Delta \hat{Q})^2 \right)
 \end{aligned} \tag{32}$$

on Γ and vanishing only if the multipliers are

$$\lambda_p = -\frac{1}{2m}, \quad \lambda_q = -\frac{\langle V''(\hat{Q}) \rangle}{2}. \tag{33}$$

Thus, the total Hamiltonian that would preserve the irregular constraints (24) with the additional compatibility condition is

$$H_{tot} = \frac{\langle \hat{P} \rangle^2}{2m} + \langle V(\hat{Q}) \rangle - \frac{\langle V''(\hat{Q}) \rangle}{2} \left[(\Delta \hat{Q})^2 - \frac{\hbar}{2m\omega} \right]. \tag{34}$$

However, this is still not satisfactory. To see this, one might observe that $\Delta(f(\hat{Q}), \hat{P}) = 0$ should hold on Γ for arbitrary $f(\hat{Q})$. The evolution generated by the total Hamiltonian (15) should yield on Γ

$$\frac{d}{dt} \Delta(f(\hat{Q}), \hat{P}) = \{\Delta(f(\hat{Q}), \hat{P}), H_{tot}\} = 0. \tag{35}$$

It turns out that the multiplier λ_q must depend on $f(\hat{Q})$, i.e., on an arbitrary function, and cannot be fixed by any means. The origin of such discrepancy is seen from

$$\langle V(\hat{Q}) \rangle = \sum_{k=0}^{\nu} \frac{V^{(k)}(\langle \hat{Q} \rangle)}{k!} \langle (\hat{Q} - \langle \hat{Q} \rangle)^k \rangle, \tag{36}$$

where possibly all moments $\langle (\hat{Q} - \langle \hat{Q} \rangle)^k \rangle$ are present (if $\nu = \infty$) and influence the dynamics (35), while our constraints (24) contain only the moment of order two. Resolution of this problem requires a number of constraints equal to the order ν of highest moment present in (36),

$$f_{q,2k-1}(X) = \langle (\hat{Q} - \langle \hat{Q} \rangle)^{2k-1} \rangle = 0, \tag{37a}$$

$$f_{q,2k}(X) = \langle (\hat{Q} - \langle \hat{Q} \rangle)^{2k} \rangle - \frac{\hbar^k (2k-1)!!}{(2m\omega)^k} = 0, \tag{37b}$$

$k = 2, 3, \dots, \lfloor (\nu+1)/2 \rfloor$. Although the constraints (37) implicitly follow from (24) and hold automatically on Γ , they must be present explicitly in the total Hamiltonian. In that case the choice of the multipliers

$$\lambda_{q,k} = -\frac{V^{(k)}(\langle \hat{Q} \rangle)}{k!} \Big|_{\Gamma} \tag{38}$$

cancel termwise the appropriate contributions of moments $\langle (\hat{Q} - \langle \hat{Q} \rangle)^k \rangle$ to the evolution (35).

We see that starting with the primary constraints (24) one would have to add a possibly infinite number of secondary constraints in order to satisfy all possible compatibility conditions (35). This is not satisfactory. Fortunately, there is an alternative procedure which starts with a different set of two primary constraints and offers a resolution.

A. More convenient primary constraints

To formulate the primary constraints in the alternative procedure, we associate with each point from \mathcal{M} denoted X_ψ a point $\alpha(\psi)$ on the coherent state manifold Γ such that

$$\alpha(\psi) = (\langle \hat{Q} \rangle_\psi, \langle \hat{P} \rangle_\psi). \tag{39}$$

By definition, the operators \hat{Q} and \hat{P} have the expectations in the coherent state $\alpha(\psi)$ the same as in the state ψ . This association of a single coherent state with the whole set of states in fact establishes an equivalence relation on \mathcal{M} that will play a crucial role in the following section.

With the notation (39) we formulate the following two constraints

$$\Phi_q = \langle V(\hat{Q}) \rangle_\psi - \langle V(\hat{Q}) \rangle_{\alpha(\psi)} = 0, \tag{40a}$$

$$\Phi_p = \langle \hat{P}^2 \rangle_\psi - \langle \hat{P}^2 \rangle_{\alpha(\psi)} = 0 \tag{40b}$$

to be imposed on the oscillator with arbitrary fixed potential $V(\hat{Q})$. The role of the constraints is to preserve during the evolution the association of the set of points $\psi(t)$ with the corresponding single coherent state $\alpha(\psi(t))$.

