Stark interaction of identical particles with the vacuum electromagnetic field as quantum Poisson process suppressing collective spontaneous emission

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The effective Hamiltonian describing resonant interaction of an ensemble of identical quantum particles with a photon-free vacuum electromagnetic field has been obtained with allowance for terms of second order in the coupling constant (the Stark interaction) by means of the perturbation theory on the basis of the unitary transformation of the system quantum state. It has been shown that in the Markov approximation the effective Hamiltonian terms of first order in the coupling constant are represented by the quantum Wiener process, whereas terms of second order are expressed by the quantum Poisson process. During the course of the investigation, it was established that the Stark interaction played a significant role in the ensemble dynamics, thus influencing the collective spontaneous decay of the ensemble of an appreciably high number of identical particles. Fundamental effects have been discovered, i.e., the excitation conservation in a sufficiently dense ensemble of identical particles and superradiance suppression in the collective decaying process of an excited ensemble with a determined number of particles.

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I. INTRODUCTION

The dynamics of excited atoms in a resonant broadband electromagnetic field is can be described by the following ideas. Atom interaction with a photon-free vacuum electromagnetic field will give rise to both the spontaneous transition from the excited to the ground state, with the photon being emitted, and a Lamb shift of atomic levels [1]. The spontaneous decay of an excited state sets the conditions for excited level broadening, and is effectively described by a two-level quantum particle model [2]. An ensemble of identical excited two-level quantum particles localized in a small volume radiates a coherent electromagnetic pulse with the time delay. The pulse intensity is proportional to the square of the particle number, and the pulse duration is inversely proportional to the particle number. As the number of particles in an ensemble increases, the pulse duration and its time delay decrease, with an enhanced intensity. Nowadays, this phenomenon, discovered by Dicke [3], is known as superradiance and is the subject of comprehensive investigations [2–4]. Various complicated theoretical considerations, such as propagation effects in extended samples, boundaries, geometry (shape) and nonhomogeneity of the samples, inhomogeneous linewidth broadening, radiation pattern and polarization of superradiance, fluctuations in the delay time and in other parameters of superradiant pulses from shot to shot, etc., have been verified experimentally. It is common knowledge that there is good agreement between superradiance theory and experiments. Nevertheless, all observational results obtained in the course of experimentations are certain to be related only to a narrow area of atomic densities. With an increase of atomic density, superradiance is not observed, which is supposed to be caused by the destruction of atomic coherence and noncollective spontaneous emission. Analogously, the conventional quantum superradiance theories describe relatively low atomic densities.

This paper represents an attempt to consider the quantum superradiance theory relative to the growth of atomic density. Special attention has been given to the fact that the increase in atomic density gives rise to the growth of the Stark interaction of collectively decaying atoms with vacuum. To the best of our knowledge, the Stark interaction of atoms with vacuum has not been taken into account in quantum superradiance theories as the Stark interaction of an ordinary single atom is small in comparison with the relevant Rabi energy defining the rate of quantum transition from the excited level to the ground level. In this context, the Stark interaction is a term of second order (as the Lamb shift), whereas the relevant Rabi energy is a term of first order in the expansion in the powers of coupling with a vacuum field.

The present paper focuses on the case where the low intensity of the Stark interaction with a vacuum field, which is characteristic of an ordinary atom, becomes high enough for an N-atom ensemble, as the Stark interaction ensemble operator is proportional to the number of atoms N. Therefore, the Stark interaction with a vacuum field is enhanced in the ensemble of identical atoms and can prove to be significant in the process of collective decay if the number of atoms is high enough. As a consequence, the collective spontaneous emission (superradiance) can be fully suppressed and the ensemble of excited atoms stops emitting if the number of atoms coincides with the critical number of atoms established in the paper. On the one hand, the result obtained offers a different treatment of the optimum conditions for superradiance observation, while, on the other hand, it presents a unique fundamental effect—the stabilization of an ensemble of excited atoms against the collective spontaneous decay. To some extent, this very stabilization effect is an apparent contradiction with superradiance.

Earlier investigations did not deal with the Stark interaction while solving superradiance and collective spontaneous emission problems. The present paper is devoted to analyzing the Stark interaction role played in the collective spontaneous emission in terms of the simplest model of the collective decay

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of the atomic ensemble localized in a small volume, the size of which is far smaller than the associated emission wavelength. Here, the generalization of the Dicke model on account of the Stark interaction has been proposed in this work. The resulting simple analytical formulas describing temporal dependence of various quantum state populations (fully excited state, single excited W state) and intensity profiles of the superradiance pulse from the number of ensemble atoms have been derived. The formulas bear a strong resemblance to the well-known Dicke forumals, but with only one exception—the presence of an additional non-Langevin factor, which is equal to unity in neglecting the Stark interaction and is equal to zero at a certain number of ensemble atoms, on account of the Stark interaction.

For instance, the W state decays exponentially with time $t\ \exp\{-\gamma\omega_{21}N_a t\}$ at a rate proportional to the ensemble atom number N_a [3,4], with no allowance for the Stark interaction. Here, $\gamma\omega_{21} = 2\omega_{21}^3 d_{21}^2/(3\hbar c^3)$ is the conventional constant of the electrodipole atomic decay (on no account of photon polarization) expressed through the transition frequency ω_{21} and its dipole moment d_{21} [3,4]. Accounting for the Stark interaction, the W-state decay is also exponential $\exp\{-\gamma\omega_{21}\gamma_{\mathrm{nL}}N_a t\}$, according to the present investigation. However, there has emerged an additional non-Langevin term $\gamma_{\mathrm{nL}} = 2[1-\cos(N_a\eta_{\mathrm{St}})]/(N_a\eta_{\mathrm{St}})^2$, $\eta_{\mathrm{St}} = \gamma|\Pi_1|\omega_{21}\hbar/d_{12}^2$, which is equal to zero at the ensemble atom number N_a, satisfying the condition $N_a\eta_{\mathrm{St}} = 2\pi n$, $n = 1,2,\ldots$, where Π_1 is a conventional parameter of optical resonance theory [5] characterizing the Stark interaction of the atom in the ground state. One can appreciate the value Π_1 with the help of the familiar presentation $|\Pi_1| \sim d_{\Gamma 1}^2/(\hbar\Delta)$, where $\Delta \ll \omega_{21}$ is the frequency detuning from the quasiresonance atomic level with the transition dipole moment $d_{\Gamma 1}$ [5]. Then the non-Langevin factor γ_{nL} is defined by the value $N_a\eta_{\mathrm{St}} \sim N_a\gamma\omega_{21}/\Delta$. It is necessary to take the quantity $\Delta/(\gamma\omega_{21})$ as being no less than 10 for an optimum estimation, since the frequency detuning Δ from the quasiresonance atomic level must be greater than the homogeneous width $\gamma\omega_{21}$ of the quasiresonant level. Then the critical number of excited atoms can appear to be of the order of 100. For a wave band $\lambda \sim 0.5$–$10\ \mu$m (from the middle visible range to the middle infrared range), it corresponds to an atomic density of the order of 10^{18}–10^{14} atoms/cm^3. Thus, appropriate requirements for the experimental research of the discovered effect differ from those for superradiance [4]. For the discovered effect to be observed, it is necessary to involve the atomic ensemble with a greater atomic density than that for a supperradiance observation.

The suppression of superradiance described in the present paper does not contradict the well-known experimental data [4], and gives a unique interpretation for the existence of the optimal range of atomic densities for the superradince to be observed. As a rule, the superradiance is observed in extended samples with high Fresnel numbers. For these cases, the superradiant effect is revealed on the background of noncoherent losses and noncollective relaxation. As the atomic density increases, the superradiant effect exceeds the noncoherent losses and noncollective relaxation, as a result of which superradiance is observed in an experimental way. A further rise in atomic density results in the disappearance of superradiance, which was conventionally attributed to the growth of the noncollective relaxation destroying the system coherence. It does not sound convincing because all effects, such as the superradiant effect (on no account of the Stark interaction) and coherent relaxation are proportional to the atomic density, might not influence greatly the interrelation between each other. In addition, coherence relaxation due to dephasing does not affect the rates of quantum transitions at all. Supposing that superradiance is suppressed by the Stark interaction, the enhancement of the noncoherent relaxation becomes obvious, thus causing the disappearance of superradiance in extended samples while the atomic density is increased.

To analyze the impact of the Stark interaction on the dynamics of an identical atom ensemble much effort should be made. First, it is due to fact that even the low-intensity Stark interaction of a single atom is enhancing, and for a sufficient number of ensemble atoms it becomes significantly high. The average value of the Stark interaction with a vacuum photon-free field is equal to zero. This is only the term of first order of the perturbation series, though. Evidently, to obtain the additional non-Langevin factor γ_{nL} in the above form, it is necessary to summarize all terms of the perturbation series, although it is not known so far how to realize this summation in terms of traditional techniques [6–8]. In addition, a rigorous account of the Stark interaction and its estimation requires that a general atom model having an arbitrary number of levels rather than a two-level model is analyzed, in contrast to all previous investigations of superradiance, dating back to Dicke's work [3]. Therefore, for the simplest Dicke model, the account of the Stark interaction leads to the involvement of rigorous mathematical methods of quantum theory.

The results shown in the paper have been obtained on the basis of the derived effective Hamiltonian with the related non-Langevin quantum stochastic differential equation (QSDE). The Stark interaction is shown to be presented by the quantum Poisson process, which is responsible for the non-Langevin type of the QSDE. Therefore, the superradiance allowing for the Stark interaction with an electromagnetic vacuum is to be referred to as non-Langevin superradiance, in contrast to the Dicke superradiance [2–4], which is of Langevin type.

The effective Hamiltonian picture allows to account for all energy levels of collectively decaying atoms and making the above-mentioned numerical estimation of the number of atoms, at which the collective spontaneous decay comes to a halt. The representation of the Stark interaction as a quantum Poisson process as being analogous to the one considered for a single quantum particle in Ref. [9] has provided the effective summation of the mentioned perturbation series due to the Hudson-Parthasarathy algebra [10] for increments of the quantum stochastic processes in the photon-free vacuum electromagnetic field.

The derivation of master equations can be made by various methods (see Refs. [6–8]), whereas the QSDE method [5,7,9–14] is not only the most elegant and straightforward, but is also an integral element of a mathematically correct description of the open systems of a particular class. (The spontaneously emitting excited atoms give an important example of open systems). The paper emphasizes the important role played by the effective Hamiltonian picture of an open system for a consistent analysis of the system dynamics.

Conventionally, the derivation of the master equation by the QSDE method was based on the Hamiltonian in the rotating frame approximation [11–13]. Such a Hamiltonian presented the initial electrodipole interaction Hamiltonian without fast oscillating terms in the Dirac picture. The indirect application of the QSDE method to the initial electrodipole interaction Hamiltonian gives rise to an unexpected and contradictory observational result [15]. The relaxation went missing in that case as if the two-level excited atom had not undergone any radiative decay. Thus, in the QSDE method, the problem of an effective Hamiltonian arises. The basic assumptions regarding the interaction of an open system with its environment must not be applied to any Hamiltonian (including any general and exact Hamiltonian) but to an effective Hamiltonian. The systematic principle of the effective Hamiltonian derivation as well as its applicability has been formulated in the present paper. Such an approach allows making a straightforward derivation of not only the rotating frame approximation Hamiltonian but to that of a basic term describing the Stark interaction of quantum particles with a broadband quantized electromagnetic field responsible for the non-Langevin type of superradiance. The very same approach imposes restrictions on further application of the master equation in the process of investigating an open system dynamics, which were neglected in several previous works, leading to incorrect results.

