Semigroup evolution in the Wigner-Weisskopf pole approximation with Markovian spectral coupling

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We establish the relation between the Wigner-Weisskopf theory for the description of an unstable system and the theory of coupling to an environment. According to the Wigner-Weisskopf general approach, even within the pole approximation, the evolution of a total system subspace is not an exact semigroup for multichannel decay unless the projectors into eigenstates of the reduced evolution generator $W(z)$ are orthogonal. With multichannel decay, the projectors must be evaluated at different pole locations $z_\alpha \neq z_\beta$, and since the orthogonality relation does not generally hold at different values of z, the semigroup evolution is a poor approximation for the multichannel decay, even for very weak coupling. Nevertheless, if the theory is generalized to take into account interactions with an environment, one can ensure orthogonality of the $W(z)$ projectors regardless of the number of poles. Such a possibility occurs when $W(z)$, and hence its eigenvectors, is independent of z, which corresponds to the Markovian limit of the coupling to the continuum spectrum.

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I. INTRODUCTION

Many physical systems demonstrate instability, i.e., a transition from a relatively stable state to a final state, which, in general, corresponds to a system with many identifiable degrees of freedom. Such occurs, for example, in particle decay or radiative atomic transitions. In many cases the process observed has the semigroup property, i.e., the operators generating the evolution on the Hilbert space of quantum states satisfy the composition law

$$Z(t_1)Z(t_2) = Z(t_1 + t_2) \tag{1}$$

for $t_1, t_2 \geqslant 0$ and for which the operators $Z(t)$ do not have an inverse and are decreasing. This relation is a general form of the well-known exponential decay law, for which Gamow [1] constructed a phenomenological Schrödinger equation with a complex energy eigenvalue of a negative imaginary part. Weisskopf and Wigner [2] formulated a basic theory that approximately reproduced the Gamow result in second-order perturbation for a single-channel decay. In their original formulation, the survival amplitude of a quantum state $|\psi\rangle$ is given as

$$U^{\rm red}(t) = \langle\psi|e^{-iHt}|\psi\rangle, \tag{2}$$

where H is the full Hamiltonian of a system consisting of an unperturbed part H_0 for which $|\psi\rangle$ is an eigenstate, and a perturbation V is understood to induce a transition to an infinite number of final states with a continuous spectrum. The Laplace transform of Eq. (2) provides an expression corresponding to the Green's function (or resolvent kernel) for the Schrödinger evolution. In a single-channel decay problem the pole approximation for the inverse Laplace transform results in an approximate semigroup property for $U^{\rm red}(t)$, which is equivalent to the perturbative analysis of Weisskopf and Wigner. However, in the case of the two- or more-channel decay, such as the neutral K meson decay (for which there is CP symmetry breaking), it has been shown that the pole approximation does not reproduce the semigroup evolution observed in Ref. [3]. To account for semigroup behavior application has been made of a quantum form [4,5] of the classical Lax-Phillips theory [6] (see also Ref. [7] and references cited therein) and its generalizations [8]. This theory achieves an exact semigroup law by imbedding the usual quantum theory in a larger Hilbert space, which consists of a direct integral of a family of Hilbert spaces of the usual type, foliated according to the time parameter. The result was particularly effective and straightforward for treating quantum-mechanical systems with Hamiltonians of unbounded spectra [9,10]. Its generalization for problems with semibounded spectra [8] made it clear that the imbedding associated with the Lax-Phillips theory effectively introduces many additional degrees of freedom. In this paper we show that semigroup evolution may be obtained by working in the framework of the usual quantum theory through an explicit coupling to many environmental degrees of freedom. Thus we provide a physical framework that accounts for the mathematical structure of the Lax-Phillips theory.

We study the Wigner-Weisskopf pole approximation theory [2] in the framework of the Lee-Friedrichs model [11]. This model consists of a subspace of discrete states interacting with a continuum subspace of states, for which there is no direct continuum-continuum interaction. It is convenient to take the discrete-discrete interaction to vanish as well. The association of the continuum with an environment constitutes the main aspect of the model. The environment here may be thought of as a distribution of a very large or even infinite number of final states into which the initial state of the system decays. The Lee-Friedrichs model [11] in Lee's construction was formulated in terms of nonrelativistic quantum field theory using a Hamiltonian for which the interaction $V^{\rm Lee} \propto a_N a_V^\dagger a_\Theta^\dagger + {\rm H.c.}$ included an annihilation operator a_N for the original (unstable) state N, multiplied by creation operators $a_V^\dagger a_\Theta^\dagger$ for the two-body final states V and Θ. These operators could have been constructed to include the creation of a many-body environment as well, maintaining

its equivalence with the Friedrichs quantum-mechanical form, with spectral coefficients coupling to a bath. In other words, the decay process may include not just the specified final states of the decay model, but also an environment. It is instructive to think of the spontaneous emission process [12,13] as the most-well-known illustration of decay into an infinity of final states, where an excited atomic state decays into a distribution of Fock space states of the radiation field (i.e., photons).

In what follows we show that by associating the continuous spectrum with an environment the reduced evolution of the discrete states subspace is defined by a spectral correlation matrix $\alpha(t)$, which is a function unifying the details of the interaction. The notion of the spectral correlation matrix is inspired by the well-studied particle-environment tensor product theories [14]. In these models the reduced evolution of the particle is obtained by tracing out the environmental degrees of freedom, leaving a stochastic dynamical equation that invokes the so-called environmental (in some examples complex) noise $z(t)$. As shown in Ref. [15], the environmental correlation function corresponds to the autocorrelation value of this noise, i.e., $\alpha(t) = \langle z^*(t)z(0)\rangle$, and, in agreement with the fluctuation-dissipation theorem, turns out to be a time-dependent memory kernel for the particle energy dissipation. Exploiting the analogous properties of the environmental correlation function and the spectral correlation matrix, we investigate the validity of the nontrivial semigroup evolution for the many-channel decay and identify the necessary conditions with the well-known Markovian limit [12,14]. In this way we consider a natural imbedding of the Wigner-Weisskopf idea into a theory of interaction with a reservoir (e.g., generalization of the theory of Anderson and Fano [16] and Lee's formulation).