The total Hamiltonian assumes the standard form

$$H_{tot} = \langle \hat{H} \rangle_\psi + \lambda_q \Phi_q + \lambda_p \Phi_p, \tag{41}$$

and the compatibility condition

$$\{\Delta(f(\hat{Q}), \hat{P}), H_{tot}\} = 0 \tag{42}$$

yields the values of Lagrange multipliers

$$\lambda_q = -1, \quad \lambda_p = -\frac{1}{2m} \quad (43)$$

independently of the function $f(\hat{Q})$, leading to

$$H_{tot} = \frac{1}{2m} \langle \hat{P}^2 \rangle_{\alpha(\psi)} + \langle V(\hat{Q}) \rangle_{\alpha(\psi)} \equiv \langle \hat{H} \rangle_{\alpha(\psi)}. \quad (44)$$

Noting that $\langle \hat{P}^2 \rangle_{\alpha(\psi)} = \langle \hat{P} \rangle_{\alpha(\psi)}^2 + m\omega\hbar/2$ and dropping the irrelevant constant we finally obtain the total constrained Hamiltonian

$$H_{tot} = \frac{1}{2m} \langle \hat{P} \rangle_{\alpha(\psi)}^2 + \langle V(\hat{Q}) \rangle_{\alpha(\psi)}, \quad (45)$$

which preserves the evolution on the manifold of the coherent states Γ . It is seen that the total Hamiltonian (45) is up to additive constant equal to the initial Hamiltonian $H \equiv \langle \hat{H} \rangle_{\psi}$ on the constrained manifold Γ . However, H_{tot} preserves constant and minimal quantum fluctuations of fundamental observables, while the evolution with H can in general make them quite large.

The important fact is that the total Hamiltonian (45) depends only on the variables $q \equiv \langle \hat{Q} \rangle_{\alpha(\psi)}$ and $p \equiv \langle \hat{P} \rangle_{\alpha(\psi)}$ that parametrize the coherent state manifold. Thus, the constrained evolution of the fields

$$\dot{\phi}(x) = \frac{\delta H_{tot}(q, p)}{\delta \pi(x)}, \quad \dot{\pi}(x) = -\frac{\delta H_{tot}(q, p)}{\delta \phi(x)}, \quad x \in \mathbb{R}^N \quad (46)$$

can be, up to the phase freedom of $\psi(x) = \phi(x) + i\pi(x)$, inferred from the Hamiltonian evolution of the coherent state parameters

$$\frac{dq}{dt} = \frac{p}{m} = \frac{\partial H_{tot}(q, p)}{\partial p}, \quad (47a)$$

$$\frac{dp}{dt} = -\langle V'(\hat{Q}) \rangle_{\alpha(\psi)} = -\frac{\partial H_{tot}(q, p)}{\partial q}; \quad (47b)$$

i.e., the dynamics of constrained system is given by the Hamiltonian dynamics of constrained manifold parameters.

Let us finally stress that the role of the constraints is to preserve during the evolution the association (39) of the set of points $\psi(t)$ with the corresponding single coherent state $\alpha(\psi(t))$. In other words, relation (39) gives an equivalence relation on the manifold \mathcal{M} . Points from $\psi \in \mathcal{M}$ which give the same expectations $\langle \hat{Q} \rangle$ and $\langle \hat{P} \rangle$ are identified with the single representative: the coherent state $\alpha(\psi)$ with the same expectations. However, the equivalence relation is not preserved by the unconstrained Schrödinger evolution of $\psi \in \mathcal{M}$. On the other hand, the constrained evolution is precisely such that it preserves the equivalence of states, which can be seen from the fact that it can be expressed entirely in terms of the expectations $q = \langle \hat{Q} \rangle$ and $p = \langle \hat{P} \rangle$ [Eqs. (47)]. In this sense the constrained evolution and the equivalence relation (39) imply each other.

B. Quantum constrained system and the classical oscillator

We shall now compare the total Hamiltonian (45) on the constrained manifold Γ of the coherent states with

$$h_{cl} = \frac{1}{2m} p^2 + V(q), \quad (48)$$

representing the Hamilton function of a classical nonlinear oscillator with the potential $V(q)$.

