Quantum theory of multiple-input–multiple-output Markovian feedback with diffusive measurements

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(Received 24 February 2011; revised manuscript received 11 May 2011; published 26 July 2011)

Feedback control engineers have been interested in multiple-input–multiple-output (MIMO) extensions of single-input–single-output (SISO) results of various kinds due to its rich mathematical structure and practical applications. An outstanding problem in quantum feedback control is the extension of the SISO theory of Markovian feedback by Wiseman and Milburn [Phys. Rev. Lett. **70**, 548 (1993)] to multiple inputs and multiple outputs. Here we generalize the SISO homodyne-mediated feedback theory to allow for multiple inputs, multiple outputs, and *arbitrary* diffusive quantum measurements. We thus obtain a MIMO framework which resembles the SISO theory and whose additional mathematical structure is highlighted by the extensive use of vector-operator algebra.

DOI: 10.1103/PhysRevA.84.012120 PACS number(s): 03.65.Ta, 42.50.Dv, 42.50.Lc, 42.50.Pq

I. INTRODUCTION

Feedback control engineering [1] is ubiquitous in modern technology [2,3]. As we further miniaturize technology, a quantum theory of feedback control can be expected to be essential [4,5]. In fact, the realization that quantum technology may benefit from modern control theory is currently driving a research program in which concepts from classical control systems [6,7] are being applied and extended to quantum systems [8–18]. This facet of quantum feedback control makes it an interdisciplinary field, attracting both engineers and physicists.

A control strategy that has been widely studied is Markovian feedback [5], which has useful applications in quantum information [19–24]. This is a continuous (in time) process which can be briefly summarized by Fig. 1. A general framework for such a process when the system has only one measurement output and one feedback input [Fig. 1(a), a case which we refer to as single-input single-output, abbreviated to SISO] mediated by homodyne detection was first put forth by Wiseman and Milburn [25]. In that work they treated feedback as an instantaneous process. A more detailed treatment that showed how to account for a feedback delay and how the limit of zero delay should be appropriately taken, giving rise to Markovian system evolution, was later given by Wiseman [26]. This is the most complete theory of Markovian feedback developed to date.

A theory of multiple-input multiple-output [MIMO; Fig. 1(b)] quantum feedback would be necessary in any situation where multiple degrees of freedom of a quantum system are monitored and controlled. The system could be a register of qubits or the different canonical momenta (or positions) of a system of quantum objects. Indeed, investigations in this direction with a few inputs and outputs have already begun [19,27–31]. With the drive to build realistic quantum computing devices where quantum information would be encoded in many qubits a general theory of MIMO control would be an valuable tool to obtain.

The extension of Ref. [26] to multiple inputs and multiple outputs would seem to be the obvious follow-up so it is natural to ask why this generalization was not made until now. There are two reasons for this. The first is related to the strategy underlying a master equation approach to open systems: changes in our distinguished system due to its interactions with other ancillary quantum systems are taken into account by including, in the master equation, parameters (numbers) which characterize these ancillary objects. The measurement step in the feedback loop shown in Fig. 1 then defines a necessary point of interaction between the system and the measuring device. A mathematical representation of the measurement is therefore necessary; without it a master equation for the controlled system *cannot* be derived. Finding this mathematical representation is nontrivial and it was not until 2001 that a representation of diffusive measurements with unit detection efficiency was found [32]. The end result is a parametrization called the unravelling matrix, generalized in 2005 to include nonunit detection efficiency [33]. In this paper we will use a different parametrization (which we refer to as the M representation (M-rep) [34]) because our results are simpler when expressed in terms of the M-rep of diffusive measurements.