Our consideration of the Stark interaction can prove to be useful for solving other similar problems in which it is necessary to take into account the operator represented by the quantum Poisson process.

The paper is organized as follows. In Sec. II the unitary Hamiltonian transformation, together with the perturbation theory, is used to derive the effective Hamiltonian and to introduce the effective Hamiltonian picture. In Sec. III the QSDE notion is involved for the Markov approximation, and the non-Langevin evolution operator is derived for a photon-free quantized electromagnetic field. Section IV is devoted to the derivation of the master equation and its representation in the Lindblad form. Section V considers the spontaneous decay of a singly excited ensemble that is symmetrical over particle permutation. Section VI is concerned with the peculiarities of the non-Langevin collective spontaneous decay of the fully excited ensemble, and the introduction of critical values at which the atomic ensemble emission is fully suppressed. Section VII considers the effective Hamiltonian picture and the QSDE as the basis of a systematic investigation of open systems, including physical systems containing fast and slow subsystems.

II. EFFECTIVE HAMILTONIAN PICTURE OF THE PROBLEM

Let us consider an ensemble of N_a identical motionless atoms interacting with a quantized broadband electromagnetic field. The atoms are localized near a point $\vec{r} = 0$ in a small volume, the size of which is much smaller than the characteristic wavelength of an electromagnetic field. An initial Hamiltonian of such a system in an electrodipole approximation

$$H^{\mathrm{Ini}} = H^A + H^F + H^{\mathrm{Int}} \tag{1}$$

consists of the sum of the Hamiltonian H^A of isolated atoms, the electromagnetic field Hamiltonian H^F, and an interaction operator of atoms with an electromagnetic field H^{Int},

$$\begin{aligned} H^A &= \sum_{i,j} E_j |E_j\rangle^{(i)} \langle E_j|^{(i)}, \qquad H^F = \sum_{\vec{q}} \hbar\omega_{\vec{q}} b^\dagger_{\vec{q}} b_{\vec{q}}, \\ H^{\mathrm{Int}} &= \sum_{\vec{q}} \Gamma_{\vec{q}} (b^\dagger_{\vec{q}} + b_{\vec{q}}) \sum_{i,kj} d_{kj} |E_k\rangle^{(i)} \langle E_j|^{(i)}, \\ \sum_j |E_j\rangle^{(i)} \langle E_j|^{(i)} &= 1^{(i)}, \quad \langle E_j|^{(i)} |E_k\rangle^{(i)} = \delta_{jk}, \end{aligned} \tag{2}$$

where $|E_j\rangle$ and $\langle E_j|$ are the ket and bra vectors describing atomic quantum nondegenerate state of energy E_j, and $d_{kj} = \langle E_k|d|E_j\rangle$ is the matrix element of the atomic dipole moment operator $d = \sum_{kj} d_{kj} |E_k\rangle\langle E_j|$. Atomic states are characterized by a definite parity, so that $\langle E_k|d|E_k\rangle = 0$. The upper index of the state vectors designates the state space of an ith atom; the sum is over all ensemble atoms. Annihilation and creation operators of photons with wave vector $\vec{q}$ and frequency $\omega_{\vec{q}}$ are given by $b_{\vec{q}}$ and $b^\dagger_{\vec{q}}$, $[b_{\vec{q}}, b^\dagger_{\vec{q}'}] = \delta_{\vec{q}\vec{q}'}$, $\omega_{\vec{q}} = qc$. The atomic coupling with a conventional three-dimensional electromagnetic field is characterized by the value $\Gamma_{\vec{q}} = (2\pi\hbar qc/\ell^3)^{1/2}$, with ℓ^3 as the quantization volume. The recoil effects and polarization photon states are neglected. The dipole-dipole interaction of identical atoms is neglected, analogous to the conventional theory of superradiance [2–4]. The justification of the electrodipole approximation for a single two-level atom can be found in Ref. [1].

The state vector $|\Psi(t)\rangle$ of the system containing atoms and a quantized electromagnetic field in the interaction (Dirac) picture obeys the Schrödinger equation

$$\begin{aligned} i\hbar \frac{d}{dt} |\Psi(t)\rangle &= H^{\mathrm{Int}}(t) |\Psi(t)\rangle, \\ |\Psi(t)\rangle &= \exp[i(H^A + H^F)t/\hbar] |\Psi\rangle, \\ H^{\mathrm{Int}}(t) &= \exp[i(H^A + H^F)t/\hbar] H^{\mathrm{Int}} \exp[-i(H^A + H^F)t/\hbar] \\ &= \sum_{\vec{q}} \Gamma_{\omega_{\vec{q}}} (b^\dagger_{\vec{q}} e^{i\omega_{\vec{q}} t} + b_{\vec{q}} e^{-i\omega_{\vec{q}} t}) \\ &\quad \times \sum_{i,kj} d_{kj} |E_k\rangle^{(i)} \langle E_j|^{(i)} e^{i\omega_{kj} t}, \\ \omega_{kj} &= (E_k - E_j)/\hbar, \end{aligned} \tag{3}$$

where $|\Psi\rangle$ is the system state vector in the Schrödinger picture.

According to unitary symmetry of quantum theory, let us make a unitary transformation,

$$|\tilde{\Psi}(t)\rangle = U(t) |\Psi(t)\rangle. \tag{4}$$

The transition from vector $|\Psi(t)\rangle$ toward vector (4) is accompanied with the Hamiltonian change

$$\tilde{H}^{\mathrm{Int}}(t) \equiv \mathbf{T}[H^{\mathrm{Int}}(t)] = U(t) H^{\mathrm{Int}}(t) U^\dagger(t) - i\hbar U(t) \frac{d}{dt} U^\dagger(t). \tag{5}$$

Here, the description of the quantum system is expressed with the help of the Schrödinger equation with a transformed Hamiltonian (5):

$$i\hbar \frac{d}{dt} |\tilde{\Psi}(t)\rangle = \tilde{H}^{\mathrm{Int}}(t) |\tilde{\Psi}(t)\rangle. \tag{6}$$

Consider a unitary operator $U(t)$ in terms of the Hermitian operator

$$U(t) = e^{-iS(t)}, \quad S^{\dagger}(t) = S(t), \tag{7}$$

in order to use the Baker-Hausdorff formula for an arbitrary operator O,

$$e^{-iS} O e^{iS} = O + \frac{(-i)}{1!}[S,O] + \frac{(-i)^2}{2!}[S,[S,O]] + \frac{(-i)^3}{3!}\{[S,[S,O]]\} + \cdots.$$

A transformed Hamiltonian (5) and operator $S(t)$ are expanded in a series of the coupling constant

$$S(t) = S^{(1)}(t) + S^{(2)}(t) + \cdots, \quad \tilde{H}^{\mathrm{Int}}(t) = \tilde{H}^{\mathrm{Int}(1)}(t) + \tilde{H}^{\mathrm{Int}(2)}(t) + \cdots, \tag{8}$$

where the upper index signifies the expansion order of the coupling constant. Substituting (7) and (8) into (5) with an account of the Baker-Hausdorff formula and equating the expression of the same order, we have

$$\tilde{H}^{\mathrm{Int}(1)}(t) = H^{\mathrm{Int}}(t) + \hbar\, d\, S^{(1)}(t)/dt, \tag{9}$$

$$\tilde{H}^{\mathrm{Int}(2)}(t) = -\frac{i}{2}[S^{(1)}(t), H^{\mathrm{Int}}(t)] - \frac{i}{2}[S^{(1)}(t), \tilde{H}^{\mathrm{Int}(1)}(t)] + \hbar \frac{dS^{(2)}(t)}{dt}. \tag{10}$$

Expansion (8) and formulas (9) and (10) define the unitary transformations (4)–(8) in a unique way by requiring the absence of fast time-varying factors in the relevant terms in the interaction picture. We characterize this transformation as the transition to an effective Hamiltonian picture. The latter, which is similar to Heisenberg and Dirac (interaction) pictures, is closed because the repetitive (or nth fold) unitary transformations $\mathbf{T}$ leave an effective Hamiltonian "fixed," $\tilde{H}^{\mathrm{Int}}(t) = \mathbf{T}\{\tilde{H}^{\mathrm{Int}}(t)\}$, since it is a fixed point of sequential identical unitary transformations $\mathbf{T}$.

Note that the unitary transition to an effective Hamiltonian picture can be made in the Schrödinger picture as well. Then the basic transformation formulas will be the following:

$$|\tilde{\Psi}\rangle = U|\Psi\rangle, \quad \tilde{H} = U H^{\mathrm{Ini}} U^{\dagger} - i\hbar U \frac{d}{dt} U^{\dagger},$$

$$i\hbar \frac{d}{dt}|\tilde{\Psi}\rangle = \tilde{H}|\tilde{\Psi}\rangle, \quad U = e^{-iS}, \quad S^{\dagger} = S,$$

$$S = S^{(1)} + S^{(2)} + \cdots, \quad \tilde{H} = \tilde{H}^{(0)} + \tilde{H}^{(1)} + \tilde{H}^{(2)} + \cdots,$$

$$\tilde{H}^{(0)} = H^A + H^F, \quad \tilde{H}^{(1)} = H^{\mathrm{Int}} - i[S^{(1)}, \tilde{H}^{(0)}] + \hbar\, dS^{(1)}/dt,$$

$$\tilde{H}^{(2)} = -\frac{i}{2}[S^{(1)}, H^{\mathrm{Int}}] - \frac{i}{2}[S^{(1)}, \tilde{H}^{(1)}] - i[S^{(2)}, \tilde{H}^{(0)}] + \hbar \frac{dS^{(2)}}{dt}. \tag{11}$$

Here, the key principle of term determination in a transformed Hamiltonian $\tilde{H}$ is the presence of appropriate fast time-varying factors in the relevant terms (escaping due to the transition to the interaction picture) in nondiagonal matrix elements of an effective Hamiltonian. The above-mentioned Hamiltonian pictures are equivalent to each other in view of the relation $\tilde{H}^{(0)} = H^A + H^F$. The unitary transformation

$$\exp[i(H^A + H^F)t/\hbar]\tilde{H} \exp[-i(H^A + H^F)t/\hbar] = \tilde{H}^{\mathrm{Int}}(t) + H^A + H^F,$$

$$\exp[i(H^A + H^F)t/\hbar] e^{-iS} \exp[-i(H^A + H^F)t/\hbar] = e^{-iS(t)},$$

realizes the indicated equivalence.

An effective Hamiltonian (5), (6), (8)–(10) [and (11)] is diagonal in the absence of any resonance. Resonant conditions of an atom-field interaction reduce an effective Hamiltonian (5) and the Schrödinger equation (6) to the closed system of an equation describing the interaction of an electromagnetic field with resonant atomic levels alone.

The peculiarities of unitary transformations (4)–(8) and (11) for the case of quantum particle interactions with classical electromagnetic fields are described in Ref. [5]. For the quantum electromagnetic field, the method was used in the author's works [5,16,17] for different conditions other than those considered in the present paper. In these works the conception of the relevant terms and the elimination of fast time-varying factors from them are illustrated with a great number of examples. The analogous method was used by Van Vleck [18] and described in textbooks [19–21]. In nonlinear optics the unitary transformation of the quantum states of the system under investigation has been practically applied, beginning with Takatsutji's works [22]. The method of specifying the unitary transformation in Refs. [18–22] differs from the above stated one. To the best of our knowledge, the "closure" property of an effective Hamiltonian picture has not been discussed up to now. The mathematical background for the perturbation theory for ordinary differential equations based on the transformation method similar to the above stated ones was developed in Ref. [23].