The quantum expectation of the potential $V(\hat{Q})$ in a coherent state α is

$$\langle V(\hat{Q}) \rangle_{\alpha} = \int_{-\infty}^{\infty} V(x) \frac{\exp\left(-\frac{(x-\langle \hat{Q} \rangle_{\alpha})^2}{2(\Delta \hat{Q})_{\alpha}^2}\right)}{(\Delta \hat{Q})_{\alpha} \sqrt{2\pi}} dx. \quad (49)$$

Using the general formula

$$\int_{-\infty}^{\infty} f(t) \frac{\exp\left(-\frac{(x-t)^2}{4q^2}\right)}{2q\sqrt{\pi}} dt = \sum_{k=0}^{\infty} \frac{q^{2k}}{k!} f^{(2k)}(x), \quad (50)$$

we see that

$$\langle V(\hat{Q}) \rangle_{\alpha} = V(q) + \sum_{k=1}^{\infty} \frac{(\Delta \hat{Q})_{\alpha}^{2k}}{2^k k!} V^{(2k)}(q), \quad (51)$$

where $q = \langle \hat{Q} \rangle_{\alpha}$ and $(\Delta \hat{Q})_{\alpha} = \sqrt{\hbar/(2m\omega)}$. Thus, the total Hamiltonian in a point $\alpha \equiv (q, p)$ on the constrained manifold is

$$\begin{aligned} H_{tot} &= \frac{p^2}{2m} + V(q) + \sum_{k=1}^{\infty} \frac{1}{2^k k!} \frac{\hbar^k V^{(2k)}(q)}{(2m\omega)^k} \\ &\equiv h_{cl} + \sum_{k=1}^{\infty} \frac{1}{2^k k!} \frac{\hbar^k V^{(2k)}(q)}{(2m\omega)^k}. \end{aligned} \quad (52)$$

In the macroscopic limit, represented as $\hbar \rightarrow 0$ [24], the terms in the sum in (52) tend to zero yielding

$$H_{tot} \rightarrow h_{cl}, \quad \hbar \rightarrow 0. \quad (53)$$

Alternatively, the dispersion $(\Delta Q)_{\alpha} = \sqrt{\hbar/(2m\omega)} \rightarrow 0$ and the exponent in the integral in (49) approaches the delta function $\delta(x - \langle \hat{Q} \rangle_{\alpha}) \equiv \delta(x - q)$ producing $\langle V(\hat{Q}) \rangle_{\alpha} \rightarrow V(q)$.

To summarize, we have formulated a consistent set of dynamical equations for an arbitrary quantum nonlinear oscillator that maintains the evolution on the coherent state manifold. Because such evolution preserves minimal fluctuations $\Delta \hat{Q}$ and $\Delta \hat{P}$, the total Hamiltonian H_{tot} on Γ differs from the Hamilton function of a classical nonlinear oscillator with the same potential $V(q)$ by the terms that are small in the macroscopic limit. At the risk of repeating ourselves, let us stress once again that during the evolution with the quantum Hamiltonian of the oscillator $\langle \hat{H} \rangle_{\psi}$ with no constraints, the quantum fluctuations $\Delta \hat{Q}$ and $\Delta \hat{P}$ can become large and thus make Hamiltonian functions $\langle \hat{H} \rangle_{\psi}$ and h_{cl} quite different even in the macroscopic limit.

For the system with more than one oscillator, which might be nonlinear and interacting, the condition that $\Delta \hat{Q}_i$ and $\Delta \hat{P}_i$ are simultaneously minimal implies that each of the oscillators is always in some pure H_4 coherent state $|\alpha_i(t)\rangle$. Thus, the total state $|\psi(t)\rangle$ is always given by the tensor product of the single

oscillator's pure coherent states $|\psi(t)\rangle = \otimes_i |\alpha_i(t)\rangle$, implying for example

$$\begin{aligned} \langle \psi(t) | \hat{Q}_1 \otimes \hat{Q}_2 | \psi(t) \rangle &= \langle \hat{Q}_1 \rangle_{\alpha_1(t)} \times \langle \hat{Q}_2 \rangle_{\alpha_2(t)} \\ &= q_1(t) \times q_2(t). \end{aligned} \quad (54)$$

Suppression of quantum fluctuations for each oscillator's degree of freedom implies that the degrees of freedom of different oscillators do not get entangled during the evolution. This is enough to generalize the results of the single-oscillator analysis to the general case of arbitrary number of interacting oscillators with constraints.