The second reason for not extending the SISO work of Ref. [26] to multiple inputs and multiple outputs earlier was due to a lack of motivation. The aforementioned research program of finding quantum-mechanical parallels of classical control has only proliferated in recent times.[1] The physics and engineering communities at the time of Ref. [26] were more or less separated and terms such as *MIMO* and *nonlinear systems* did not mean much to physicists. Control engineers have long been interested in generalizing various SISO results to the MIMO case, due to both its mathematical structure and the prospects of practical applications that MIMO systems can offer [38–43]. It remains to be an active line of research today in the engineering community [44–47]. So a second motivation for constructing a MIMO theory of feedback is to allow quantum control to benefit from the works of engineers and, more generally, aid in the broader program of drawing

[1]It is interesting to note that some engineers were already curious about such questions much earlier [35–37].

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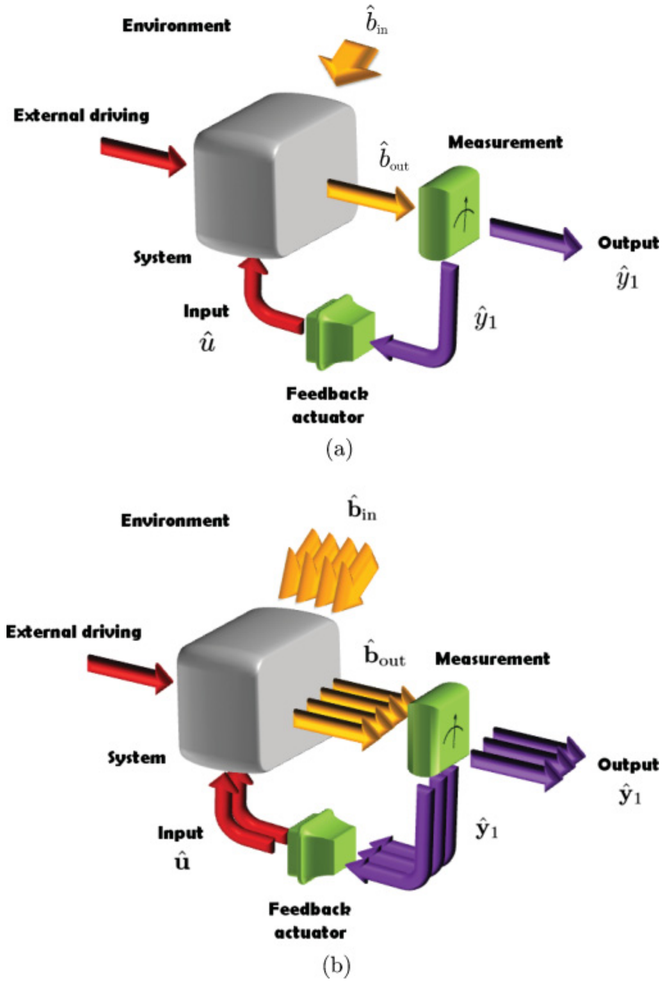

FIG. 1. (Color online) Markovian feedback in the case of (a) SISO and (b) MIMO. Note the number of $\hat{b}_{\rm in}$ fields is not part of the definition of MIMO. For generality we will take there to be L such inputs. We take the environment to be a collection of (bosonic) harmonic oscillators. The system interacts with the bath field $\hat{\mathbf{b}}_{\rm in}$ and this process turns $\hat{\mathbf{b}}_{\rm in}$ into $\hat{\mathbf{b}}_{\rm out}$ which then gets detected. Here we are defining an N-component vector operator $\hat{\mathbf{a}}$ as $\hat{\mathbf{a}} \equiv (\hat{a}_1,\hat{a}_2,\ldots,\hat{a}_N)^\top$ (see Appendix A of Ref. [34]). The detection process produces a current, modelled by $\hat{\mathbf{y}}_1$, which is then fed back into a feedback actuator. The actuator uses the information in the measured current to implement a control $\hat{\mathbf{u}}$ on the system. The measurement ouput $\hat{\mathbf{y}}_1$ is usually referred to as just the "output" and the control vector operator $\hat{\mathbf{u}}$ as the "input." It is possible to allow the number of outputs to differ from the number of inputs, but for simplicity (and without loss of generality, see Appendix A) we will let these be the same, equal to R. Markovian feedback may then be defined by $\hat{\mathbf{u}} = \hat{\mathbf{y}}_1$.

analogies between classical and quantum theories of feedback control.