Assume that the electromagnetic field does not have any photons. Atoms can populate either a ground (lower) energy level $|E_1\rangle$ or an excited (upper) energy level $|E_2\rangle$, thus forming an optically allowed transition $E_2 \to E_1$. The interaction with the electromagnetic field causes transitions from the excited level to the ground one only. The characteristic frequency of the electromagnetic field of such processes is determined by the frequency ω_{21} of the indicated transition $E_2 \to E_1$, $\omega_{21} = (E_2 - E_1)/\hbar$. Therefore, a resonant interaction arises between atoms and the electromagnetic field. For the resonant interaction, the operator $H^{\mathrm{Int}}(t)$ in the interaction picture only has the following slow time-varying terms which determine $\tilde{H}^{\mathrm{Int}(1)}(t)$:

$$\tilde{H}^{\mathrm{Int}(1)}(t) = \sum_{i,\vec{q}} \Gamma_{\vec{q}} b_{\vec{q}}^{\dagger} d_{12} e^{i(\omega_{\vec{q}} - \omega_{21})t} |E_1\rangle^{(i)} \langle E_2|^{(i)} + \sum_{i,\vec{q}} \Gamma_{\vec{q}} b_{\vec{q}} d_{21} e^{-i(\omega_{\vec{q}} - \omega_{21})t} |E_2\rangle^{(i)} \langle E_1|^{(i)}. \tag{12}$$

This equation allows to write down an equation for the operator $S^{(1)}(t)$ following from Eq. (9) and containing fast time-varying terms with factors $e^{\pm i(\omega_{\vec{q}} + \omega_{21})t}$ eliminated from $\tilde{H}^{\mathrm{Int}(1)}(t)$. We solve this equation by making a conventional

assumption that the "atom-field" interaction is switched on adiabatically, and we find the operator $S^{(1)}(t)$ in the form

$$S^{(1)}(t) = i\sum_{\vec{q}}\Gamma_{\vec{q}}b^{\dagger}_{\vec{q}}\sum_{i,kj}{}'|E_k\rangle^{(i)}\langle E_j|^{(i)}\frac{e^{i(\omega_{\vec{q}}+\omega_{kj})t}d_{kj}}{\hbar(\omega_{\vec{q}}+\omega_{kj})} - i\sum_{\vec{q}}\Gamma_{\vec{q}}b_{\vec{q}}\sum_{i,kj}{}'|E_k\rangle^{(i)}\langle E_j|^{(i)}\frac{e^{-i(\omega_{\vec{q}}-\omega_{kj})t}d_{kj}}{\hbar(\omega_{\vec{q}}-\omega_{kj})}. \tag{13}$$

Formula (13) contains both resonant and nonresonant atomic states. The prime sign in the sum means the absence of resonant denominators $\omega_{\vec{q}} - \omega_{21}$.

Substituting $S^{(1)}(t)$ and $\tilde{H}^{(1)}(t)$ into (10) and retaining only slow time-varying terms in the commutators, we obtain $\tilde{H}^{\mathrm{Int}(2)}$ in the form

$$\begin{aligned}\tilde{H}^{\mathrm{Int}(2)}(t) = {}&\sum_{\vec{q}}\Gamma_{\vec{q}}b^{\dagger}_{\vec{q}}\sum_{\vec{q}'}\Gamma_{\vec{q}'}b_{\vec{q}'}e^{-i(\omega_{\vec{q}'}-\omega_{\vec{q}})t}\\ &\times\sum_{i,k}\tfrac{1}{2}[\Pi_k(\omega_{\vec{q}})+\Pi_k(\omega_{\vec{q}'})]|E_k\rangle^{(i)}\langle E_k|^{(i)}\\ &+\sum_{\vec{q}}\Gamma^2_{\vec{q}}\sum_{i,kj}\frac{|d_{kj}|^2}{\hbar(\omega_{kj}-\omega_{\vec{q}})}|E_k\rangle^{(i)}\langle E_k|^{(i)}\\ &-\sum_{\vec{q}}\Gamma^2_{\vec{q}}\sum_{i\neq i',kj}{}'|E_k\rangle^{(i)}\langle E_j|^{(i)}|E_j\rangle^{(i')}\langle E_k|^{(i')}\\ &\times\frac{|d_{kj}|^2}{\hbar(\omega_{\vec{q}}-\omega_{kj})},\end{aligned} \tag{14}$$

where the conventional parameters of the optical resonance theory are introduced [5],

$$\Pi_k(\omega) = \sum_j\frac{|d_{kj}|^2}{\hbar}\left(\frac{1}{\omega_{kj}+\omega}+\frac{1}{\omega_{kj}-\omega}\right).$$

The rest of the terms in formula (10) [after separating $\tilde{H}^{\mathrm{Int}(2)}(t)$ in the form (14)] define the operator $S^{(2)}(t)$. However, the operator $S^{(2)}(t)$ is not required and omitted in the following.

The operator (14) is the sum of three operators,

$$\tilde{H}^{\mathrm{Int}(2)}(t) = H^{\mathrm{Stark}}(t) + H^{\mathrm{Lamb}} + V^{\mathrm{Ex}},$$

$$\begin{aligned}H^{\mathrm{Stark}}(t) = {}&\sum_{\vec{q}}\Gamma_{\vec{q}}b^{\dagger}_{\vec{q}}\sum_{\vec{q}'}\Gamma_{\vec{q}'}b_{\vec{q}'}e^{-i(\omega_{\vec{q}'}-\omega_{\vec{q}})t}\\ &\times\sum_{i,k}\frac{1}{2}[\Pi_k(\omega_{\vec{q}})+\Pi_k(\omega_{\vec{q}'})]|E_k\rangle^{(i)}\langle E_k|^{(i)},\end{aligned}$$

$$\begin{aligned}H^{\mathrm{Lamb}} = {}&\sum_{\vec{q}}\Gamma^2_{\vec{q}}\sum_{i,kj}\frac{|d_{kj}|^2}{\hbar(\omega_{kj}-\omega_{\vec{q}})}|E_k\rangle^{(i)}\langle E_k|^{(i)},\\ V^{\mathrm{Ex}} = {}&-\sum_{\vec{q}}\Gamma^2_{\vec{q}}\sum_{i\neq i',kj}{}'|E_k\rangle^{(i)}\langle E_j|^{(i)}|E_j\rangle^{(i')}\langle E_k|^{(i')}\\ &\times\frac{|d_{kj}|^2}{\hbar(\omega_{\vec{q}}-\omega_{kj})}.\end{aligned}$$

The first one, H^{Lamb}, is regarded as the Lamb operator. It describes the Lamb level shifts of a single atom [1]. The second operator, $H^{\mathrm{Stark}}(t)$, is referred to as the Stark interaction operator. The third operator, $V^{\mathrm{Ex}}(t)$, describes the interaction between atoms of dipole-dipole type with an excitation exchange.

The Lamb operator is diagonal and can be eliminated by a unitary transformation,

$$|\tilde{\tilde{\Psi}}(t)\rangle = \exp(iH^{\mathrm{Lamb}}t/\hbar)|\tilde{\Psi}(t)\rangle, i\hbar\frac{d}{dt}|\tilde{\tilde{\Psi}}(t)\rangle = \tilde{\tilde{H}}^{\mathrm{Int}}(t)|(t)\rangle,$$

$$\begin{aligned}\tilde{\tilde{H}}^{\mathrm{Int}}(t) = {}&\exp(iH^{\mathrm{Lamb}}t/\hbar)\tilde{H}^{\mathrm{Int}}(t)\exp(-iH^{\mathrm{Lamb}}t/\hbar)\\ &-i\hbar\exp(iH^{\mathrm{Lamb}}t/\hbar)\frac{d}{dt}\exp(-iH^{\mathrm{Lamb}}t/\hbar)\\ = {}&\exp(iH^{\mathrm{Lamb}}t/\hbar)\tilde{H}^{\mathrm{Int}}(t)\exp(-iH^{\mathrm{Lamb}}t/\hbar) - H^{\mathrm{Lamb}},\\ \tilde{H}^{\mathrm{Int}}(t) = {}&\tilde{H}^{\mathrm{Int}(1)}(t)+\tilde{H}^{\mathrm{Int}(2)}(t).\end{aligned}$$

We now rewrite the effective Hamiltonian in terms of resonant and nonresonant atomic levels, and present it as the sum of four operators,

$$\begin{aligned}\tilde{\tilde{H}}^{\mathrm{Int}}(t) = {}&\sum_{i,\vec{q}}\Gamma_{\vec{q}}b^{\dagger}_{\vec{q}}d_{12}e^{i(\omega_{\vec{q}}-\omega'_{21})t}\,|E_1\rangle^{(i)}\,\langle E_2|^{(i)}\\ &+\sum_{i,\vec{q}}\Gamma_{\vec{q}}b_{\vec{q}}d_{21}e^{-i(\omega_{\vec{q}}-\omega'_{21})t}\,|E_2\rangle^{(i)}\,\langle E_1|^{(i)}\\ &+\sum_{\vec{q}}\Gamma_{\vec{q}}b^{\dagger}_{\vec{q}}\sum_{\vec{q}'}\Gamma_{\vec{q}'}b_{\vec{q}'}e^{-i(\omega_{\vec{q}'}-\omega_{\vec{q}})t}\\ &\times\sum_{i,k}\tfrac{1}{2}[\Pi_k(\omega_{\vec{q}})+\Pi_k(\omega_{\vec{q}'})]\,|E_k\rangle^{(i)}\,\langle E_k|^{(i)}+V(t)\\ = {}&H^{\mathrm{Int-TL}}(t)+H^{\mathrm{Nonres}}(t)+V^{\mathrm{TL-Ex}}+V^{\mathrm{Nonres}}.\end{aligned}$$

The first one, $H^{\mathrm{Int-TL}}(t)$, describes two resonant atomic levels and transitions between them due to the interaction with the electromagnetic field,

$$\begin{aligned}H^{\mathrm{Int-TL}}(t) = {}&\sum_{i,\vec{q}}\Gamma_{\vec{q}}b^{\dagger}_{\vec{q}}d_{12}e^{i(\omega_{\vec{q}}-\omega'_{21})t}\,|E_1\rangle^{(i)}\,\langle E_2|^{(i)}\\ &+\sum_{i,\vec{q}}\Gamma_{\vec{q}}b_{\vec{q}}d_{21}e^{-i(\omega_{\vec{q}}-\omega'_{21})t}\,|E_2\rangle^{(i)}\,\langle E_1|^{(i)}\\ &+\sum_{\vec{q}}\Gamma_{\vec{q}}b^{+}_{\vec{q}}\sum_{\vec{q}'}\Gamma_{\vec{q}'}b_{\vec{q}'}e^{-i(\omega_{\vec{q}'}-\omega_{\vec{q}})t}\\ &\times\sum_{\substack{i,\\ k=1,2}}\tfrac{1}{2}[\Pi_k(\omega_{\vec{q}})+\Pi_k(\omega_{\vec{q}'})]\,|E_k\rangle^{(i)}\,\langle E_k|^{(i)}.\end{aligned}$$

The second operator, $H^{\mathrm{Nonres}}(t)$, characterizes the Stark interaction of nonresonant levels,

$$\begin{aligned}H^{\mathrm{Nonres}}(t) = {}&\sum_{\vec{q}}\Gamma_{\vec{q}}b^{\dagger}_{\vec{q}}\sum_{\vec{q}'}\Gamma_{\vec{q}'}b_{\vec{q}'}e^{-i(\omega_{\vec{q}'}-\omega_{\vec{q}})t}\\ &\times\sum_{\substack{i,\\ k\neq 1,2}}\frac{1}{2}[\Pi_k(\omega_{\vec{q}})+\Pi_k(\omega_{\vec{q}'})]\,|E_k\rangle^{(i)}\,\langle E_k|^{(i)}.\end{aligned}$$

The third and fourth operators, V^{TL-Ex} and V^{Nonres}, represent terms of the atom-atom interaction operator V. The interaction between the resonant atomic states $|E_1\rangle$ and $|E_2\rangle$

is described by $V^{\mathrm{TL-Ex}}$,

$$V^{\mathrm{TL-Ex}} = -\sum_{\vec{q}} \Gamma_{\vec{q}}^2 \sum_{i\neq i'} |E_1\rangle^{(i)} \langle E_2|^{(i)} |E_2\rangle^{(i')} \langle E_1|^{(i')} \times \frac{|d_{21}|^2}{\hbar(\omega_{\vec{q}} + \omega_{21})}.$$

The rest of terms in V (after separating V^{TL-Ex}) are designated as V^{Nonres}.