We have formulated the constrained evolution of a quantum system of oscillators with the corresponding constraints. In general, the Hilbert space of a quantum system represents the space of an irreducible representation of the corresponding dynamical algebra \mathfrak{g} , which need not be the Heisenberg algebra as it is in the case of oscillators. Nevertheless, one could study the evolution of such a system with the constraints analogous to (23). The constraint manifold of such a system with a Lie dynamical algebra \mathfrak{g} should coincide with the manifold of the corresponding \mathfrak{g} -generalized coherent states [25–27].

IV. EQUIVALENCE RELATION AMONG THE QUANTUM STATES

The fundamental quantum observables \hat{Q}_i, \hat{P}_i ($i = 1, 2, \dots, n$) define $2n$ functions $\langle X | \hat{Q}_i | X \rangle, \langle X | \hat{P}_i | X \rangle$ on \mathcal{M} . Values that these functions take on the coherent states parametrize the $2n$ -dimensional manifold of the coherent states Γ . Thus, the set of fundamental quantum observables and the constrained manifold are seen to be in a one-to-one relation.

We use the coherent states or the elementary quantum observables \hat{Q}_i, \hat{P}_i to define an equivalence relation on \mathcal{M} . Two general quantum states $X_1 \in \mathcal{M}$ and $X_2 \in \mathcal{M}$ are defined to be equivalent, or *physically indistinguishable*, if each fundamental quantum observable takes the same value in X_1 as in X_2 . Thus, $X_1 \sim X_2$ if $q_i(X_1) = q_i(X_2), p_i(X_1) = p_i(X_2)$ ($i = 1, 2, \dots, n$). An equivalent definition is that the states $X_{1,2}$ are equivalent if there is a coherent state (q, p) such that $q_i(X_{1,2}) = q_i(q, p) = q_i, p_i(X_{1,2}) = p_i(q, p) = p_i$ ($i = 1, 2, \dots, n$). Each equivalence class contains one and only one coherent state, i.e., a state from the constraint manifold Γ .

The quantum phase space \mathcal{M} appears as a bundle over the constraint manifold $\Gamma = \mathcal{M}/\sim$. Γ is even dimensional and is parametrized by the values of only $2n$ independent variables (q_i, p_i) , $i = 1, 2, \dots, n$. Γ inherits a symplectic structure ω which is the pullback of the symplectic structure Ω on \mathcal{M} . In fact Γ is finite-dimensional symplectic manifold and (q_i, p_i) , $i = 1, 2, \dots, n$ are canonical coordinates. Thus, the constraint manifold Γ is the phase space of a classical system of n oscillators. The constraints (40) then force the quantum dynamics onto the classical phase space and thus generate a Hamiltonian dynamical system on Γ . This is the way in which the phase space of a classical mechanical system and the Hamiltonian dynamics on it appear from the structure of the quantum mechanics. The

resulting Hamiltonian system on Γ is to be compared with the system of classical oscillators with physically the same potential.

We have seen that (a) the constrained manifold Γ is related to a certain equivalence relation on full quantum phase space \mathcal{M} and (b) Γ has the phase-space structure of a finite Hamiltonian dynamical system. We can distinguish two dynamical systems on Γ defined by Hamilton functions H_{tot} restricted on Γ and h_{cl} . Since $\Delta \hat{Q} = \sqrt{\hbar/(2m\omega)} = \text{const}$ during the evolution defined by H_{tot} such evolution differs from the dynamics generated by h_{cl} by the terms which are small in the macroscopic limit.

V. DISCUSSION

The presented picture where the constraints are seen as the equivalence relation imposed on the quantum states suggests a physical interpretation of the constrained Hamiltonian system $(\Gamma, \omega, H_{tot}|_{\Gamma})$. The equivalence classes of quantum states determine the corresponding quantum observables that can be considered as physically distinguishable. Thus, in the Hamiltonian system with constraints only functions defined on Γ are considered as physically distinguishable. In other words, if two functions on \mathcal{M} correspond to two different operators but generate the same function on Γ , the two operators should be considered as physically indistinguishable. We see that imposing the constraint on the quantum system in fact provides the mathematical representation of a coarse-grained description of the quantum system.