We now foreshadow some of our key results and outline the organization of the paper to help simplify its reading. In Sec. II we introduce the theory of quantum measurements in the Heisenberg picture and discuss how such a model can be extended to include feedback.

The theory of Sec. II is then used to describe the unconditional evolution of the system in Sec. III. The unconditional dynamics can be described by either a master equation for the system state ρ or a Heisenberg equation of motion for an arbitrary vector operator $\hat{\mathbf{s}}$ on the joint system-bath Hilbert space. Strictly speaking, the system state ρ will obey only a *Markovian* master equation if the feedback delay τ is negligible. This will be the case if τ is much smaller than the characteristic time scale of any other system dynamics. We will first derive the MIMO Markovian feedback master equation by first allowing a nonzero delay and taking the zero-delay limit in the end. This gives us Result 1 in Fig. 2. We will derive the master equation by working in the Heisenberg picture and then transforming to the Schrödinger picture in the end. We will also take a shortcut to Result 1 by deriving only the feedback contribution of the Heisenberg equation for $\hat{\mathbf{s}}$ [see (26)]. This point will be expounded on in Appendix C. The full Heisenberg equation of motion, which corresponds to the master equation in the Markovian limit (i.e., $\tau \to 0$), is shown as Result 3 in Fig. 2. Result 3 is described by Fig. 1(b) where the measurement output is represented by a vector operator $\hat{\mathbf{y}}_1$, as opposed to a c number. The unconditional evolution in the Markovian limit can also be described by assuming the feedback to be instantaneous ($\tau = 0$) from the start. We also take this approach, and as one would hope for, obtain an equation of motion for $\hat{\mathbf{s}}$ (Result 4) which can also be derived from Result 3 as shown in Fig. 2.

In Sec. IV we consider time evolution with conditioning on the fed-back current. The MIMO stochastic feedback master equation (shown as Result 5 in Fig. 2) and two-time correlation function of the measured current are derived in this section using Result 4. In Sec. V we show how our theory of MIMO feedback correctly reproduces previously known results in the limiting cases of homodyne- and heterodyne-mediated feedback. We then conclude with a discussion in Sec. VI.

At this point we would like to refer the reader to our exposition of vector-operator algebra in the appendix of Ref. [34] as this is used extensively in this paper. We also mention that for convenience we will not necessarily reflect the multicomponent nature of vectors or vector operators in our language when they are referred to, such as in "the field $\hat{\mathbf{a}}$" or "the current $\hat{\mathbf{y}}$," as opposed to using plurals as in "the fields $\hat{\mathbf{a}}$" or "the currents $\hat{\mathbf{y}}$."

II. REVIEW OF HEISENBERG-PICTURE DYNAMICS

A. Open quantum systems

To set the premise of our theory we refer to Fig. 1 but in the absence of the feedback actuator (i.e., $\hat{\mathbf{u}} = \mathbf{0}$). The system and environment can be considered as one closed system whose time evolution is described by

$$\hat{H} = \hat{H}_0 + \hat{H}_1 + \hat{H}_{\rm m}, \tag{1}$$

where $\hat{H}_0$ consists of the free Hamiltonians for the system and bath. Evolution due to external driving or, for example, the extra Lamb shift that is often dropped in quantum optics [48] are accounted for by $\hat{H}_1$. The environment is assumed to be a free bosonic field in one dimension (i.e., specified by a space-time coordinate) in the vacuum state and the system interacts with the environment by exchanging energy quanta with the bath field. We model this by the coupling Hamiltonian

$$\hat{H}_{\rm m} = i(\hat{\mathbf{b}}_{\rm in}^\dagger\,\hat{\mathbf{c}} - \hat{\mathbf{c}}^\dagger\,\hat{\mathbf{b}}_{\rm in}), \tag{2}$$