Sections II and III include a review of essential results from the Wigner-Weisskopf theory and the derivation of the Markovian limit, provided with a brief summary of situations where this limit may be realized exactly or approximately (readers familiar with these concepts are welcome to quickly leaf through). In Sec. IV, which includes the main point of the paper, we discuss the effect of the Markovian limit on the Wigner-Weisskopf pole approximation and explain how the semigroup evolution law is achieved for the many-channel decay as well.

II. WIGNER-WEISSKOPF METHOD

In this section we summarize a functional formulation of the Wigner-Weisskopf pole approximation theory [2] and its generalization to many-channel decay [7]. Following the usual model for the decay of an unstable system, we consider a Hamiltonian of the form

$$H = H_0 + V, \tag{3}$$

where the spectrum of H_0 consists of a finite number N of discrete eigenvalues $\{\lambda_\alpha\}$, embedded into a continuum $\{\lambda \geqslant 0\}$ with spectral weight $dE(\lambda) = |\lambda\rangle\langle\lambda|d\lambda$. We study, in particular, the Lee-Friedrichs model [11], for which the interaction V couples the discrete states to the continuum, but does not couple continuum states or discrete states among themselves. The fact that the continuum subspace is associated with the products of the decay process presents an opportunity for the introduction of an environment. We assume that the initial unstable state of the system is given by a superposition of the discrete eigenstates $\{|\phi_\alpha\rangle\}$ of H_0:

$$|\psi_0\rangle = \sum_{\alpha=1}^{N} c_\alpha|\phi_\alpha\rangle, \quad H_0|\phi_\alpha\rangle = \lambda_\alpha|\phi_\alpha\rangle. \tag{4}$$

The reduced evolution, i.e., the evolution of the discrete subspace, is governed by the reduced propagator $R(z)$. This propagator in the Laplace transform is defined (for $\mathrm{Im}z > 0$) by the projection of the total system propagator

$$U(z) = \int_0^\infty dt\, e^{izt}e^{-iHt} = \frac{i}{z-H} \tag{5}$$

on the discrete subspace. The $\alpha\beta$ matrix element of the reduced propagator $R(z)$ in the Laplace domain is

$$R_{\alpha\beta}(z) \equiv -i\langle\phi_\alpha|U(z)|\phi_\beta\rangle = \left\langle\phi_\alpha\left|\frac{1}{z-H}\right|\phi_\beta\right\rangle. \tag{6}$$

Equation (6) may be written using $N \times N$ matrix notation [defining $W(z)$]

$$R(z) = \frac{1}{z - W(z)}, \tag{7}$$

which is confined to the discrete subspace and where, by comparing with Eq. (5), $W(z)$ is understood as the Laplace space reduced evolution generator. The reduced evolution in the time domain, dictated by $U^{\mathrm{red}}(t)$, is given by the inverse Laplace transform of Eq. (7),

$$U^{\mathrm{red}}(t) = \frac{1}{2\pi i}\int_C R(z)e^{-izt}dz, \tag{8}$$

where the contour of the integration C, shown in Fig. 1, runs slightly above the real line on the z plane from $+\infty$ to zero (the bottom of the positive spectrum) and then around the branch point from zero back to $+\infty$ slightly below the real line.

Since, in the general case, the exact calculation of Eq. (8) is difficult, one is interested in a useful approximation. We first note that there can be no pole for $\mathrm{Im}z \neq 0$ in Eq. (7) on the first Riemann sheet (see the Appendix). However, we may

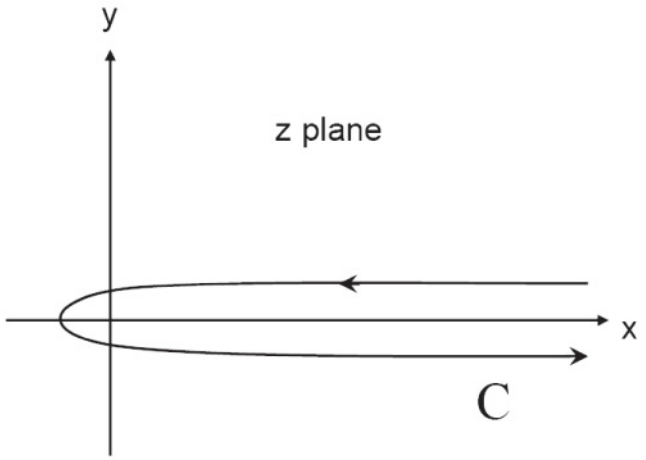

FIG. 1. Inverse Laplace transform contour C in Eq. (8).

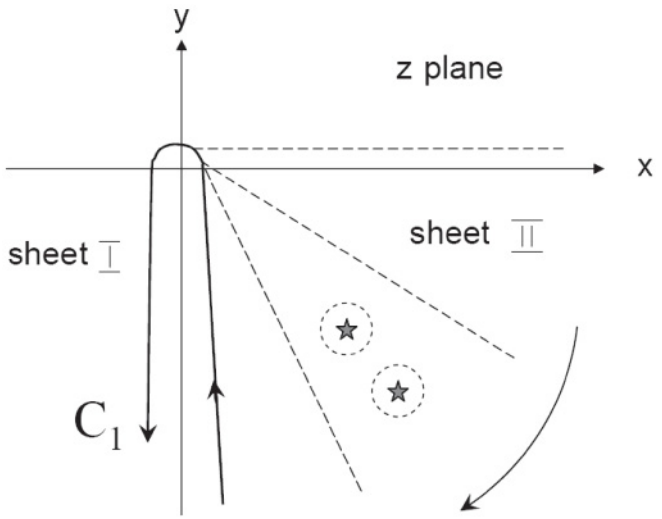

FIG. 2. Modified inverse Laplace transform contour C_1 in Eq. (10). The poles are denoted schematically by the stars.

