Conclusive discrimination among N equidistant pure states

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We find the allowed complex overlaps for N equidistant pure quantum states. The accessible overlaps define a petal-shaped area on the *Argand plane*. Each point inside the petal represents a set of N linearly independent pure states and each point on its contour represents a set of N linearly dependent pure states. We find the optimal probabilities of success of discriminating unambiguously in which of the N equidistant states the system is. We show that the phase of the involved overlap plays an important role in the probability of success. For a fixed overlap modulus, the success probability is highest for the set of states with an overlap with phase equal to zero. In this case, if the process fails, then the information about the prepared state is lost. For states with a phase different from zero, the information could be obtained with an *error-minimizing measurement* protocol.

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I. INTRODUCTION

Discriminating among different nonorthogonal quantum states becomes a fundamental issue in quantum information and computation theory [1–4]. There are two schemes for discriminating in which pure state a system was prepared. One scheme is the *unambiguous quantum states discrimination* (UQSD) about which has been written many interesting works, some generic [5–8] and others concerned with particular applications [9–12]. The UQSD scheme requires a set of outcomes which allow us to infer the prepared state without error, i.e., unambiguously. This can be performed at the expense of having a nonzero probability of an inconclusive outcome. An *optimum UQSD* protocol holds when the probability for success is maximum. For discriminating one among N nonorthogonal and linearly independent (LI) pure states, it is required to map or to represent them onto two orthogonal subspaces which can be called the *conclusive subspace* and the *inconclusive subspace*. In the conclusive subspace, each state has no null component on only one state belonging to an orthonormal basis. In the inconclusive subspace, each state has no null components on N states of a linearly dependent (LD) set. Thus, the overlaps among the states are completely in the inconclusive subspace. Another scheme for state discrimination is called the *error-minimizing* measurement protocol (EMMP). EMMP tolerates error, which means that a given outcome says which is the most probable state but without discarding absolutely the other ones. An optimum measurement process minimizes the probability of making a wrong guess about the prepared state [13,14,16–18]. The advantage of this procedure is that it can be applied for extracting partial information about LD states.

The article is organized as follows: In Sec. II, we characterize the allowed complex numbers associated with the overlap of the N equidistant pure quantum states. We find that the permitted area on the *Argand plane* has the form of a petal, where LI sets are inside the petal and the LD sets are on its contour. In Sec. III, we study the protocol for discriminating unambiguously one among N equidistant states. We find the success probability of achieving the discrimination when the N states have been prepared with equal *a priori* probabilities. In the last section, we summarize our results.

II. N EQUIDISTANT STATES

The modulus of the overlap of two normalized states is a measure of the separation or distance between them [20]. The separation goes from 0 to 1, where 0 corresponds to orthogonal states and 1 refers to parallel states. Two nonparallel states, i.e., whose separation is less than 1, are always LI. For more than two states, the fact that their distances are different from 1 does not guarantee that they are linearly independent. Thus, an interesting question arises in this issue: given the set $\mathtt{A}_N(\alpha)$ of N normalized and equidistant states,

$$\mathtt{A}_N(\alpha) \doteq \{|\alpha_1\rangle,|\alpha_2\rangle,\ldots,|\alpha_N\rangle : \langle\alpha_k|\alpha_{k'}\rangle = \alpha,\ \forall\ k<k'\},$$

what are the α overlaps for which $\mathtt{A}_N(\alpha)$ is a LI or a LD set?

The set $\mathtt{A}_N(\alpha)$ is LI if and only if the equation

$$\sum_{k=1}^{N} A_k|\alpha_k\rangle = \mathbf{null} \tag{1}$$

implies that the N unknown coefficients $A_k = 0$, otherwise it is a LD set [19]. This criteria is equivalent to considering the Gram determinant

$$\mathcal{D}_{N\times N} = \det\begin{pmatrix} 1 & \alpha & \alpha & \cdots & \alpha \\ \alpha^* & 1 & \alpha & \cdots & \alpha \\ \alpha^* & \alpha^* & 1 & \cdots & \alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha^* & \alpha^* & \alpha^* & \cdots & 1 \end{pmatrix}_{N\times N} \tag{2a}$$

$$= \frac{\alpha(1-\alpha^*)^N - \alpha^*(1-\alpha)^N}{\alpha-\alpha^*}, \tag{2b}$$

which is higher than or equal to zero. $\mathcal{D}_{N\times N}$ is not zero for all LI sets and is equal to zero for each LD set [19].