The coarse-grained description gives a system with the kinematic properties of a classical Hamiltonian mechanical system. Furthermore, dynamics of the constrained system is such that the quantum fluctuations of fundamental observables are constant and simultaneously minimal during the evolution. In fact, one can identify a class of classical Hamiltonian dynamical systems that is generated by the constrained quantum system and that preserves the quantum fluctuations. The systems in this class differ from each other by terms that are arbitrarily small in the macroscopic limit. On the contrary, the corresponding terms in the quantum Hamiltonian system with no constraints, i.e., in the full-detail picture without the coarse-graining, necessarily become large during the evolution. They are responsible for the creation of typical quantum superpositions.

It is well known that a generic Hamiltonian dynamical system is not structurally stable; i.e., small perturbations of the Hamilton function typically induce nonequivalent phase portraits [28–30]. Thus, one can expect qualitative differences between the quantum systems for large values of the classicality parameter and the classical system. However, this is the problem of any Hamiltonian theory as a framework for robust modeling of dynamical phenomena, and is not strictly related to the QC relation.

We see that the classical system appears because of (a) the coarse-grained description of the quantum system and then (b) the classical, i.e., macroscopic, limit. It is important to note that the two factors, i.e., the coarse-graining and the macroscopic limit, are independent and both are necessary

(please see also [5,31]). The two factors, one leading to the suppression of, i.e., impossible to observe, dynamically created quantum coherences and the other involving the macroscopic limit, also appear in other explanations of the appearance of the classical world from the quantum, such as in the theory of environmentally induced decoherence [2,3].

Finally, let us illustrate the independent roles played by the macroscopic limit and the coarse-grained observation using one more example. Consider a large collection of $1/2$ spins $\hat{\sigma}^i$, $i = 1, 2, \dots, N$. One can define collective quantum observables $\hat{m}_x = \sum_i \hat{\sigma}_x^i/N$, $\hat{m}_y = \sum_i \hat{\sigma}_y^i/N$, $\hat{m}_z = \sum_i \hat{\sigma}_z^i/N$. The macroscopic limit corresponds in this case to the limit of large N . However, the macroscopic magnetizations $m_{x,y,z} = \langle \hat{m}_{x,y,z} \rangle$ in general do not behave as classical variables. Even if the initial state is such that $\Delta m_x/m_x$, $\Delta m_y/m_y$, and $\Delta m_z/m_z$ are all small, the evolution might be such that quite quickly these ratios become large, i.e., close to unity [32]. This occurs if the Hamiltonian includes long-range interactions, for example if $H_{int} = \sum_{i,j} \hat{\sigma}^i \hat{\sigma}^j$. Thus, the macroscopic limit alone does not imply the classical behavior even for the selected set of global observables. This has been nicely illustrated in [5,31] (please see also [32]). A coarse-graining analogous to the one discussed in this paper is also needed. One declares that the only states that are physically measurable are necessarily such that Δm_x , Δm_y , Δm_z are simultaneously minimal. The states satisfying this condition are the SU(2) coherent states of the N -term direct product representation. Such coarse-graining is equivalent to the evolution constrained on the submanifold of these coherent states so that all three dispersions are small during such evolution. Notice that the coarse-graining also implies that the eigenstates of the quantum collective variables $\hat{m}_{x,y,z}$ are not among the physically measurable states. Equally, the states corresponding to a superposition of states with very different values of the macroscopic variables $\langle \hat{m}_{x,y,z} \rangle$ are not physical.

VI. SUMMARY

We have used the formulation of quantum dynamics in the form of a Hamiltonian dynamical system to study the relation between quantum and classical systems of nonlinear interacting oscillators. The classical system has finite dimensional phase space and the quantum system viewed as the Hamiltonian system is infinite dimensional in an essential way. Kinematical and dynamical properties of the classical system are obtained from the quantum one via the two-step procedure consisting of (a) coarse-graining and (b) macroscopic limit $\hbar \rightarrow 0$. The coarse-graining is mathematically treated as an equivalence relation on the set of quantum states, and as a result emerges the classical phase space. The equivalence relation imposes a constraint on the Hamiltonian dynamics of the quantum system. The effect of the constraints is to preserve constant and minimal quantum fluctuations of the canonical observables. The formulation of the most appropriate finite set of constraints that fulfill the goal is not straightforward, and involves the nonlinear potential. The resulting constrained Hamiltonian system on the constrained manifold represents the coarse-grained description of the quantum system of oscillators. The system differs from the classical system with the same potential only in the terms that are arbitrarily small in the macroscopic limit.

The procedure can be generalized to obtain other classical systems from the corresponding coarse-grained quantum systems in the corresponding macroscopic limit.

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