explicitly continue the integration in Eq. (8) analytically to the second Riemann sheet [using Eq. (6)] [7]. Doing so yields

$$R^{\mathrm{II}}(z) = \frac{1}{z - W^{\mathrm{II}}(z)}, \tag{9}$$

which allows one to deform the contour of integration. The eigenvalues of $W^{\mathrm{II}}(z)$ (see below) determine the poles of $R^{\mathrm{II}}(z)$ in the lower half plane. The procedure, described in detail in Ref. [7], results in the following alternative expression for the reduced propagator:

$$U^{\mathrm{red}}(t) = \frac{1}{2\pi i}\int_{C_1} R^{\mathrm{I,II}}(z)e^{-izt}dz - 2\pi i \times \sum_j e^{-iz_j t}\mathrm{Res}[R^{\mathrm{II}}(z_j)], \tag{10}$$

where $\mathrm{Res}[R^{\mathrm{II}}(z_j)]$ [$R^{\mathrm{II}}(z)$ on the right and $R(z)$, which we denote above by $R^{\mathrm{I}}(z)$, on the left] is the residue of $R^{\mathrm{II}}(z)$ at the pole position z_j and the contour of the integration C_1, shown in Fig. 2, now runs around the branch point along the negative imaginary axis. The integration along the contour C_1 carries the factor e^{-izt} for z in the lower half plane. Hence, for $t > 0$ and not too small,[1] one can consider neglecting this term, called the background contribution. (There is a very long time contribution from the neighborhood of the branch point that we do not consider here.) Doing so, the evaluation of the reduced propagator reduces to the summation of the contributions of the residues of the poles of $R^{\mathrm{II}}(z)$ in the lower half plane. The assumed dominance of the these contributions is called the pole approximation.

[1]The scale of the value necessary to go beyond the nonexponential region of decay curve, often called the Zeno time [17], is determined by the dispersion of the Hamiltonian, as discussed in Ref. [18] and below [Eqs. (37) and (38)]. The onset of the exponential behavior generally occurs after the curve of steepest descent, which rotates clockwise in time [18], passes the first pole, which then dominates the time dependence.

To obtain an expression describing the residues of $R^{\mathrm{II}}(z)$, we focus on the reduced generator $W^{\mathrm{II}}(z)$. Even though $W^{\mathrm{II}}(z)$ is not generally self-adjoint, Eq. (9) may be represented in a way analogous to the spectral theorem, as the sum of normalized projectors $\{Q_\alpha(z)\}$, so that

$$R^{\mathrm{II}}(z) = \sum_{\alpha=1}^{N} \frac{Q_\alpha(z)}{z - \omega_\alpha(z)}, \tag{11}$$

where

$$Q_\alpha(z) = \frac{|\alpha,z\rangle_{RL}\langle\alpha,z|}{{}_L\langle\alpha,z|\alpha,z\rangle_R} \tag{12}$$

are made of the left and right eigenvectors of $W^{\mathrm{II}}(z)$ (corresponding to appropriate linear combinations of the eigenvectors of H_0, sometimes called decay eigenstates):

$$W^{\mathrm{II}}(z)|\alpha,z\rangle_R = \omega_\alpha(z)|\alpha,z\rangle_R,$$

$${}_L\langle\alpha,z|_R W^{\mathrm{II}}(z) = \omega_\alpha(z){}_L\langle\alpha,z|. \tag{13}$$

Clearly,

$$\begin{aligned} {}_L\langle\beta,z|W^{\mathrm{II}}(z)|\alpha,z\rangle_R &= \omega_\alpha(z){}_L\langle\beta,z|\alpha,z\rangle_R \\ &= \omega_\beta(z){}_L\langle\beta,z|\alpha,z\rangle_R \end{aligned} \tag{14}$$

may be valid for $\omega_\alpha(z) \neq \omega_\beta(z)$ (for any z) only if

$${}_L\langle\beta,z|\alpha,z\rangle_R = 0. \tag{15}$$

This orthogonality relation for the eigenvectors of $W^{\mathrm{II}}(z)$ provides the orthogonality of the appropriately normalized projectors

$$Q_\alpha(z)Q_\beta(z) = Q_\alpha(z)\delta_{\alpha\beta}, \tag{16}$$

at each point z. Equation (11) follows from Eq. (9) by using the spectral representation $W^{\mathrm{II}}(z) = \sum_\alpha^N \omega_\alpha(z)|\alpha,z\rangle_{RL}\langle\alpha,z|$ and the orthogonality properties of $Q_\alpha(z)$.

Equation (16) plays a crucial role in the examination of the semigroup property of the reduced evolution. Applying the pole approximation procedure, we neglect the background contribution and approximate the reduced propagator by the sum of the residues of $R^{\mathrm{II}}(z)$ [Eq. (11)], which for weak coupling V may be well approximated [7] to yield

$$U^{\mathrm{red}}(t) \cong \sum_j e^{-iz_j t}Q_{\alpha_j}(z_j), \tag{17}$$

where α_j corresponds to the singularity at $z_j = \omega_{\alpha_j}(z_j)$. Repeated application of this reduced evolution is then

$$U^{\mathrm{red}}(t_2)U^{\mathrm{red}}(t_1) \cong \sum_{j,k} e^{-iz_j t_2}e^{-iz_k t_1}Q_{\alpha_j}(z_j)Q_{\alpha_k}(z_k). \tag{18}$$

Although for the single-channel problem, if there is just one pole, the projectors' product on the right-hand side of Eq. (18) is trivially unity and Eq. (18) shows semigroup decay, for the many-channel decay with many poles the projectors $Q_{\alpha_j}(z_j)$ and $Q_{\alpha_k}(z_k)$ are generally evaluated at different pole locations on the Laplace plane. For $z_j \neq z_k$ the orthogonality relation [Eq. (16)] can no longer ensure $Q_{\alpha_j}(z_j)Q_{\alpha_k}(z_k) = 0$. Thus,

even in the pole approximation, the semigroup evolution is generally not valid [19,20], i.e.,

$$U^{\rm red}(t_2)U^{\rm red}(t_1) \neq U^{\rm red}(t_1+t_2) = \sum_j e^{-iz_j(t_1+t_2)}Q_{\alpha_j}(z_j). \tag{19}$$

In spite of this conclusion, many experiments display semigroup decay to high accuracy [3], while the estimates [19,20] have shown that the deviations predicted by the Wigner-Weisskopf theory would exceed the experimental error [3].