We denoted ω'_{21} as the resonant transition frequency allowing for the Lamb shifts,

$$\omega'_{21} = \omega_{21} + \sum_{\vec{q}} \Gamma_{\vec{q}}^2 \sum_j \frac{|d_{2j}|^2}{\hbar^2(\omega_{2j} - \omega_{\vec{q}})} - \sum_{\vec{q}} \Gamma_{\vec{q}}^2 \sum_j \frac{|d_{1j}|^2}{\hbar^2(\omega_{1j} - \omega_{\vec{q}})}.$$

Furthermore, the prime at the resonant transition frequency will be omitted.

The accepted assumption related to allowed level populations and quantum transitions permits developing the above stated theory based only on the operator $H^{\mathrm{Int-TL}}(t) + V^{\mathrm{TL-Ex}}$.

Thus, the dynamics of the ensemble of identical atoms in the photon-free electromagnetic field is reduced to the dynamics of the ensemble of two-level identical atoms and the quantized electromagnetic field presented by the state vector $|\Psi^{\mathrm{TL}+F}(t)\rangle$, obeying the equations

$$i\hbar \frac{d}{dt}|\Psi^{\mathrm{TL}+F}(t)\rangle = \{H^{\mathrm{Tr}}(t) + H^{\mathrm{St}}(t) + V^{\mathrm{TL-Ex}}\}|\Psi^{\mathrm{TL}+F}(t)\rangle, \tag{15}$$

$$H^{\mathrm{Tr}}(t) = \sum_{\vec{q}} \Gamma_{\vec{q}} b_{\vec{q}}^\dagger d_{12} e^{i(\omega_{\vec{q}} - \omega_{21})t} R_- + \sum_{\vec{q}} \Gamma_{\vec{q}} b_{\vec{q}} d_{21} e^{-i(\omega_{\vec{q}} - \omega_{21})t} R_+,$$

$$H^{\mathrm{St}}(t) = \sum_{\vec{q}} \Gamma_{\vec{q}} b_{\vec{q}}^\dagger \sum_{\vec{q}'} \Gamma_{\vec{q}'} b_{\vec{q}'} e^{-i(\omega_{\vec{q}'} - \omega_{\vec{q}})t} \times \left\{ \Pi_+(\omega_{\vec{q}}, \omega_{\vec{q}'}) \frac{N_a}{2} + \Pi_-(\omega_{\vec{q}}, \omega_{\vec{q}'}) R_3 \right\},$$

$$V^{\mathrm{TL-Ex}} = -\sum_{\vec{q}} \frac{\Gamma_{\vec{q}}^2 |d_{21}|^2}{\hbar(\omega_{\vec{q}} + \omega_{21})} (R_- R_+ + R_+ R_- - N_a).$$

The Stark interaction parameters $\Pi_\pm(\omega,\omega')$ were involved as in

$$\Pi_\pm(\omega,\omega') = \tfrac{1}{2}\{\Pi_2(\omega) + \Pi_2(\omega') \pm [\Pi_1(\omega) + \Pi_1(\omega')]\}.$$

The operators $R_\pm$ and R_3 take the form

$$R_3 = \frac{1}{2}\sum_i (|E_2\rangle^{(i)}\langle E_2|^{(i)} - |E_1\rangle^{(i)}\langle E_1|^{(i)}),$$

$$R_- = \sum_i |E_1\rangle^{(i)}\langle E_2|^{(i)}, \quad R_+ = \sum_i |E_2\rangle^{(i)}\langle E_1|^{(i)},$$

and obey the commutation relation of the su(2) algebra

$$[R_3, R_\pm] = \pm R_\pm, \quad [R_+, R_-] = 2R_3.$$

We will consider the initial states of two-level atoms $|\Psi_0^{\mathrm{TL}}\rangle$ and electromagnetic field $|\Psi_0^F\rangle$ to be noncorrelated with each other, $|\Psi_0^{\mathrm{TL}+F}\rangle = |\Psi_0^{\mathrm{TL}}\rangle \otimes |\Psi_0^F\rangle$. The field states corresponding to different wave vectors are also noncorrelated and are photon-free:

$$\langle\Psi_0^F|b_{\vec{q}}^\dagger b_{\vec{q}'}|\Psi_0^F\rangle = 0, \quad \langle\Psi_0^F|b_{\vec{q}} b_{\vec{q}'}^\dagger|\Psi_0^F\rangle = \delta_{\vec{q}\vec{q}'}, \tag{16}$$

$$\langle\Psi_0^F|b_{\vec{q}} b_{\vec{q}'}|\Psi_0^F\rangle = \langle\Psi_0^F|b_{\vec{q}}^\dagger b_{\vec{q}'}^\dagger|\Psi_0^F\rangle = 0. \tag{17}$$

Besides, $\langle\Psi_0^F|b_{\vec{q}}|\Psi_0^F\rangle = \langle\Psi_0^F|b_{\vec{q}}^\dagger|\Psi_0^F\rangle = 0$. Thus, the electromagnetic field involved is a photon-free bath.

The solution to Eq. (15) is presented with the help of the evolution operator $U(t)$ (I is a unity operator):

$$|\Psi^{\mathrm{TL}+F}(t)\rangle = U(t)\left|\Psi_0^{\mathrm{TL}+F}\right\rangle, \quad U(0) = I,$$
$$i\hbar\frac{d}{dt}U(t) = [H^{\mathrm{Tr}}(t) + H^{\mathrm{St}}(t) + V^{\mathrm{TL-Ex}}]U(t). \tag{18}$$

Equations (15)–(18) represent the basis of analyzing collective spontaneous emission with allowance for the Stark interaction by any known method.

III. THE MARKOV APPROXIMATION AND THE RELATED QUANTUM STOCHASTIC DIFFERENTIAL EQUATION FOR THE SYSTEM EVOLUTION OPERATOR

Now, let us express the main equations (15)–(18) in a form suitable for further application of the QSDE method. First, we will write Eqs. (15)–(18) in a dimensionless form. The resonant transition frequency ω_{21} will serve as the characteristic frequency, and the value of ω_{21}^{-1} will be treated as the characteristic time. The value of d_{12} will be considered to be real. We introduce the dimensionless time $\tau = \omega_{21}t$ and frequencies $\nu = \omega_{\vec{q}}/\omega_{21}$, $\nu\prime = \omega_{\vec{q}'}/\omega_{21}$. The wave vector $\vec{q}$ is presented with the help of the unity vector $\vec{n}$, $\vec{q} = \vec{n}\nu\omega_{21}/c$. We replace the summation with an integration,

$$\sum_{\vec{q}} \to \left(\frac{\ell\omega_{21}}{2\pi c}\right)^3 \int_0^\infty 4\pi\nu^2\, d\nu \int \frac{d\Omega_{\vec{n}}}{4\pi}.$$

We will denote by $\int d\Omega_{\vec{n}} \equiv \int d\Omega_{\vec{q}}$ as the integration over various wave-vector orientations. The following dimensionless values and operators are introduced:

$$|\Psi^{\mathrm{TL}+F}(\tau)\rangle \equiv \left|\Psi^{\mathrm{TL}+F}\left(\tau\omega_{21}^{-1}\right)\right\rangle, \quad b_\nu = \mu\frac{\sqrt{\ell^3}}{\pi\sqrt{2}}\left(\frac{\omega_{21}}{c}\right)^{3/2} \times \nu\int\frac{d\Omega_{\vec{n}}}{4\pi} b_{\vec{n}\nu\omega_{21}/c}, \quad U(\tau) \equiv U\left(\tau\omega_{21}^{-1}\right),$$

$$H^{\mathrm{Tr}}(\tau) = \frac{1}{\sqrt{2\pi}}\int_0^\infty d\nu\, b_\nu^+ e^{i(\nu-1)\tau}\chi(\nu)R_- + \frac{1}{\sqrt{2\pi}}\int_0^\infty d\nu\, b_\nu e^{-i(\nu-1)\tau}\chi(\nu)R_+, \tag{19}$$

$$H^{\mathrm{St}}(\tau) = \frac{1}{2\pi}\int_0^\infty d\nu\, b_\nu^\dagger e^{i(\nu-1)\tau}\int_0^\infty d\nu'\, b_{\nu'} e^{-i(\nu'-1)\tau} \times \left\{\eta_+(\nu,\nu')\frac{N_a}{2} + \eta_-(\nu,\nu')R_3\right\}, \tag{20}$$

$$V = -\frac{\kappa}{2}(R_-R_+ + R_+R_- - N_a),$$

$$\chi(\nu)=\frac{\sqrt{2}\omega_{21}d_{12}}{\mu c^{3/2}\sqrt{\hbar}}\nu,\ \ \eta_{\pm}(\nu,\nu')=\chi(\nu)\chi(\nu')\frac{\Pi_{\pm}(\omega_{21}\nu,\omega_{21}\nu')}{d_{12}^2/(\hbar\omega_{21})},$$

$$\kappa=\sum_{\vec{q}}\frac{\Gamma_{\vec{q}}^2\,|d_{21}|^2}{\hbar^2(\omega_{\vec{q}}+\omega_{21})\omega_{21}}.$$

The correction parameter μ was introduced for the following reason. The value $\mu=1$ corresponds to the replacement of the integration over a quantization cube to a solid sphere in the sequence of transformations

$$\begin{aligned}
[b_{\vec{q}},b_{\vec{q}'}^{\dagger}]&=\delta_{\vec{q}\vec{q}'}=\frac{1}{\ell^3}\int d\vec{r}\,e^{i(\vec{q}-\vec{q}')\vec{r}},\\
&\left[\int\frac{d\Omega_{\vec{q}}}{4\pi}b_{\vec{q}},\int\frac{d\Omega_{\vec{q}'}}{4\pi}b_{\vec{q}'}^{\dagger}\right]\\
&=\frac{1}{\ell^3}\int\frac{d\Omega_{\vec{q}}}{4\pi}\int\frac{d\Omega_{\vec{q}'}}{4\pi}\int d\vec{r}\,e^{i(\vec{q}-\vec{q}')\vec{r}}\\
&=\frac{1}{\ell^3}\int 4\pi r^2\,dr\int\frac{d\Omega_{\vec{q}}}{4\pi}e^{iqr\,\cos\theta}\\
&\quad\times\int\frac{d\Omega_{\vec{q}'}}{4\pi}e^{-iq'r\,\cos\theta'}\\
&=\frac{1}{\ell^3}\int 4\pi\,dr\frac{1}{i2q}(e^{iqr}-e^{-iqr})\\
&\quad\times\frac{1}{i2q'}(e^{iq'r}-e^{-iq'r})\to\frac{1}{\ell^3}\frac{2\pi^2}{qq'}\\
&\quad\times\delta(q-q'),\quad \ell\to\infty,\quad q>0,\quad q'>0.
\end{aligned}$$

Parameter χ will be seen to determine the rate of the Langevin spontaneous decay at $\eta_{\pm}=0$, whereas parameters $\eta_{\pm}$ tend to define the non-Langevin factor of spontaneous emission suppression. In Sec. V we will obtain the spontaneous decay rate for a singly excited atomic ensemble, which, for the case of a single particle, will differ from the conventional rate for the Langevin spontaneous emission [1] by a numerical factor. The value $\mu=\sqrt{3}$ corrects this divergence. For all applications of the QSDE to spontaneous emission (e.g., Refs. [11–14]) parameter χ was treated as a phenomenological one. (However, a detailed examination of this divergence is not the subject of the present paper.) The above-mentioned divergence does not distort the theory if we think the values χ and $\eta_{\pm}$ to be phenomenological parameters according to the QSDE theory of the Langevin spontaneous emission.