We recall that $\mathcal{D}_{N\times N} > 0$ is a real hypervolume in the Hilbert space defined by the N equidistant states. $\mathcal{D}_{N\times N} = 0$ corresponds to a hypersurface, which means that at least one state can be written as a superposition of the other ones.

First, we consider $\alpha = x$ real. It is easy to show from Eq. (2b) that, in this case, the N states $\{|\alpha_k\rangle\}$ are LI if and only if

$$-\frac{1}{N-1} < x < 1.$$

For $x = 1$, the N states become only one and for $x = -1/(N-1)$ the N states are LD and form a symmetric structure in a $(N-1)$-dimensional subspace. The range $-1 \leqslant x < -1/(N-1)$ is not accessible for N equidistant states. For instance, three equidistant states are LD for $x = -1/2$, which means that they lie on a 2-dimensional plane on which they are *separated* by an angle of $2\pi/3$. That family of three states is the well-known trine set [14,16]. Four equally separated states become LD for $x = -1/3$, i.e., they are at a 3-dimensional subspace and form a tetrahedron.

Now, we consider the imaginary case $\alpha = iy$. From Eq. (2b), we find that $\mathtt{A}_N(iy)$ is a LI set for

$$-\frac{\sin\frac{\pi}{2N}}{\cos\frac{\pi}{2N}} < y < \frac{\sin\frac{\pi}{2N}}{\cos\frac{\pi}{2N}}.$$

The extreme values $\pm\sin(\pi/2N)/\cos(\pi/2N)$ correspond to two sets of N equidistant LD states. The range $\sin(\pi/2N)/\cos(\pi/2N) < |y| \leqslant 1$ is not accessible.

In general, the overlap $\alpha = |\alpha|e^{i\theta}$ of N equidistant LI states satisfies the constraint

$$0 \leqslant |\alpha| < \left|\langle\alpha_k^{LD}(\theta)|\alpha_{k'}^{LD}(\theta)\rangle\right|, \quad (3)$$

with

$$\left|\langle\alpha_k^{LD}(\theta)|\alpha_{k'}^{LD}(\theta)\rangle\right| = \frac{\sin\frac{\pi-\theta}{N}}{\sin\left(\theta + \frac{\pi-\theta}{N}\right)}. \quad (4)$$

This function must be evaluated with θ inside the interval $[0,2\pi]$ only; we mean that if $\theta = -\pi/4$, then the Eq. (4) function has to be evaluated at $\theta = 7\pi/4$. We obtain the previous results for $\theta = 0$, $\theta = \pi$ (α real), and $\theta = \pm\pi/2$ (α imaginary) by means of a limit process. On the contour defined by Eq. (4), the N equidistance states are LD. The LI allowed equidistant states have overlap α, which is inside the region defined by Eqs. (3) and (4). In other words, each point inside the Eq. (4) contour represents a LI set and each point on the contour represents a LD set. Notice that for a given phase θ there are an infinite number of LI sets $\mathtt{A}_N(|\alpha|e^{i\theta})$ and only one LD set $\{|\alpha_k^{LD}(\theta)\rangle, k = 1,2,\ldots N\}$. Figure 1 shows with grey-degradation the allowed region in the *Argand plane* for different values of N: (a) 3, (b) 7, (c) 11, and (d) 31. The white means the inaccessible values of α. It is worth emphasizing that on a radius (θ fixed) the modulus of the overlaps go from zero up to the maximum allowed value of Eq. (4) where the N states become LD. These LD states lie symmetrically on a $(N-1)$-dimensional subspace. We can notice that the allowed surface in the *Argand plane* decreases as N increases, being a circle for $N = 2$, petal-shaped for $N > 2$, and a narrow petal around the positive real axis for $N \ggg 2$.