III. ASSOCIATION OF THE SPECTRAL DENSITY FUNCTION WITH AN ENVIRONMENT

In this section we sketch a treatment for the reduced system dynamics familiar in the field of quantum optics [12,13] and condensed-matter physics [14,16]. We adopt the Lee-Friedrichs model of the system described in the preceding section for the total Hamiltonian given by Eq. (3). For our current purpose it is not necessary to make any preliminary assumptions (other than, for simplicity, the lack of degeneracy) regarding the structure of the continuous part of H_0; it may or may not be bounded from below. The interaction picture propagator, defined through the total system propagator $U(t_2-t_1)$ and the unperturbed total system propagator $U_0(t-t_0)=e^{-iH_0(t-t_0)}$, is given by

$$\tilde{U}(t_2,t_1) = U_0^{-1}(t_2-t_0)U(t_2-t_1)U_0(t_1-t_0) \tag{20}$$

and obeys the equation

$$i\frac{d}{dt}\tilde{U}(t,t_0) = \tilde{V}(t)\tilde{U}(t,t_0), \tag{21}$$

where

$$\tilde{V}(t) = U_0^{-1}(t-t_0)VU_0(t-t_0) \tag{22}$$

is the interaction picture Hamiltonian. Integrating Eq. (21) and iterating it one time, we get the exact equation

$$\tilde{U}(t,t_0) = 1 - i\int_{t_0}^{t}\tilde{V}(\tau)d\tau - \int_{t_0}^{t}\int_{t_0}^{\tau}\tilde{V}(\tau) \times \tilde{V}(\tau')\tilde{U}(\tau',t_0)d\tau' d\tau. \tag{23}$$

Projecting Eq. (23) on the discrete subspace, we find

$$\tilde{U}^{\rm red}_{\alpha\beta}(t,t_0) \equiv \langle\phi_\alpha|\tilde{U}(t,t_0)|\phi_\beta\rangle = \delta_{\alpha\beta} - \sum_\gamma \int_{t_0}^{t}\int_{t_0}^{\tau}\int_\lambda e^{i\lambda_\alpha(\tau-t_0)}\langle\phi_\alpha|V|\lambda\rangle e^{-i\lambda(\tau-t_0)} \times e^{i\lambda(\tau'-t_0)}\langle\lambda|V|\phi_\gamma\rangle e^{-i\lambda_\gamma(\tau'-t_0)} \times \tilde{U}^{\rm red}_{\gamma\beta}(\tau',t_0)d\lambda\, d\tau' d\tau. \tag{24}$$

Here the first-order term $\langle\phi_\alpha|\tilde{V}(\tau)|\phi_\beta\rangle$ has vanished, because by assumption V does not couple the discrete states among themselves, and the last term was obtained using Eq. (22). Next we differentiate Eq. (24) with respect to t and obtain the reduced, i.e., projected into the unstable subspace, master equation

$$\frac{d}{dt}\tilde{U}^{\rm red}_{\alpha\beta}(t,t_0) = -\sum_\gamma e^{i\lambda_\alpha(t-t_0)}\int_{t_0}^{t}\int_\lambda e^{-i\lambda(t-\tau')}\omega_{\alpha\gamma}(\lambda)d\lambda \times e^{-i\lambda_\gamma(\tau'-t_0)}\tilde{U}^{\rm red}_{\gamma\beta}(\tau',t_0)d\tau', \tag{25}$$

where the matrix elements

$$\omega_{\alpha\gamma}(\lambda) \equiv \langle\phi_\alpha|V|\lambda\rangle\langle\lambda|V|\phi_\gamma\rangle \tag{26}$$

form the so-called spectral density matrix. The integral transform of the latter,

$$\tilde{\alpha}_{\alpha\gamma}(t,\tau') \equiv e^{i\lambda_\alpha(t-t_0)}\left(\int e^{-i\lambda(t-\tau')}\omega_{\alpha\gamma}d\lambda\right)e^{-i\lambda_\gamma(\tau'-t_0)}, \tag{27}$$

defines the elements of the spectral correlation matrix within the interaction representation. Using Eqs. (26) and (27), Eq. (25) may be written as

$$\frac{d}{dt}\tilde{U}^{\rm red}_{\alpha\beta}(t,t_0) = -\sum_\gamma\int_{t_0}^{t}\tilde{\alpha}_{\alpha\gamma}(t,\tau')\tilde{U}^{\rm red}_{\gamma\beta}(\tau',t_0)d\tau'. \tag{28}$$

Transforming back to the Schrödinger representation, we use the inverse version of Eq. (20). Taking into account that the spectral correlation matrix $\tilde{\alpha}(t,\tau')$ transforms analogously to $\tilde{U}(t_2,t_1)$, that $\frac{d}{dt}\tilde{U}(t,t_0) = iH_0U_0^{-1}(t-t_0)U(t,t_0) + U_0^{-1}(t-t_0)\frac{d}{dt}U(t,t_0)$, and that H_0 and $U_0(t_2-t_1)$ are diagonal in the reduced basis $\{|\phi_\alpha\rangle\}$ representation, in terms of matrix notation we obtain

$$\frac{d}{dt}U^{\rm red}(t,t_0) = -iH_0^{\rm red}U^{\rm red}(t,t_0) - \int_{t_0}^{t}\alpha(t-\tau')U^{\rm red}(\tau',t_0)d\tau'. \tag{29}$$

Here

$$H_0^{\rm red} = \sum_{\alpha=1}^{N}\lambda_\alpha|\phi_\alpha\rangle\langle\phi_\alpha| \tag{30}$$

is the discrete part of the unperturbed Hamiltonian H_0 and the spectral correlation matrix defined as

$$\alpha(t) \equiv \int_\lambda e^{-i\lambda t}\omega(\lambda)d\lambda \tag{31}$$

is the Fourier transform of the spectral density matrix

$$\omega(\lambda) \equiv \sum_{\alpha,\gamma=1}^{N}\omega_{\alpha\gamma}(\lambda)|\phi_\alpha\rangle\langle\phi_\gamma|. \tag{32}$$

We argue that $\alpha(t)$ defined by Eq. (31) can be associated with the noisy environmental correlation function $\alpha(t) = \langle z^*(t)z(0)\rangle$ mentioned earlier since the microscopic definition [14,15] of the latter evidently coincides with Eq. (31) up to an obvious generalization. The correlation matrix $\alpha(t)$ [Eq. (31)] represents all the microscopic details of the interaction, whose properties determine the type of reduced evolution, as will be clear from the following.