Now we can write down the dimensionless form of the main equations as

$$\frac{d}{d\tau}|\Psi^{\mathrm{TL}+F}(\tau)\rangle=-i(H^{\mathrm{Tr}}(\tau)+H^{\mathrm{St}}(\tau)+)|\Psi^{\mathrm{TL}+F}(\tau)\rangle, \tag{21}$$

$$\frac{d}{d\tau}U(\tau)=-i(H^{\mathrm{Tr}}(\tau)+H^{\mathrm{St}}(\tau)+V)U(\tau),\quad U(0)=I, \tag{22}$$

$$\left\langle\Psi_0^F\right|b_{\nu}^{\dagger}b_{\nu'}\left|\Psi_0^F\right\rangle=\left\langle\Psi_0^F\right|b_{\nu}b_{\nu'}\left|\Psi_0^F\right\rangle=\left\langle\Psi_0^F\right|b_{\nu}^{\dagger}b_{\nu'}^{\dagger}\left|\Psi_0^F\right\rangle=0,$$
$$\left\langle\Psi_0^F\right|b_{\nu}b_{\nu'}^{\dagger}\left|\Psi_0^F\right\rangle=\delta(\nu-\nu').$$

For an ordinary atom, the Stark interaction parameters $\eta_{\pm}(\nu,\nu\prime)$ are small compared with parameters $\chi(\nu)$ determining the Rabi frequency as seen from

$$\eta_{\pm}(\nu,\nu')=\chi(\nu)\chi(\nu')\frac{\Pi_{\pm}(\omega_{\vec{q}},\omega_{\vec{q}'})}{d_{21}^2/(\hbar\omega_{21})}.$$

Then, $\eta_{\pm}\ll\chi\ll 1$ as $\Pi_{\pm}\sim d_{12}^2/(\omega_{21}\hbar)$ and $\chi\ll 1$ for the expansion (8) to be reasonable. However, the relation $\Pi_{\pm}\gg d_{12}^2/(\omega_{21}\hbar)$ can be true if the anomalous smallness of the resonant transition dipole moment is possible due to any reason. As a result, one can expect the values to be $\eta_{\pm}\sim 1$. Several models of two-photon transitions [9] are also characterized by $\eta_{\pm}\sim 1$. Moreover, the Stark interaction parameter η_{+} is contained in (20), together with the number of atoms in ensemble N_a as the multiplicand. Therefore, the Stark interaction parameters $\eta_{\pm}$ can make an impact on the atomic dynamics in the case of $N_a\gg 1$, or under some special conditions mentioned above. Therefore, the value $\eta_{+}N_a$ will be considered to be of order of a unit, while $\chi\ll 1$.

Finally, we present the formal solution to Schrödinger equation (22) for the evolution operator $U(\tau)$ in terms of the $\overleftarrow{T}$ exponent

$$\begin{aligned}
U(\tau)&=I+(-i)\int_0^{\tau}[H^{\mathrm{Tr}}(\tau')+H^{\mathrm{St}}(\tau')+V]d\tau'\\
&\quad+(-i)^2\int_0^{\tau}\int_0^{\tau'}[H^{\mathrm{Tr}}(\tau')+H^{\mathrm{St}}(\tau')+V]\\
&\quad\times[H^{\mathrm{Tr}}(\tau'')+H^{\mathrm{St}}(\tau'')+V]d\tau'\,d\tau''+\cdots\\
&=\overleftarrow{T}\exp\left(-i\int_0^{\tau}[H^{\mathrm{Tr}}(\tau')+H^{\mathrm{St}}(\tau')+V]d\tau'\right).
\end{aligned} \tag{23}$$

Now we will introduce quantum stochastic processes and make unique basic assumptions characterizing the QSDE method.

$$\begin{aligned}
b(\tau)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\nu\,e^{-i(\nu-1)\tau}\,b_{\nu},\\
b^{\dagger}(\tau)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\nu\,e^{i(\nu-1)\tau}\,b_{\nu}^{\dagger},\\
B(\tau)&=\int_0^{\tau}d\tau'\,b(\tau'),\ \ B^{\dagger}(\tau)=\int_0^{\tau}d\tau'\,b^{\dagger}(\tau'),\\
\Lambda(\tau)&=\int_0^{\tau}d\tau'\,b^{\dagger}(\tau')b(\tau'),
\end{aligned} \tag{24}$$

supposing that the integration limits in $b(\tau)$ range from $-\infty$ to $+\infty$, rather than from 0 to $+\infty$. This noteworthy assumption leads to the following relations:

$$[b(\tau),b^{\dagger}(\tau')]=\delta(\tau-\tau'),\quad [B(\tau),B^{\dagger}(\tau)]=\tau,$$
$$[B(\tau_1),B^{\dagger}(\tau_2)]=\int_0^{\tau_1}d\tau'\int_0^{\tau_2}d\tau''\,\delta(\tau'-\tau'')=\min(\tau_1,\tau_2).$$

Assume that the parameters of the direct transition $\chi(\nu)$ and the Stark level shifts $\eta_{\pm}(\nu,\nu')$ are not affected by the frequency ω,

$$\begin{aligned}
&\nu=1,\quad \chi(\nu)=\mathrm{const}=\chi(1)\equiv\chi,\\
&\eta_{\pm}(\nu,\nu')=\mathrm{const}=\eta_{\pm}(1,1)\equiv\eta_{\pm}.
\end{aligned} \tag{25}$$

The values introduced in (24) as well as assumptions (25) allow writing the interaction operators (19) and (20) in the form

$$H^{\mathrm{Tr}}(\tau)d\tau = \chi R_+ dB(\tau) + \chi R_- dB^\dagger(\tau), \tag{26}$$

$$H^{\mathrm{St}}(\tau)d\tau = \left(\eta_+ \frac{N_a}{2} + \eta_- R_3\right) d\Lambda(\tau), \tag{27}$$

$$\begin{aligned} dB(\tau) &= B(\tau + d\tau) - B(\tau), \\ dB^\dagger(\tau) &= B^\dagger(\tau + d\tau) - B^\dagger(\tau), \\ d\Lambda(\tau) &= \Lambda(\tau + d\tau) - \Lambda(\tau). \end{aligned} \tag{28}$$

The applied approximations are the Markov conditions, namely, the dynamics of the electromagnetic broadband field (16) and (17) is determined by the field state at a point time and is not affected by the field state of the previous moments of time [1,11,13,14]. The approximations were used in all previous works [11–17] with no account of the Stark interaction. Conditions (25), along with the correction parameter $\mu = \sqrt{3}$ in the definition of χ, provide the correct value of the spontaneous decay rate through the resonant atomic transition characteristics (with no allowance for photon polarization).

For the Markov conditions, Eqs. (21)–(23) have proved to be mathematically incorrect [11,13]. It is evident from the detailed consideration of integrals contained in formula (23). We will consider these integrals to be taken in the Ito form:

$$\int_0^\tau \varphi(\tau')dB^\dagger(\tau') = \lim_{N\to\infty} \sum_{i=1}^{N} \varphi(\tau_{i-1})[B^\dagger(\tau_i) - B^+(\tau_{i-1})],$$

where the limit is taken as the mean-square one [11,13,14]. Here, $0 < \tau_1 < \tau_2 < \cdots < \tau_{N-1} < \tau$, with $\tau_0 = 0$ and $\tau_N = \tau$, and the maximum of the time intervals $\tau_i - \tau_{i-1}$ tends to zero, with the number of time interval N tending to infinity. The values $\varphi(\tau)$ are nonanticipating, i.e., statistically independent of the subsequent behavior of $B(\tau)$ and $B^\dagger(\tau)$. In the mathematical way, it is expressed as

$$[\varphi(\tau), dB(\tau)] = [\varphi(\tau), dB^\dagger(\tau)] = [\varphi(\tau), d\Lambda(\tau)] = 0.$$

The Ito quantum stochastic differential equation

$$\begin{aligned} d\varphi(\tau) = {} & \alpha(\varphi(\tau),\tau)dB(\tau) + \beta(\varphi(\tau),t)dB^\dagger(\tau) \\ & + \varepsilon(\varphi(\tau),\tau)d\Lambda(\tau) + \gamma(\varphi(\tau),\tau)d\tau \end{aligned}$$

is the equation for which the integral relation is true,

$$\begin{aligned} \varphi(\tau) - \varphi(0) = {} & \int_0^\tau \alpha(\tau')dB(\tau') + \int_0^\tau \beta(\tau')dB^\dagger(\tau') \\ & + \int_0^\tau \varepsilon(\tau')d\Lambda(\tau') + \int_0^\tau \gamma(\tau')d\tau', \end{aligned}$$

where the stochastic integrals are interpreted in terms of the Ito form. The differentials $d\varphi(\tau)$, $dB(\tau)$, $dB^\dagger(\tau)$, and $d\Lambda(\tau)$ are referred to as the Ito differentials, or the Ito increments.

Hudson and Parthasarathy [10] (see also Refs. [24] and [25]) concluded that the Ito differentials (28) satisfy the below stated algebra:

$$\begin{aligned} &d\Lambda(\tau)d\Lambda(\tau) = d\Lambda(\tau), \quad d\Lambda(\tau)dB^\dagger(\tau) = dB^\dagger(\tau), \\ &dB(\tau)d\Lambda(\tau) = dB(\tau), \quad dB(\tau)dB^\dagger(\tau) = d\tau, \\ &d\Lambda(\tau)dB(\tau) = d\Lambda(\tau)d\tau = dB^\dagger(\tau)d\Lambda(\tau) \\ &\qquad = dB^\dagger(\tau)d\tau = dB(\tau)d\tau = 0. \end{aligned} \tag{29}$$

The operators $B(\tau)$, $B^\dagger(\tau)$, and $\Lambda(\tau)$ define the Wiener quantum process $Q(\tau)$ and the Poisson quantum process $N(\tau)$ according to [24,25]

$$Q(\tau) = B(\tau) + B^\dagger(\tau), \quad N(\tau) = \Lambda(\tau) + i(B^\dagger(\tau) - B(\tau)).$$

The operators $dB(\tau)$, $dB^\dagger(\tau)$, and $d\Lambda(\tau)$ are the increments of annihilation, creation, and counting processes determining the Wiener quantum process $Q(\tau)$ and the Poisson quantum process $N(\tau)$. Furthermore, we will think of the Wiener quantum processes as the operators $B(\tau)$ and $B^\dagger(\tau)$ [or $dB(\tau)$ and $dB^\dagger(\tau)$], and the Poisson quantum process as $\Lambda(\tau)$ [or $d\Lambda(\tau)$], which does not lead to any misunderstanding.