These features are illustrated in the following example. By choosing an arbitrary orthonormal basis $\{|0\rangle,|1\rangle,|2\rangle,|3\rangle\}$ in a four-dimensional Hilbert space, a set of four equidistant states can be represented as follows:

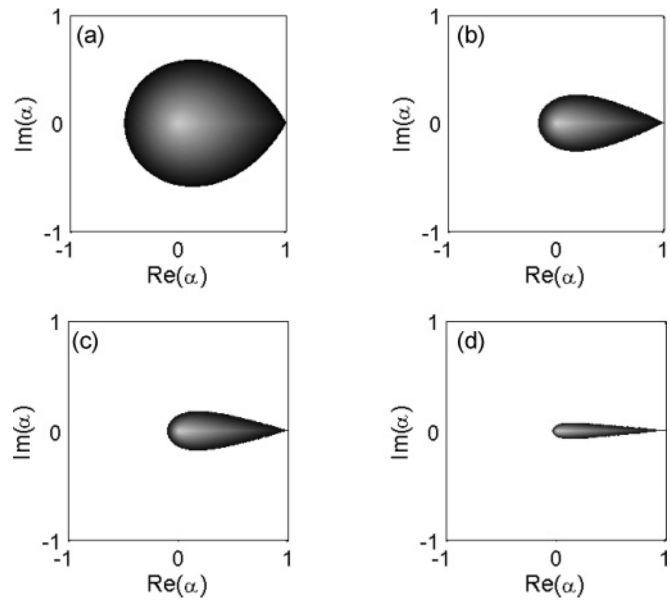

FIG. 1. *Argand plane*: grey-degradation represents the allowed values of α and white means the inaccessible values of α for N equidistant states. Specifically, for (a) $N = 3$, (b) $N = 7$, (c) $N = 11$, and (d) $N = 31$. The linear grey degradation represents the value of the success probability of Eq. (8) going from 1 (white color at $|\alpha| = 0$) down to 0 (black color at $|\alpha| = |\langle\alpha_k^{LD}(\theta)|\alpha_{k'}^{LD}(\theta)\rangle|$).

$$|\alpha_1\rangle = |0\rangle$$
$$|\alpha_2\rangle = \alpha|0\rangle + \sqrt{\mathcal{D}_{2\times2}}|1\rangle$$
$$|\alpha_3\rangle = \alpha|0\rangle + \frac{\alpha - |\alpha|^2}{\sqrt{\mathcal{D}_{2\times2}}}|1\rangle + \sqrt{\frac{\mathcal{D}_{3\times3}}{\mathcal{D}_{2\times2}}}|2\rangle$$
$$|\alpha_4\rangle = \alpha|0\rangle + \frac{\alpha - |\alpha|^2}{\sqrt{\mathcal{D}_{2\times2}}}|1\rangle + \frac{\mathcal{D}_{3\times3} - (1-\alpha)\mathcal{D}_{2\times2}}{\sqrt{\mathcal{D}_{2\times2}\mathcal{D}_{3\times3}}}|2\rangle + \sqrt{\frac{\mathcal{D}_{4\times4}}{\mathcal{D}_{2\times2}\mathcal{D}_{3\times3}}}|3\rangle.$$

We see explicitly that only for $\theta \neq 0$ the four states converge to four different LD states as $|\alpha|$ goes to $|\langle\alpha_k^{LD}|\alpha_{k'}^{LD}\rangle| < 1$. For $\theta = 0$, the four states converge to the $|0\rangle$ single state as α goes to $|\langle\alpha_k^{LD}|\alpha_{k'}^{LD}\rangle| = 1$.

It is important to point out that a nonorthogonal equidistant basis is a generalization of a equidistant orthogonal one. On the other hand, a nonorthogonal equidistant basis becomes a set of symmetric states only for α real because the elements $\langle\alpha_k|\alpha_{k'}\rangle$ form a circulant matrix [21].