A. Equivalence of exact Markovian coupling and globally flat and unbounded spectral density matrix

Note that in an ideal case for which $\omega_{\alpha\gamma}(\lambda)$ [Eq. (26)] are independent of λ and the continuous spectrum of H_0 is unbounded (a physical example of such a situation could occur for a Stark-type interaction with a bath, which induces a shot noise) the correlation matrix [Eq. (31)] reduces to a delta function of time:

$$\alpha(t-\tau) = \omega\int_{-\infty}^{\infty} e^{-i\lambda(t-\tau)}d\lambda = 2\pi\omega\delta(t-\tau). \tag{33}$$

Substituting this into Eq. (29), we find

$$\frac{d}{dt}U^{\rm red}(t-t_0) = \left(-iH_0^{\rm red} - 2\pi\omega\right)U^{\rm red}(t-t_0). \tag{34}$$

Equation (33) is a local in time first-order differential equation with constant evolution generator, i.e., an equation describing semigroup evolution, also called the Markov equation, since it describes a Markovian stochastic process [21]. Its solution for $t > t_0$ is

$$U^{\rm red}(t-t_0) = e^{(-iH_0^{\rm red}-\Gamma)(t-t_0)}, \tag{35}$$

where the decay matrix Γ is given by

$$\Gamma \equiv \int_{t_0}^{t}\alpha(\tau)d\tau = \int_{t_0}^{t} 2\pi\omega\delta(t-\tau')d\tau = 2\pi\omega. \tag{36}$$

The well-known demonstration of the Zeno effect for very short times t relies on an expansion of the survival probability $P(t)$ of an unstable state $|\psi\rangle$ in a series [18]

$$\begin{aligned} P(t) &= |\langle\psi|e^{-iHt}|\psi\rangle|^2 \\ &\simeq |\langle\psi|1 - iHt - \tfrac{1}{2}H^2t^2 + \cdots|\psi\rangle|^2 \simeq 1 - t^2\Delta H^2, \end{aligned} \tag{37}$$

where

$$\Delta H^2 = \langle\psi|H^2|\psi\rangle - (\langle\psi|H|\psi\rangle)^2 \tag{38}$$

is the dispersion of the total Hamiltonian H in the state $|\psi\rangle$. The standard argument leading one to a conclusion about the principle impossibility of pure semigroup evolution is the presumable possibility to cut off the above expansion [Eq. (37)] after the second order in t. Note, however, that for the Lee-Friedrichs model in the Markovian limit the coefficient ΔH^2 diverges. If, for example, $|\psi\rangle = |\psi_\alpha\rangle$, Eq. (38) becomes

$$\begin{aligned} (\Delta H^2)^{\rm Markov} &= \langle\phi_\alpha|V^2|\phi_\alpha\rangle \\ &= \int_{-\infty}^{\infty} d\lambda\,\omega_{\alpha\alpha}(\lambda) \Rightarrow \infty. \end{aligned} \tag{39}$$

Hence the truncation of the expansion is invalid together with the physically interpreted conclusion. It should be stressed that under our assumptions regarding the structure of the coupling, there is no Zeno (see also Ref. [22]) effect and the result [Eq. (35)] is exact.

Now we briefly review how such assumptions may be realized. For this purpose we adopt the usual approach to the original definition of the spectral density function, i.e., as the continuum limit of a quasicontinuous spectrum. We assume that the unperturbed system, described by H_0, is confined to a large box with some standard boundary conditions. The spectrum of H_0 then consists of N discrete eigenstates embedded in a quasicontinuous spectrum of the environment including the decay products. Each element $\omega_{\alpha\gamma}(\lambda)$ of the spectral density matrix $\omega(\lambda)$ [Eq. (32)] is defined as the product of the interaction amplitudes $\langle\phi_\alpha|V|\lambda\rangle\langle\lambda|V|\phi_\gamma\rangle$, weighted by the local degeneracy of the environmental states

$$\omega_{\alpha\gamma}(\lambda) = \langle\phi_\alpha|V|\lambda\rangle\langle\lambda|V|\phi_\gamma\rangle D(\lambda), \tag{40}$$

where the density of states $D(\lambda)$ serves as an effective coarse graining. When the spectrum is truly continuous $D(\lambda)$ may be absorbed in $\langle\phi_\alpha|V|\lambda\rangle\langle\lambda|V|\phi_\gamma\rangle d\lambda$. The interaction amplitudes $\langle\phi_\alpha|V|\lambda\rangle\langle\lambda|V|\phi_\gamma\rangle$ are determined by the microscopic details, so their functional dependence on λ is dictated by the nature of the interaction. In contrast, the density of the environmental states $D(\lambda)$ is determined by H_0 and, in particular, by the boundary conditions to which the system is confined. Thus $D(\lambda)$ is at our disposal. Designing the geometry of the space, we may find a $D(\lambda)$ that suppresses, at least approximately, the λ dependence of $\langle\phi_\alpha|V|\lambda\rangle\langle\lambda|V|\phi_\gamma\rangle$ and yields a semigroup evolution with a desirable accuracy.[2]

B. Approximate Markovian coupling vs weak interaction and resonance

In the preceding subsection we have assumed a continuous spectrum on the whole real line in order to obtain an exact semigroup evolution. However, if there is a resonance and the coupling is weak, the leading behavior of the system is dictated by small regions of the continuous spectrum where the effect of the interaction is sharply enhanced. Such regions occur at λ sufficiently close to the resonances, which for small coupling are close to the eigenvalues $\{\lambda_\alpha\}$ of the discrete subspace. To show this we return to Eq. (25) and see whether it is possible to utilize it approximately, even if $\omega(\lambda)$ is not constant and the continuous part of the H_0 spectrum is bounded from below. Inspecting the integrand of the last term on the right hand-side of Eq. (25), we note that the collapse of the memory kernel may result from rather symmetrical manipulations with respect to the spectral or the time variables. For the perfectly Markovian coupling $\omega(\lambda)$ is constant and unbounded and the integral over λ yields $\delta(t-\tau')$ and admits the semigroup evolution of Eq. (34). Conversely, if it is justified to neglect the difference between $\tilde{U}^{\rm red}_{\gamma\beta}(\tau',t_0)$ and $\tilde{U}^{\rm red}_{\gamma\beta}(t,t_0)$ and to stretch the limits of the integration over τ' to $\pm\infty$, it also yields a Markovian equation of the form of Eq. (33) [and results in a factor $\delta(\lambda-\lambda_\gamma)$ as well]. The error caused by the replacement $\tilde{U}^{\rm red}_{\gamma\beta}(\tau',t_0) \to \tilde{U}^{\rm red}_{\gamma\beta}(t,t_0) \propto O(V^2)$ is negligible in case the interaction is sufficiently weak, because the last term of Eq. (25) is already second order in V. Such an