The Hudson-Parthasarathy algebra (29) allows us to provide a correct mathematical expression for the evolution operator equation. Consider the Ito differential $dU(\tau)$:

$$dU(\tau) \equiv U(\tau + d\tau) - U(\tau).$$

If Eq. (23) is taken in the form

$$\begin{aligned} U(\tau) = {} & \lim_{N\to\infty} \exp\left(\frac{H^{\mathrm{Tr}}(\tau_{N-1}) + H^{\mathrm{St}}(\tau_{N-1}) + V}{i}(\tau_N - \tau_{N-1})\right) \\ & \times \cdots \exp\left(\frac{H^{\mathrm{Tr}}(\tau_0) + H^{\mathrm{St}}(\tau_0) + V}{i}(\tau_1 - \tau_0)\right), \end{aligned}$$

then

$$\begin{aligned} dU(\tau) = {} & \left\{ \exp\left[-i\left(\chi R_+ dB(\tau) + \chi R_- dB^\dagger(\tau)\right.\right.\right. \\ & \left.\left.\left. + \left(\eta_+ \frac{N_a}{2} + \eta_- R_3\right)d\Lambda(\tau) + V\, d\tau\right)\right] - 1 \right\} U(\tau). \end{aligned}$$

This expression shows the unitary property of the evolution operator and the Ito differentiation rule,

$$\begin{aligned} d[U(\tau)U^\dagger(\tau)] = {} & [dU(\tau)]U^\dagger(\tau) + U(\tau)dU^\dagger(\tau) \\ & + [dU(\tau)][dU^\dagger(\tau)]. \end{aligned}$$

Expanding the exponent in series and applying the Hudson-Parthasarathy algebra (29), we obtain the Ito equation for the evolution operator,

$$\begin{aligned} dU(\tau) = {} & A_0\, d\tau\, U(\tau) + A_+\, dB(\tau)U(\tau) + A_-\, dB^\dagger(\tau)U(\tau) \\ & + A_\Lambda\, d\Lambda(\tau)U(\tau) - iV\, d\tau\, U(\tau), \end{aligned} \tag{30}$$

$$\begin{aligned} dU^\dagger(\tau) = {} & U^\dagger(\tau)A_0^\dagger\, d\tau + U^\dagger(\tau)dB^\dagger(\tau)A_+^\dagger + U^\dagger(\tau)dB(\tau)A_-^\dagger \\ & + U^\dagger(\tau)d\Lambda(\tau)A_\Lambda^\dagger + iU^\dagger(\tau)V\, d\tau, \end{aligned}$$

$$\begin{aligned} A_0 = {} & \chi^2 R_+ \\ & \times \frac{\exp\left[-i\left(\eta_+ \frac{N_a}{2} + \eta_- R_3\right)\right] - 1 + i\left(\eta_+ \frac{N_a}{2} + \eta_- R_3\right)}{\left(\eta_+ \frac{N_a}{2} + \eta_- R_3\right)^2} R_-, \end{aligned}$$

$$A_- = \frac{\exp\left[-i\left(\eta_+ \frac{N_a}{2} + \eta_- R_3\right)\right] - 1}{\eta_+ \frac{N_a}{2} + \eta_- R_3} \chi R_-,$$

$$A_+ = \chi R_+ \frac{\exp\left[-i\left(\eta_+\frac{N_a}{2}+\eta_- R_3\right)\right]-1}{\eta_+\frac{N_a}{2}+\eta_- R_3},$$

$$A_\Lambda = \exp\left[-i\left(\eta_+\frac{N_a}{2}+\eta_- R_3\right)\right]-1.$$

The operators

$$\frac{\exp\left[-i\left(\eta_+\frac{N_a}{2}+\eta_- R_3\right)\right]-1+i\left(\eta_+\frac{N_a}{2}+\eta_- R_3\right)}{\left(\eta_+\frac{N_a}{2}+\eta_- R_3\right)^2},$$

$$\frac{\exp\left[-i\left(\eta_+\frac{N_a}{2}+\eta_- R_3\right)\right]-1}{\eta_+\frac{N_a}{2}+\eta_- R_3}$$

are interpreted as the Taylor series of the corresponding functions of x: $(\eta_+\frac{N_a}{2}+\eta_- R_3)\to x$, with subsequent reverse substitution $x\to(\eta_+\frac{N_a}{2}+\eta_- R_3)$.

In absence of the Stark interaction $\eta_\pm = 0$ for a single quantum particle $N_a = 1$, $V = 0$, Eq. (30) coincides with the familiar case and describes the Langevin atomic relaxation [13]. The Langevin relaxation type is determined by the Langevin (or Wiener) form of the quantum stochastic differential equation defined by the Wiener quantum processes $dB(t)$ and $dB^\dagger(t)$ alone, in the absence of the Poisson quantum process $d\Lambda(t)$. Dependence of the evolution operator on $d\Lambda(t)$ is a sign of the non-Langevin (or non-Wiener) process manifestation. At $\eta_+ = 0$ and $N_a = 1$, Eq. (30) coincides with the non-Langevin equation derived in Ref. [9].

IV. MASTER EQUATION FOR THE ATOMIC ENSEMBLE DENSITY MATRIX

The master equation for the density matrix of atomic ensemble and electromagnetic field $\rho(\tau) = U(\tau)|\Psi_0^{TL+F}\rangle\langle\Psi_0^{TL+F}|U^+(\tau)$ is derived from the quantum stochastic differential equation for the evolution operator (30) by calculating the increment

$$\begin{aligned}d\rho(\tau) &= \rho(\tau+d\tau)-\rho(\tau)\\ &= dU(\tau)\left|\Psi_0^{TL+F}\right\rangle\left\langle\Psi_0^{TL+F}\right| U^\dagger(\tau)+U(\tau)\left|\Psi_0^{TL+F}\right\rangle\\ &\quad\times\left\langle\Psi_0^{TL+F}\right| dU^\dagger(\tau)+dU(\tau)\left|\Psi_0^{TL+F}\right\rangle\\ &\quad\times\left\langle\Psi_0^{TL+F}\right| dU^\dagger(\tau).\end{aligned}$$

The use of the Hudson-Parthasarathy algebra gives rise to

$$\begin{aligned}&d\rho(\tau)\\ &= -i[V,\rho(\tau)]d\tau + A_0\,d\tau\,\rho(\tau)+A_+\,dB(\tau)\rho(\tau)\\ &+A_-\,dB^\dagger(\tau)\rho(\tau)+A_\Lambda\,d\Lambda(\tau)\rho(\tau)\\ &+\rho(\tau)A_0^\dagger\,d\tau+\rho(\tau)dB^+(\tau)A_+^\dagger+\rho(\tau)dB(\tau)A_-^\dagger\\ &+\rho(\tau)d\Lambda(\tau)A_\Lambda^\dagger+A_+\,dB(\tau)\rho(\tau)dB^\dagger(\tau)A_+^\dagger\\ &+A_+\,dB(\tau)\rho(\tau)dB(\tau)A_-^\dagger+A_+\,dB(\tau)\rho(\tau)d\Lambda(\tau)A_\Lambda^\dagger\\ &+A_-\,dB^\dagger(\tau)\rho(\tau)dB^\dagger(\tau)A_+^\dagger+A_-\,dB^\dagger(\tau)\rho(t)dB(\tau)A_-^\dagger\\ &+A_-\,dB^\dagger(\tau)\rho(\tau)d\Lambda(\tau)A_\Lambda^\dagger+A_\Lambda\,d\Lambda(\tau)\rho(\tau)dB^\dagger(\tau)A_+^\dagger\\ &+A_\Lambda\,d\Lambda(\tau)\rho(\tau)dB(\tau)A_-^\dagger+A_\Lambda\,d\Lambda(\tau)\rho(\tau)d\Lambda(\tau)A_\Lambda^\dagger.\end{aligned}$$

The master equation for the atomic ensemble only $\rho^{TL}(\tau) = \mathrm{Tr}_F\rho(\tau)$ on account of the relations

$$\mathrm{Tr}_F[\rho(\tau)dB(\tau)] = \mathrm{Tr}_F[\rho(\tau)dB^\dagger(\tau)] = \mathrm{Tr}_F[\rho(\tau)d\Lambda(\tau)] = 0$$

is derived in the form

$$\begin{aligned}\frac{d\rho^{TL}}{d\tau} &= -i[V,\rho^{TL}]+\chi^2 a_-^{NL}(\eta_+,\eta_-,R_3)R_-\rho^{TL}\\ &\quad\times R_+ a_+^{NL}(\eta_+,\eta_-,R_3)-\frac{\chi^2}{2}\{R_+[a_0^{NL}(\eta_+,\eta_-,R_3)\\ &\quad-ia_s^{NL}(\eta_+,\eta_-,R_3)]R_-\rho^{TL}\\ &\quad+\rho^{TL}R_+[a_0^{NL}(\eta_+,\eta_-,R_3)+ia_s^{NL}(\eta_+,\eta_-,R_3)]R_-\}.\end{aligned} \tag{31}$$

This equation describes the non-Langevin collective spontaneous decay of ensemble of identical two-level atoms in a photon-free vacuum field. The non-Langevin operators are involved,

$$a_0^{NL}(\eta_+,\eta_-,R_3) = 2\frac{1-\cos\left(\eta_+\frac{N_a}{2}+\eta_- R_3\right)}{\left(\eta_+\frac{N_a}{2}+\eta_- R_3\right)^2},$$

$$a_s^{NL}(\eta_+,\eta_-,R_3) = 2\frac{\eta_+\frac{N_a}{2}+\eta_- R_3-\sin\left(\eta_+\frac{N_a}{2}+\eta_- R_3\right)}{\left(\eta_+\frac{N_a}{2}+\eta_- R_3\right)^2},$$

$$\begin{aligned}a_\pm^{NL}(\eta_+,\eta_-,R_3) &= \frac{\cos\left(\eta_+\frac{N_a}{2}+\eta_- R_3\right)-1}{\eta_+\frac{N_a}{2}+\eta_- R_3}\\ &\quad\pm i\frac{\sin\left(\eta_+\frac{N_a}{2}+\eta_- R_3\right)}{\eta_+\frac{N_a}{2}+\eta_- R_3}.\end{aligned}$$

In the absence of the Stark interaction $\eta_\pm = 0$, the non-Langevin operators are proportional to the unity operator I,

$$a_0^{NL}(\eta_+,\eta_-,R_3) = I,\quad a_s^{NL}(\eta_+,\eta_-,R_3) = 0,$$
$$a_\pm^{NL}(\eta_+,\eta_-,R_3) = \pm iI,$$

and the master equation (31) agrees with the familiar master equation describing collective Langevin atomic decay [13].