III. PROBABILITY FOR UNAMBIGUOUS DISCRIMINATION

Let us suppose that the system of interest is prepared in the state $|\alpha_k\rangle$ with *a priori* probability p_k. The $|\alpha_k\rangle$ belong to $\mathtt{A}_N(\alpha)$, i.e., they are a set of equidistant states. In order to discriminate the state in which the system of interest was prepared, we couple it to an auxiliary system by means of a joint unitary operation $\hat{U}$. We consider that the ancillary system is initially in a known and normalized state $|\Lambda\rangle_a$ and assume that $\hat{U}$ transforms as follows:

$$\hat{U}|\alpha_k\rangle|\Lambda\rangle_a = \sqrt{1-|s_k|^2}|k\rangle|\perp\rangle_a + s_k\big|\alpha_k^{LD}\big\rangle|\vdash\rangle_a, \quad (5)$$

where $\{|\perp\rangle_a,|\vdash\rangle_a\}$ is an orthonormal basis of the auxiliary system. The set $\{|k\rangle\}_{k=1,2,\ldots,N}$ is an orthogonal basis of the unambiguous subspace of the system of interest. The $\{|\alpha_k^{LD}\rangle\}$ are N LD states lying on the ambiguous subspace of the system of interest. In this transformation, the $\{|\alpha_k^{LD}\rangle\}$ set preserves the phase θ and the s_k probability amplitudes preserve the modulus of the overlap. The unambiguous state discrimination can be achieved probabilistically by performing a von Neumann measurement on the auxiliary system. Thus, the system of interest is mapped onto the unambiguous subspace with probability of success

$$P_s = 1-\sum_{k=1}^{N} p_k|s_k|^2, \quad (6a)$$

$$= 1-|s|^2 \quad (6b)$$

and on the ambiguous subspace with probability $P_f = 1 - P_s$. In the equality Eq. (6b), we have considered that the possible states are prepared with equal *a priori* probabilities $p_k = 1/N$. In this case, due to the symmetry of the equidistant states, the s_k probability amplitudes do not depend on k.

Since the overlap is preserved under unitary transformations, we get from Eq. (5),

$$\langle\alpha_k|\alpha_{k'}\rangle = |s|^2\big\langle\alpha_k^{LD}(\theta)\big|\alpha_{k'}^{LD}(\theta)\big\rangle,$$

$$|\alpha|e^{\pm i\theta} = |s|^2\big|\big\langle\alpha_k^{LD}(\theta)\big|\alpha_{k'}^{LD}(\theta)\big\rangle\big|e^{\pm i\theta},$$

with minus sign for $k < k'$ and plus for $k > k'$. From here we obtain

$$|s|^2 = \frac{|\alpha|}{\big|\big\langle\alpha_k^{LD}(\theta)\big|\alpha_{k'}^{LD}(\theta)\big\rangle\big|}. \quad (7)$$

Inserting this expression of $|s|$ and Eq. (4) into Eq. (6b), we obtain the probability of success for discriminating unambiguously the prepared equidistant state:

$$P_s = 1-\frac{|\alpha|}{|\langle\alpha_k^{LD}(\theta)|\alpha_{k'}^{LD}(\theta)\rangle|},$$

$$= 1-|\alpha|\frac{\sin\left(\theta+\frac{\pi-\theta}{N}\right)}{\sin\left(\frac{\pi-\theta}{N}\right)}. \quad (8)$$

In the case of $N = 2$, the success probability becomes the Peres formula $P_s = 1 - |\alpha|$ [3]. The Peres formula is also obtained for all N when $\theta = 0$. We also note that the success probability depends on the phase θ for all $N > 2$ and on the modulus $|\alpha|$ for all N. Specifically, P_s is linear with respect to $|\alpha|$ having its maximum value 1 at $|\alpha| = 0$ and its minimum value, 0, at $|\alpha| = |\langle\alpha_k^{LD}(\theta)|\alpha_{k'}^{LD}(\theta)\rangle|$, the petal contour. This is in agreement with the fact that orthogonal states can be discriminated with certainty and a set of LD states cannot be unambiguously discriminated. These characteristics are illustrated in the grey degradation inside the petal shape of Fig. 1. In Fig. 2, the left panel shows such linear behavior of the success probability as functions of $|\alpha|$ for $N = 7$ and for different values of θ: 0 (solid), $\pi/11$ (dashes), $\pi/5$ (dots), and π (dash-dots). The right panel shows P_s as functions of θ for the different values of $|\alpha|$: $1/17$ (solid), $1/8$ (dashes), $1/5$ (dots), and $1/3$ (dash-dots). The discontinuities of the dotted and dash-dotted lines hold because these values of $|\alpha|$ and θ are outside the petal-shaped area. For $N \neq 2$, we observe that for an allowed fixed $|\alpha| \leqslant 1/(N-1)$ the phase has no restriction, whereas for $1/(N-1) < |\alpha| < |\langle\alpha_k^{LD}(\theta)|\alpha_{k'}^{LD}(\theta)\rangle|$, the range of the phase is restricted having a range of values inaccessible for N equidistant states.