[2]There exists a class of phenomenological descriptions of the spectral density functions $\omega_{\alpha\gamma}(\lambda) \propto \eta_s\lambda_c^{1-s}\lambda^s e^{-\lambda/\lambda_c}$ [14], where η_s is the viscosity constant, the exponential factor provides a smooth cutoff modulated at the frequency λ_c, and $0 < s < 1$ and $s > 1$ describe the so-called subohmic and superohmic interactions, respectively. The slower the dependence of the spectral density function on λ, the closer the reduced evolution is to the Markovian limit. The boundary case of $s = 1$ corresponds to the ohmic interaction and is associated with an approximate Markovian spectral coupling.

approximation yields a solution that up to the second order in V is exact and provides a direct interpretation of the original perturbative computation of Weisskopf and Wigner [2] in the resolvent formalism. Making the weak-coupling assumption, we approximate the reduced master equation [Eq. (25)] by

$$\frac{d}{dt}\tilde{U}^{\rm red}_{\alpha\beta}(t,t_0) = -\sum_{\gamma}\int_{\lambda} e^{i\lambda_\alpha(t-t_0)}e^{i\lambda_\gamma t_0}e^{-i\lambda t}\omega_{\alpha\gamma}(\lambda) \times \int_{t_0}^{t} e^{-i(\lambda_\gamma-\lambda)\tau'}d\tau' d\lambda \tilde{U}^{\rm red}_{\gamma\beta}(t,t_0), \quad (41)$$

and focus on the integration over time. Since the interaction V is time independent, one may select the time origin such that $t_0 = -T/2$ and $t = T/2$. Then the integration over τ' in Eq. (41) yields [12]

$$\int_{-T/2}^{T/2} e^{-i(\lambda_\gamma-\lambda)\tau'}d\tau' = \frac{1}{\pi}\frac{\sin[(\lambda-\lambda_\gamma)T/2]}{(\lambda-\lambda_\gamma)}. \quad (42)$$

This is the well-known sinc function, which in the limit $T \to \infty$ is one of the definitions of the Dirac δ function. The fact that the sinc function [Eq. (42)] is sharply peaked around $\lambda = \lambda_\gamma$ and falls quickly when $\lambda - \lambda_\gamma > \frac{2\pi}{T}$ gives rise to the notion of the resonance: substituted back into the integral over λ in Eq. (41); the sinc function suppresses all the values of the spectral density function $\omega_{\alpha\gamma}(\lambda)$ outside this region. Thus we observe that stretching the limits of integration over τ' in Eq. (41) to infinity implies the replacement of the actual sinc function [Eq. (42)] by the Dirac δ function $\delta(\lambda-\lambda_\gamma)$. The error induced by this approximation is legitimate to neglect for $t = T/2$ sufficiently large and then Eq. (41) becomes the further approximated reduced master equation

$$\frac{d}{dt}\tilde{U}^{\rm red}_{\alpha\beta}(t-t_0) \approx -2\pi\sum_{\gamma} e^{i(\lambda_\alpha-\lambda_\gamma)(t-t_0)}\omega_{\alpha\gamma}(\lambda_\gamma)\tilde{U}^{\rm red}_{\gamma\beta}(t-t_0). \quad (43)$$

Transforming Eq. (43) back to the Schrödinger representation, we regain in matrix notation Eqs. (34) and (35), except that instead of the constant density matrix ω we have the resonant spectral density matrix

$$\omega^{\rm res} = \sum_{\alpha,\gamma=1}^{N} \omega_{\alpha\gamma}(\lambda=\lambda_\gamma)|\phi_\alpha\rangle\langle\phi_\gamma|, \quad (44)$$

whose elements are given by $\omega_{\alpha\gamma}(\lambda)$ [Eq. (26)] evaluated at the discrete eigenvalues $\lambda = \{\lambda_\gamma\}$ of H_0, where the subscript $\gamma \in [1,N]$ corresponds to the column index. Thus, for large enough $t = T/2$, the contribution of the spectral density over the sharp resonances is very much enhanced, the structure of the rest of the continuous spectrum of H_0 is quite unimportant for the reduced dynamics, and the error involved in the approximate Markovian equation [Eq. (43)] is small.

From the above considerations it is clear that the resonance Markovian assumption fails if (i) the interaction is strong and (ii) $\omega(\lambda)$ vanishes or undergoes significant changes near the resonant values $\lambda \cong \{\lambda_\gamma\}$. Nevertheless, the Markovian approximation is usually very good and suits a large variety of natural environments. To see why, let us restore the Planck constant $\hbar$, taken so far to be equal to unity. Doing so shows [12] that the region of the non-negligible values of the sinc function, i.e., the resonance area, is equal to $\frac{2\pi\hbar}{T}$. Since, for some reasonable T, the δ-function approximation of the sinc function is clearly a good one, the smoothness requirement of $\omega(\lambda)$ refers to a finite number of small regions in the spectrum. The requirements of the approximation in Eq. (43) are then not highly restrictive. Therefore, it would be experimentally difficult to detect any deviation from the semigroup evolution, which occurs only on atomic time scales. In order to do this, one may need to deliberately destroy the smoothness of the environmental spectrum (by careful choice of the boundary conditions), which on such small scales might be difficult [23,24]. If $\omega(\lambda)$ is changing rapidly (or vanishes) near the resonances, the Markovian approximation would be invalidated and the lifetime of the Zeno effect would be lengthened.

IV. WIGNER-WEISSKOPF POLE APPROXIMATION IN THE MARKOVIAN LIMIT

In the preceding section we saw that the general integro-differential equation [Eq. (29)] for the reduced propagator $U^{\rm red}(t,t_0)$ expresses the point that the dynamics of the open system may depend on its history. Yet semigroup evolution for the open system can be achieved if due to some special circumstances $U^{\rm red}(t,t_0)$ may reduce to a dynamics approximated by the form of Eq. (34). Since, in contrast to the closed system, the energy of the open system need not be conserved, the reduced evolution generator is not necessarily Hermitian and may have effective complex eigenvalues responsible for the decay.

As mentioned, Eq. (29) may reduce to a Markovian form either exactly or approximately. The former case occurs when the spectral density matrix $\omega(\lambda)$ [Eq. (26)] is constant and unbounded. The latter, much more realistic case yields the desired effect approximately through a combination of the second-order weak-coupling perturbation and resonances. In either case, the dramatic mutation of the time-dependent spectral correlation matrix $\alpha(t)$ [Eq. (31)] into a delta-correlated operator leads to the equation of the form of Eq. (34), which, independently of the dimension of the reduced subspace, results in a semigroup solution of the form of Eq. (35) and thus permits a semigroup decay also for the many-channel problem, as shown in our analysis of the Wigner-Weisskopf method.