The non-Langevin operators obey the relation

$$a_+^{NL}(\eta_+,\eta_-,R_3)a_-^{NL}(\eta_+,\eta_-,R_3) = a_0^{NL}(\eta_+,\eta_-,R_3).$$

One can introduce the Lindblad operators

$$L_- = a_-^{NL}(\eta_+,\eta_-,R_3)R_-,\quad L_+ = R_+a_+^{NL}(\eta_+,\eta_-,R_3),$$
$$L_+L_- = R_+a_0^{NL}(\eta_+,\eta_-,R_3)R_-,$$

and give the master equation of the Lindblad form [26]

$$\begin{aligned}\frac{d\rho^{TL}}{d\tau} &= \chi^2 L_-\rho^{TL}L_+-\frac{\chi^2}{2}(L_+L_-\rho^{TL}+\rho^{TL}L_+L_-)\\ &\quad-i[(H^{NL-ST}+V),\rho^{TL}],\end{aligned} \tag{32}$$

where

$$H^{NL-ST} = -\frac{\chi^2}{2}R_+a_s^{NL}(\eta_+,\eta_+,R_3)R_-.$$

The operator H^{NL-ST} defines the atomic-level shifts caused by the Stark interaction. These same shifts do differ both from the Lamb shifts included into the frequency ω_{21} and the Stark

shifts represented as $\langle\Psi_0^F|H^{\rm Stark}(t)|\Psi_0^F\rangle$, and equal to zero in a photon-free electromagnetic field. For a single atom, this shift was described in Ref. [9].

Note that the operator V also leads to level shifts due to the excitation exchange in atom-atom interactions of dipole-dipole type. Therefore, the operator V can be regarded as an excitation-exchange operator. It provides the dephasing effect, leading to coherence relaxation, but it does not affect the rates of quantum transitions.

The next two sections will provide solutions to Eq. (32) for the cases of singly and fully excited atomic ensembles.

V. NON-LANGEVIN DECAY OF SINGLY EXCITED ATOMIC ENSEMBLE

Collective spontaneous emission of identical atoms in the common vacuum field is not easy to be completely investigated. Even a two-atom system with originally factorized different quantum states appears to be entangled as a result of spontaneous decay [27–30]. The main reason for the system behavior is due to the existence of the decoherence-free subspace [31,32]. Below we will consider simpler cases of collective spontaneous emission with the initial Dicke state [3]. For a singly excited atomic ensemble, these states will reduce to W states [33], which are important in quantum information processing.

Let the initial atomic ensemble state at $\tau = 0$ be the following:

$$|\Psi_0^{TL}\rangle = \frac{1}{\sqrt{N_a}}\{|E_2\rangle^{(1)}|E_1\rangle^{(2)}\cdots|E_1\rangle^{(N_a)} + |E_1\rangle^{(1)}|E_2\rangle^{(2)} \cdots|E_1\rangle^{(N_a)} + \cdots + |E_1\rangle^{(1)}|E_1\rangle^{(2)}\cdots|E_2\rangle^{(N_a)}\}.$$

It is quite essential to make clear how the number of atoms N_a affects the non-Langevin decay rate. Let $N_a = 2r$. Because of the symmetry properties of $|E_0^{TL}\rangle$ with respect to permutations of atoms, $|\Psi_0^{TL}\rangle$ can be expressed by $|r, -r+1\rangle$ of $(N_a + 1)$-dimensional space of irreducible representation of su(2) algebra,

$$R_+|r,m-1\rangle = \sqrt{(r+m)(r-m+1)}\,|r,m\rangle\,,$$
$$R_-|r,m\rangle = \sqrt{(r+m)(r-m+1)}\,|r,m-1\rangle\,,$$
$$R_3|r,m\rangle = m\,|r,m\rangle\,, \quad -r \leqslant m \leqslant r,$$

with the Casimir operator $R^2 = R_+R_- + R_3^2 - R_3 = R_-R_+ + R_3^2 + R_3$: $R^2|r,m\rangle = r(r+1)|r,m\rangle$. The excitation exchange operator V has the simplest form, $V = -\kappa(r^2 - R_3^2)$, for the irreducible representation space.

The equation for the density matrix $\rho^{TL}_{-r+1,-r+1}$, describing the decay of a singly excited atomic ensemble, is derived from (32),

$$\frac{d\rho^{TL}_{-r+1,-r+1}}{dt} = -4\chi^2 r\,\frac{1-\cos(r\eta_+ - r\eta_-)}{(r\eta_+ - r\eta_-)^2}\,\rho^{TL}_{-r+1,-r+1}. \tag{33}$$

The singly excited atomic ensemble decays exponentially so that the population of excited state is given by

$$\rho^{TL}_{-r+1,-r+1}(\tau) = \exp\left\{-4\chi^2 r\,\frac{1-\cos(r\eta_+ - r\eta_-)}{(r\eta_+ - r\eta_-)^2}\,\tau\right\}.$$

For one atom $r = 1/2$ and $\eta = (\eta_+ - \eta_-)/2$, Eq. (32) coincides with the master equation obtained in Ref. [9], although non-Langeven spontaneous emission was investigated in Ref. [9] on the basis of a simpler model.

If the Stark interaction is absent or negligibly small, $\eta_\pm = 0$, Eq. (32) describes the Langevin decay of the singly excited atomic ensemble

$$\rho^{TL}_{-r+1,-r+1}(\tau) = \exp\{-2\chi^2 r\tau\}.$$

Thus, the constant $2r\chi^2$ may be regarded as the Langevin decay rate of the singly excited atomic ensemble. It is directly proportional to the number of ensemble atoms.

The Stark interaction of atoms with a vacuum photon-free electromagnetic field produces a decrease of the spontaneous emission rate of the atomic ensemble (in comparison with the Langevin case) at any intensity of the Stark interaction. That is the consequence of quantum interference of both spontaneous transition from the excited level to the ground one, with one photon being emitted, and virtual transitions with returning to the excited level with no photon emission. As the number of ensemble atoms increases, the total intensity of the Stark interaction increases also. Despite the fact that the Langevin spontaneous emission rate constantly increases as the number of atoms rises, $N_a = 2r$, there exists the critical value of the number of atoms $N_a^{\rm cr} = 4\pi/(\eta_+ - \eta_-)$ or parameter r,

$$r^{\rm cr} = 2\pi/(\eta_+ - \eta_-), \tag{34}$$

at which the spontaneous decay is completely suppressed and

$$\rho^{TL}_{-r+1,-r+1}(\tau) = \rho^{TL}_{-r+1,-r+1}(0) = \text{const}.$$

Let us note that the dimensionless combination of parameters $\eta_+ - \eta_-$ is determined by the value $\Pi_1(\omega)$, characterizing the Stark shift of the ground level, so

$$N_a^{\rm cr} = \frac{2\pi\, d_{12}^2}{\chi^2\,|\Pi_1|\,\omega_{21}\hbar}.$$

Reference [9] was devoted to analyzing phenomena where the Stark interaction growth of a single quantum particle could be expected to get substantial. With spontaneous emission being the basic reason for decoherence producing mechanism and hindering the quantum operations, the search for situations where the Stark interaction cannot be neglected is of interest for quantum information processing. However, it is not a subject for discussion in the present paper.

It is important to emphasize that the quantum Poisson process $\Lambda(\tau)$ describing the Stark interaction acts as if it were an original "interaction accumulator" according to the relation $d\Lambda(\tau)d\Lambda(\tau) = d\Lambda(\tau)$. Instead of making a quantum transition, with a photon emitting, a particle is involved in perpetual virtual transitions and returns to an excited level with no photon emission. It gives rise to stabilization of an excited state. To calculate such an effect in a different way [6–8] other than the quantum SDE method, it is necessary to summarize an infinite series where only the first terms have been accounted for previously. The same "interaction accumulator" effect has been revealed in the photon counting while the radiating particle is being continuously measured [12]. The summation of the above-mentioned infinite series is automatically carried

out in the QSDE method due to the Hudson-Partasarathy algebra (29). It is the basic advantage of the QSDE method.

VI. NON-LANGEVIN DECAY OF FULLY EXCITED ATOMIC ENSEMBLE

One more relatively simple case of spontaneous decay of an excited atomic ensemble with the Stark interaction manifestation is the spontaneous decay of fully excited two-level particles. At $\tau = 0$ the initial state of such an atomic ensemble is

$$\left|\Psi_0^{TL}\right\rangle = |E_2\rangle^{(1)}\,|E_2\rangle^{(2)}\cdots|E_2\rangle^{(N_a)}.$$

The relaxation dynamics of this system is also given by Eq. (32) in the case of state space spanned by vectors $|r,m\rangle$, $N_a = 2r$, with the initial state $|\Psi_0^{TL}\rangle = |r,r\rangle$. For the diagonal matrix elements, it is easy to obtain the equation from (32),

$$\frac{d\rho_{mm}^{TL}}{dt} = -2\chi^2 g_{m\,m-1}\frac{1-\cos[r\eta_+ + \eta_-(m-1)]}{[r\eta_+ + \eta_-(m-1)]^2}\rho_{mm}^{TL} + 2\chi^2 g_{m+1m}\frac{1-\cos(r\eta_+ + \eta_- m)}{(r\eta_+ + \eta_- m)^2}\rho_{m+1m+1}^{TL}, \quad (35)$$

where

$$g_{mm-1} = \langle m|R_+|m-1\rangle\langle m-1|R_-|m\rangle = (r+m)(r-m+1).$$

The decay of the excited atomic ensemble is of non-Langevin type if the number of ensemble atoms is sufficiently high. Here, we can also think of the existence of the critical value of the number of atoms in the ensemble

$$r^{\rm cr} = \frac{2\pi + \eta_-}{\eta_+ + \eta_-} \cong \frac{2\pi}{\eta_+ + \eta_-}, \quad (36)$$

where $\rho_{rr}^{TL}(\tau) = \text{const} = 1$, and the atomic ensemble of fully excited two-level atoms is not decayed as a result of the excitation stabilization caused by the Stark interaction. It is a curious notion that the critical value (36) is different from the critical value (34) for the singly excited atomic ensemble, and is defined by the Stark interaction parameter characterizing the excited atomic level alone. It can be seen from the corresponding expression given for the critical number of atoms,

$$N_a^{\rm cr} = \frac{2\pi\, d_{12}^2}{\chi^2\,|\Pi_2|\,\omega_{21}\hbar}.$$

Lastly, there are critical values $m^{\rm cr} \cong (2\pi - r\eta_+)/\eta_-$ of atomic ensemble excitations when the collective spontaneous emission of the fully excited atomic ensemble stops and the ensemble is stabilized in the excited state with $m^{\rm cr} + r$ excited atoms, $0 \leqslant m^{\rm cr} + r \leqslant N_a$.

Depending on the meaning of the Stark interaction parameters $\eta_\pm$, all the above indicated critical values "break" the atomic ensemble dynamics into predetermined domains where the non-Langevin factors exert various impacts on the spontaneous emission rates, which can either increase or decrease during the course of photon emission.