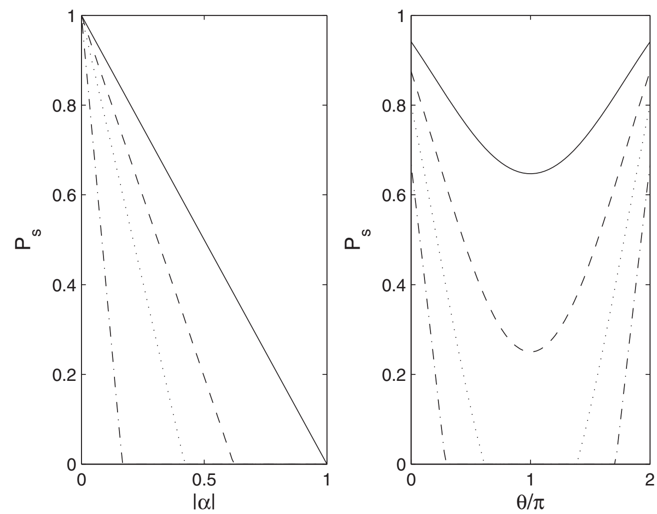

FIG. 2. Left panel: P_s success probability as a function of $|\alpha|$ for different values of the phase θ: 0(solid), $\pi/11$ (dashes), $\pi/5$ (dots), and π (dash-dots); Right panel: P_s as a function of θ for the different values of $|\alpha|$: 1/17 (solid), 1/8 (dashes), 1/5 (dots), and 1/3 (dash-dots). For both plots $N = 7$.

Here we observe that for a given modulus $|\alpha|$, the maximal probability of discriminating conclusively N (> 2) equidistant states holds for the phase $\theta = 0$. Curiously, in this case, the ambiguous subspace is 1-dimensional, since the N equidistant LD states, $\{|\alpha_k^{LD}(\theta)\rangle\}$, are equal. In other words, when $\theta = 0$, all the removable information about the prepared states is unambiguously acquired. On the other hand, since for $\theta \neq 0$, each set at the contour of the petal shape has N different equidistant LD states, then in the ambiguous subspace there is partial information about the prepared state. Even though that partial information cannot be unambiguously obtained, it can be ambiguously extracted by means of the *LD states error-minimizing measurement* protocol [13,14,16]. Thus, when partial information remains in the ambiguous subspace, the success probability is smaller than when all the information can be extracted from that subspace. We would like to point out that for performing the UQSD protocol on N equidistant states, it is required a $(2N-1)$-dimensional Hilbert space when $\theta \neq 0$ and a $(N+1)$-dimensional one when $\theta = 0$ [7].

IV. SUMMARY

We have characterized all the sets of N equidistant states finding the allowed values of the involved overlap. We find that the permitted surfaces in the *Argand plane* decrease as N increases, being a circle for $N = 2$, petal-shaped for $N > 2$, and a narrow petal around the positive real axis for $N \ggg 2$.

In addition, we studied the unambiguous discrimination of N equidistant pure quantum states, finding the probability of success in the case of equal *a priori* probabilities. The success probability depends on both the modulus and the phase of the overlap. We also find that the success probability for UQSD of N equidistant pure states reaches the maximal value when there is not information about them in the ambiguous subspace. If the phase of the overlap is different from zero and the UQSD protocol fails, then an EMMP can be applied since in this case some information remains in a set of N different equidistant LD states. In this way, for N equidistant LI pure states with phase different from zero we could apply a *complete quantum pure states discrimination* (CQPSD) scheme, which allows obtaining all the possible information about the prepared state of the system of interest. The CQPSD scheme should consist in applying first an optimal UQSD protocol and, if it fails, then an EMMP can be implemented.

Finally, we would like to emphasize that nonorthogonal equidistant states are a generalization of an equidistant orthogonal basis and, in addition to their symmetry, they are characterized by only one complex physical parameter.

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