To clarify the impact of the Markovian assumption on the pole approximation approach we first review its basics for the idealized Markovian limit. Note that in the case in which the continuum spectrum is unbounded, there is no branch point and the integration path C in the fundamental equation [Eq. (8)] goes above the whole real line (see Fig. 3). An immediate consequence of this is that the modified integration path C_1 of the background term $\int_{C_1} R^{\rm II}(z)e^{-izt}dz$ in Eq. (10) may be taken parallel to the real axis $\mathrm{Im}z = 0$ and if there is no obstacle to dragging it down to $\mathrm{Im}z \to -\infty$, the background contribution vanishes for all positive times. Further, we focus on the contributions of the pole terms. Recall Eq. (7) and note that assuming this reduced equation is put in the Markovian form implies that the evolution generator $W(z)$ appearing in the denominator of Eq. (7) [similarly to the total system propagator

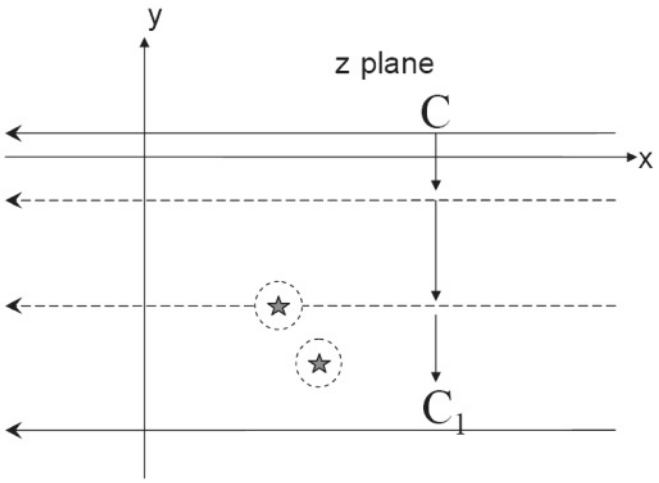

FIG. 3. Inverse Laplace transform contour C of Eq. (8) in the case in which the unbounded perfect Markovian spectral coupling is modified into the contour C_1. The poles are denoted schematically by the stars.

$U(z) = \mathcal{L}_{t\to iz}e^{-iHt} = i/(z-H)]$ is z independent, though not essentially Hermitian. The z independence of $W(z)$ automatically implies the z independence of its eigenvectors and eigenvalues defined in Eq. (13) and hence guarantees the orthogonality of the projectors $Q_{\alpha_j}(z_j)$ and $Q_{\alpha_k}(z_k)$ if $\omega_{\alpha_j} \neq \omega_{\alpha_k}$ for any z_j and z_k. Therefore, the idealized Markovian limit allows pure semigroup evolution with no Zeno effect [22] independently of the position or the number of poles.

To prove our last conclusion explicitly we establish the connection between the reduced generator $W^{\rm II}(z)$ and the spectral correlation matrix in the lower half of the Laplace plane. Combining Eqs. (9), (A7), and (A8), we find

$$W^{\rm II}(z) = H_0^{\rm red} + i\alpha(z). \tag{45}$$

Generally, the spectral correlation matrix $\alpha(t)$ [Eq. (31)], standing for the memory kernel of the integro-differential equation [Eq. (29)], may be any spread function of time. The essence of the Markovian limit consists of collapsing (either exactly or approximately) this memory kernel into a δ-correlated matrix. For the artificially unbounded spectrum $\lambda \in (-\infty,\infty)$ we have

$$\alpha^{\rm Markov}(t)\int_{-\infty}^{\infty} d\lambda\,\omega(\lambda)e^{-i\lambda t} = 2\pi\omega\delta(t), \tag{46}$$

and hence

$$i\alpha^{\rm Markov}(z) = i2\pi\omega\int_{-\infty}^{0} dt\,\delta(t)e^{izt} - 2\pi i\omega = -\pi i\omega. \tag{47}$$

where the spectral density matrix ω is a constant. Substituting this back into Eq. (45) yields

$$W^{\rm Markov\,II}(z) = H_0^{\rm red} + i\alpha^{\rm Markov}(z) = \text{const}. \tag{48}$$

What is left is to return to the original formulation of the Wigner-Weisskopf theory with the semibounded continuous spectrum and explain how the effects of weak coupling and resonance explain the high precision of semigroup evolution for either the single- or the many-channel decay. In reaching the above result, it was assumed that $\omega(\lambda)$ is supported in $(-\infty,\infty)$. We prove in the Appendix that $h^{\rm II}(z)$ may have zero determinant for some values of z in the lower half plane. The leading behavior of the reduced evolution is therefore dictated only by a finite number of small regions corresponding to the poles of $R^{\rm II}(z)$, where the contribution of the spectral density matrix $\omega(\lambda)$ is strongly enhanced, in agreement with resonance Markovian approximation. In such a case, the integral over λ in Eq. (A9) contributes primarily to the set of λ's close to the real part of the poles, whereas a fictitious extension of the integration to $-\infty$ will not change appreciably the value of the integral. Hence it is straightforward to repeat the derivations leading to the conclusion on the simultaneous vanishing of the background contribution and the orthogonality of the pole term projectors. The only difference is that instead of the constant spectral density matrix ω in Eqs. (46) and (47) we shall have the resonant λ-independent matrix $\omega^{\rm res}$ given by Eq. (44). To conclude, the result Eq. (48) is quite general in the case in which the resonances are fairly sharp (i.e., the poles in the second sheet are close to the real line). Even though not strictly exact, this Markovian limit approximation should be valid for many experimental conditions and becomes even better for weaker coupling.

V. SUMMARY

By associating the spectral weights of the Wigner-Weisskopf model for unstable system decay with the statistical properties of coupling to an environment, we were able to characterize the reduced evolution in the subspace of unstable states in terms of the spectral correlation matrix. Exploiting the properties of the latter, we showed that the Markovian limit distribution is sufficient to account for semigroup behavior for an arbitrary number of the decay channels, observed in experiment, such as in Ref. [3]. Besides the important impact of the spectral correlation matrix notion for understanding the reduced evolution of the multichannel decays, such as the two-channel K meson decay [19,20], it is clear that the association of the coupling to a spectral continuum with an environment may be similarly useful for any analogously modeled theory.