Now we will consider the average intensity of collective spontaneous emission $\bar{I}(t)$, which is directly proportional to energy losses of the atomic ensemble

$$\bar{I}(t) = -\alpha\frac{d}{dt}\mathrm{Tr}(\hbar\omega_{21}R_3\rho^{TL}),$$

where the geometrical factor α is introduced and level shifts are neglected. Then

$$\bar{I}(t) = \alpha\sum_{m=-r}^{r}\hbar\omega_{21}2\chi^2 G_{mm-1}\rho_{mm}^N \equiv \alpha'\sum_{m=-r}^{r}G_{mm-1}\rho_{mm}^N,$$

$$G_{mm-1} = g_{mm-1}\frac{1-\cos[r\eta_+ + \eta_-(m-1)]}{[r\eta_+ + \eta_-(m-1)]^2}.$$

In the simplest case $\eta_- = 0$, Eq. (35) coincides with the basic equations of the conventional superradiance theory [4], which allows us to apply the well-known analytical results to a large number of excited atoms $r \gg 1$:

$$\bar{I}(t) \approx \bar{\gamma}r^2\,\mathrm{sech}^2[\bar{\gamma}r(t-t_D)],\ \bar{\gamma} = 2\chi^2\alpha\hbar\omega_{21}\frac{1-\cos(\eta_+ r)}{(\eta_+ r)^2},$$
$$t_D = (2\bar{\gamma}r)^{-1}\ln 2\bar{\gamma}r. \quad (37)$$

Here again one can introduce the critical value of the atom number $N_a^{\rm cr}$ (or parameter $r^{\rm cr} = N_a^{\rm cr}/2$), when the decay is fully suppressed,

$$r^{\rm cr} = 2\pi/\eta_+. \quad (38)$$

At different values of the number of ensemble atoms, the collective spontaneous emission for $\eta_- = 0$ is similar to the conventional (Langevin) superradiance [4]. For the conventional superradiance, however, the larger is the number of excited atoms, the shorter is the pulse duration and time delay of superradiance. At the same time for the non-Langevin superradiance described by formula (37), one can observe an increase of superradiance duration and time delay as the number of ensemble atoms rises due to the non-Langevin factor $[1-\cos(\eta_+ r)]/(\eta_+ r)^2$, renormalizing the rate $\bar{\gamma}$ of the collective spontaneous emission, and establishing the main difference of the collective spontaneous emission from the conventional case under the strong Stark interaction.

To define the minimal number of atoms at which the superradiance suppression is expected to happen, it is to be noted that the maximal value of parameter Π_2 is typical for atoms with the appropriate quasiresonance level $|E_q\rangle$, with the latter being $E_2 < E_q$ and the value $|E_q - E_2 - \hbar\omega_{21}|$ being estimated as minimally feasible. The choice of such atoms needs a more detailed study. It is necessary to add that the value $|E_q - E_2 - \hbar\omega_{21}|$ represents the detuning of the central frequency of quantum noise source (in resonance with the transition $E_2 \to E_1$) from the frequency of the transition $E_q \to E_2$. This same detuning must be far greater than the homogeneous width $\hbar\gamma\omega_{q2}$ of the quasiresonance level $|E_q\rangle$ due to expansion (11). Neglecting polarization, one can obtain $\gamma\omega_{q2} = 2\omega_{21}^3 d_{q2}^2/(3\hbar c^3)$ for the homogeneous level width, where d_{q2} is the dipole moment of the transition $E_q \to E_2$. Then the following estimation is given as

$$N_a^{\rm cr} = 2\pi\left|E_q - E_2 - \hbar\omega_{21}\right|/(\hbar\gamma\omega_{q2}) \sim 2\pi\cdot 10 \sim 10^2.$$

For the parameter Π_1 of the ground level, a similar quasiresonance level may not exist, so that the inequality $|\Pi_1| \ll |\Pi_2|$ was considered to be true. Therefore, the critical number of resonant atoms, at which the atomic ensemble with a small excitation number gets stabilized with respect to the collective spontaneous decay, will be much greater than the above indicated estimation for $N_a^{\rm cr}$.

On condition $|\Pi_1| \ll |\Pi_2|$, the approximate equality $\eta_+ \approx \eta_- = \eta$ is usual, so that $m^{\rm cr} \cong 2\pi/\eta - r = N_a^{\rm cr} - r$, $N_a^{\rm cr} \leqslant 2r$.

The presented theory has been developed for atoms in a small volume but not for extended media, which is why it is not possible to offer a rigorous description of a superradiant effect with allowance for the Stark interaction in extended media. To the best of our knowledge, the non-Langevin QSDE method for extended medium is known to be nonexistent. Nevertheless, the results obtained can be taken into account for extended media also. Even though excited atoms become stabilized with respect to the collective spontaneous decay, the noncollective decay typical of extended media returns excited atoms to the ground state, which can be regarded as the coherence relaxation impact on superradiance destruction in the conventional experiments. The experiments investigating the superradiance destruction, while atomic density is enhancing, are not known to the author. If research of this kind were done, the superradiance duration dependence from atomic density should be studied in the framework of the critical parameters to make an indirect experimental verification in extended media. According to Eq. (37), the superradiance duration increase must be observed as the atomic density enhances.

The critical densities of resonant atoms for the extended media of length $L = 10^2$ cm will be estimated for the superradiance wavelength $\lambda_{21} = 3.41$ μm. The superradiance suppression takes place at the critical density of resonant atoms $n_a^{\rm cr} = N_a^{\rm cr}/(L\lambda_{21}^2) \sim 10^7$ cm^{-3} in the case of atoms with a maximally feasible value of Π_2. This value is as much as 10^3 times less than the density range of sodium vapor (10^9–10^{10} cm^{-3}), where the superradiance was spotted in experiment [34] at the indicated wavelength (transition $5S$-$4P$). For the sodium atom, however, the above-mentioned estimation is not relevant since the sodium atom does not possess the quasiresonance level $|E_q\rangle$ with minimally feasible detuning $|E_q - E_2 - \hbar\omega_{21}|$. In the absence of knowledge of Π_2 (for the $5S$ state), use will be made of $\eta_\pm$ through the Lamb shift $\delta\omega^{\rm Lamb}$ as both parameters define the terms derived from the commutation of creation and annihilation operators of resonant photons. Assuming $\eta_\pm \sim 2\pi\delta\omega^{\rm Lamb}/\omega_{21}$ and applying the typical Lamb shift value of the order of one GHz, we obtain $N_a^{\rm cr} \sim 10^5$ which provides for the critical density of resonant atoms $n_a^{\rm cr} \sim 10^{10}$ cm^{-3}, i.e., the value of sodium vapor density beyond the limit of which the superradiance was not registered in experiment [34].

Emphasis is to be placed on the fact that for extended media the above values are of rough approximation in view of the idea that the theory of the non-Langevin superradiance in extended media needs to be developed.

The derived regularities of collective spontaneous emission of an identical atom ensemble can serve as a starting point for further detailed investigations with allowance for different atomic excitation distributions, different atomic positions, inhomogeneous level broadening, etc. In the present paper our aim was to attract the reader's attention to the phenomenon of the enhancement of the Stark interaction with vacuum while increasing the particle number in the atomic ensemble. As a result, the Stark interaction, in turn, gives rise to radical changes in the collective spontaneous emission, invoking its suppression and stabilization of the excited atomic ensemble.

VII. CONCLUSION

The main difference in the present investigation from previous works devoted to spontaneous emission is the allowance for the Stark interaction with a vacuum electromagnetic field. In the absence of such an interaction, spontaneous emission is of Langevin type and is defined by the quantum Wiener process. Then, the higher is the interaction intensity with a field and/or the particle number in the atomic ensemble, the higher is the spontaneous emission rate. The Stark interaction is described by the quantum Poisson process, which determines the non-Langevin type of spontaneous emission. The role played by the Stark interaction increases as the number of atoms taking part in the collective spontaneous emission rises. The enhancement of the Stark interaction intensity and particle number in the atomic ensemble gives rise to a nonlinear decrease of the spontaneous emission rate as compared to the Langevin decay. It has been found that there is a set of critical values for the number of ensemble atoms taking part in collective spontaneous emission when the spontaneous emission is fully suppressed and the atomic ensemble is stabilized in an excited state.

One more distinctive feature of the present work is the formulation of the effective Hamiltonian picture along with the QSDE for the evolution operator. This approach allows deriving the atomic master equation in a straightforward and elegant manner owing to both an effective Hamiltonian picture and the Hudson-Partasarathy algebra (29) for the increment of the quantum stochastic processes.

The research done in the field of spontaneous emission has provided an example of the analysis of an open system in the Markov approximation in terms of combining the effective Hamiltonian of the open system and QSDE for the evolution operator. The QDSEs are formulated in the framework of the factorization of the initial state of the open system and its environment, independence of different environment modes [Eqs. (16) and (17)], and the homogeneous interaction (15) combined with the definitions of quantum stochastic processes in the Markov approximation [Eq. (25)]. All these requirements laid down above are related not to the initial Hamiltonian and state vector but to the transformed, or effective, Hamiltonian and transformed initial state vector [Eqs. (5) and (4)]. As a result, the master equation in the Lindblad form has been derived for the atomic ensemble. The Lindblad operator has been shown to consist of the non-Langevin operators such as $a_\pm^{NL}$. The master equation contains no parameter $\Delta = \omega_\Gamma - \omega_{21}$ characterizing the feasible detuning of the central frequency ω_Γ of a broadband quantized electromagnetic field from the resonant transition frequency ω_{21} in view of singular conditions (16) and (17), and the uncertainty equation for the evolution operator. Involvement of such a parameter and analysis of the case when it becomes large enough, e.g., compared with the spontaneous emission

rate, are impossible due to the basic requirement for the effective Hamiltonian, i.e., the lack of fast time-varying terms in the interaction picture. The allowance for a parameter similar to Δ means that at a transition to the transformed Hamiltonian, not all fast time-varying terms in the interaction picture are excluded, so this Hamiltonian cannot be regarded as the effective Hamiltonian. Therefore, in such circumstances one should transform the Hamiltonian so that all fast time-varying terms, including the ones related to the parameter Δ, are to be excluded. Only such a Hamiltonian can be thought of as the effective Hamiltonian. This requirement differs from those of previous works [18–22] describing the unitary transformation for the Hamiltonian simplification. Attention has been attracted to the stated principle obedience in Ref. [17], where the dispersive limit for the master equation for atoms in the low Q-cavity (with losses) was considered.

The dispersive limit means that the detuning $\Delta = \omega_c - \omega_{21}$ from the resonance of the cavity mode frequency ω_c is great in comparison with the detuning-free quantum transition rate. The effective Hamiltonians for small and large values of the parameter Δ have proved to be different from each other and not correlated to each other by the limiting transition as Δ grows. Each master equation being obtained from its corresponding effective Hamiltonian is also different from each other and not related to each other by the limiting transition. Thus, the above stated principle of the absence of fast time-varying factors in the relevant terms defines the unique effective Hamiltonian for each physical condition, as in the example of the strong resonant interaction of the cavity mode with atoms or its dispersive limit. Each effective Hamiltonian in the Markov approximation determines its own QSDE and associated quantum noise sources, as well as the master equation. The transition to the effective Hamiltonian picture unambiguously allows representing the external broadband field as a series of independent quantum noise sources. It is a distinctive feature of a unqiue approach toward the investigation of open systems, the notions of which were developed in Refs. [5,16,17,27]. The present paper provides an essential development in this approach. Now the quantum Poisson process and the terms of second order in the coupling with the environment have been introduced into the effective Hamiltonian and its QSDE. The approach can be successfully applied to the formulation and solution to unqiue problems in the field of nonlinear and quantum optics and open system theory, in particular, allowing for various two-quantum radiating processes.

In addition, the above stated principle of the absence of fast time-varying factors in the relevant terms should be taken into account in the process of investigating the dynamics of open systems started from the Lindblad-type master equation. The Lindblad-type master equation is the general form of dissipative dynamics controlled by the continuous quantum dynamic semigroup [26]. However, there is no apparent restriction to the fast time-varying terms in the Lindblad-type master equation. Assuming the existence of fast and slow subsystems in the open system being described by the Lindblad-type master equation, it is necessary to obtain unique Lindblad operators by the above stated approach.

The different effects of collective spontaneous emission suppression presented in the current paper may appear to be useful in quantum information processing, providing the decoherence-free excited atomic states.

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