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We thank Dr. Y. Strauss for discussion of analogous results obtained by Nagy-Foias-Kolmogorov dilation [25], in which the quantum-mechanical problem is imbedded in a larger space as in the work of Maassen [26], who constructs a Fock space resulting in the generation of a noisy environment of the type we have discussed.

APPENDIX: SECOND SHEET PROPERTIES OF $R(z)$

In this appendix we discuss the structure of the reduced propagator in the complex Laplace plane. We specifically show that the propagator has poles only in the lower half plane near the real axis and also provide the background needed for establishing the connection between the reduced system

evolution generator and the spectral density matrix [Eq. (45)]. Following a well-known procedure [7], we rewrite Eq. (7) as

$$R(z) = \frac{1}{h(z)}. \tag{A1}$$

Here the operator

$$h(z) = z - H_0^{\rm red} - \int d\lambda \frac{\omega(\lambda)}{z-\lambda}, \tag{A2}$$

with $H_0^{\rm red}$ and $\omega(\lambda)$ given by Eqs. (30) and (32), respectively, is found straightforwardly from the projection of the total system propagator on the discrete subspace. The last term of Eq. (A2) can be recognized as the Laplace transform (for $\mathrm{Im}z > 0$) of the spectral correlation matrix $\alpha(t)$ [Eq. (31)]:

$$\begin{aligned} i\alpha(z) &\equiv \int d\lambda \frac{\omega(\lambda)}{z-\lambda} = i\int_0^\infty dt\, \alpha(t) e^{izt} \\ &= i\int_0^\infty dt \int d\lambda\, \omega(\lambda) e^{-i\lambda t} e^{izt}. \end{aligned} \tag{A3}$$

To obtain a singularity in $R(z)$ [Eq. (A1)] we need that the determinant of $h(z)$ [Eq. (A2)] vanishes. These non-Hermitian matrices can be put into Jordan canonical form with a unitary transformation, with eigenvalues along the diagonal. A vanishing eigenvalue implies a vanishing determinant. We therefore can ask whether $h(z)$ has a vanishing eigenvalue, i.e.,

$$\begin{aligned} &h(z)|\chi(z)\rangle_R \\ &\quad = \left(z - H_0^{\rm red} - \int d\lambda \frac{\omega(\lambda)}{z-\lambda}\right)|\chi(z)\rangle_R = 0, \end{aligned} \tag{A4}$$

where $|\chi(z)\rangle_{\rm R}$ is the right eigenvector of $h(z)$ with the presumed zero eigenvalue. Taking the scalar product with the eigenvector ${}_L\langle\chi(z)|$ from the left yields

$$\begin{aligned} &z|\chi(z)\rangle|^2 - {}_L\langle\chi(z)|H_0^{\rm red}|\chi(z)\rangle_R \\ &\quad - \int d\lambda \frac{{}_L\langle\chi(z)|\omega(\lambda)|\chi(z)\rangle_R}{z-\lambda} = 0. \end{aligned} \tag{A5}$$

In general, for the convergence of the integral in Eq. (A5), we must assume that $\omega(\lambda)$ decreases better than an inverse power of λ (no matter how small) since the integral contains $\sim\frac{d\lambda}{\lambda}$ for very large λ. Now we consider the imaginary part of Eq. (A5). Since $H_0^{\rm red}$ is Hermitian, its expectation value in the state $|\chi(z)\rangle_R$ is real and we are left with

$$\mathrm{Im}z\left(|\chi(z)\rangle|^2 + \int d\lambda \frac{{}_L\langle\chi(z)|\omega(\lambda)|\chi(z)\rangle_R}{|z-\lambda|^2}\right) = 0, \tag{A6}$$

i.e., $\mathrm{Im}z$ times a positive quantity. This can never be zero for $\mathrm{Im}z \neq 0$ on the first Riemann sheet. We are therefore required to go to the second sheet:

$$R^{\rm II}(z) = \frac{1}{h^{\rm II}(z)}, \tag{A7}$$

$$h^{\rm II}(z) = z - H_0^{\rm red} - i\alpha(z). \tag{A8}$$

Here $R^{\rm II}(z)$ and $h(z)^{\rm II}$ are the smooth continuations of $R(z)$ and $h(z)$ into the lower half plane and

$$i\alpha(z) = i\int_{-\infty}^0 dt \int d\lambda\, \omega(\lambda) e^{-i\lambda t} e^{izt} - 2\pi i\omega(z) \tag{A9}$$

is the analytic continuation of the spectral correlation matrix $\alpha(z)$ [Eq. (A3)], where $\omega(z)$ is the analytic continuation of $\omega(\lambda)$ into the lower half plane. The condition for a vanishing eigenvalue [Eq. (A5)] now reads

$$\begin{aligned} &z|\chi(z)\rangle|^2 - {}_L\langle\chi(z)|H_0^{\rm red}|\chi(z)\rangle_R \\ &\quad - \int d\lambda \frac{{}_L\langle\chi(z)|\omega(\lambda)|\chi(z)\rangle_R}{z-\lambda} \\ &\quad + 2\pi i\, {}_L\langle\chi(z)|\omega(z)|\chi(z)\rangle_R = 0. \end{aligned} \tag{A10}$$

Taking the imaginary part as before gives

$$\begin{aligned} &\mathrm{Im}z\left(|\chi(z)\rangle|^2 + \int d\lambda \frac{{}_L\langle\chi(z)|\omega(\lambda)|\chi(z)\rangle_R}{|z-\lambda|^2}\right) \\ &\quad + 2\pi i\, {}_L\langle\chi(z)|\omega(z)|\chi(z)\rangle_R = 0, \end{aligned} \tag{A11}$$

so that there may be a solution for $\mathrm{Im}z$ negative since $\omega(\lambda)$ is positive definite. It is easy to see that the expectation of $\omega(\lambda)$ is the sum of absolute squares; the analytic continuation of $\omega(\lambda)$ to the lower half plane for small imaginary part of z, enough to reach a resonance pole, is assumed to remain approximately real since it is smoothly connected to real values of $\omega(\lambda)$ for λ on the real line. In the Markovian limit this function is taken to approach a constant.

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