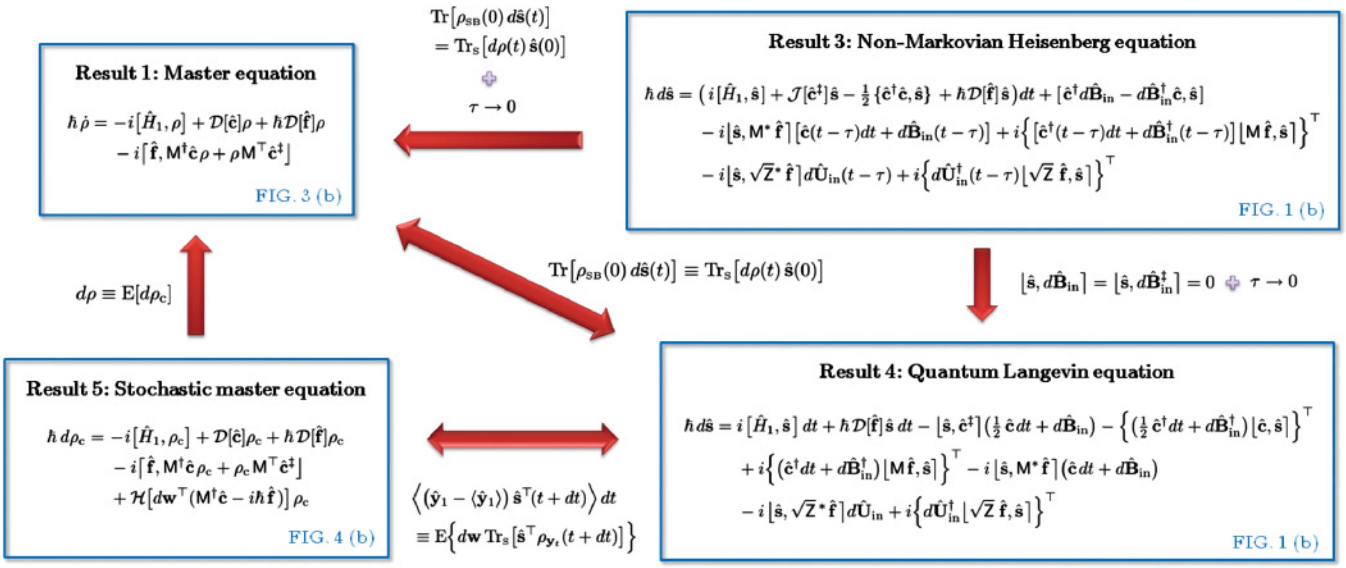

FIG. 2. (Color online) Preview of some of the major results of this paper and their relations with each other. Note that Results 2 and 6 are not shown here. The red arrows indicate that one result may be obtained from another by using the equations next to the arrows. An equation next to an arrow written with an $\equiv$ means that equation is always true, i.e., it is an identity in the context implied by the results connected. If the equation is written with an $=$ (or an $\to$) then it is an assumption (explained below). All the results shown here describe diffusion-mediated MIMO Markovian feedback so for simplicity we do not qualify these results by referring, for example, to Result 1 as a "diffusion-mediated MIMO Markovian feedback master equation." The only exception is Result 3, which is non-Markovian. Result 1 is derived first in the paper, which is a master equation. This is an equation of motion for the state of the system ρ and is thus a result in the Schrödinger picture. One may also describe feedback in the Heisenberg picture, and the counterpart to Result 1 is the quantum Langevin equation given by Result 4. Result 1 can be obtained from Result 3 as indicated but note the assumption that $\hat{\mathbf{s}}(0)$ is a vector operator on the system Hilbert space. Result 4 can also be derived directly from Result 3 and the assumption made in doing so is that $\hat{\mathbf{s}}$ commutes with $d\hat{\mathbf{B}}_{\mathrm{in}}$ and $d\hat{\mathbf{B}}_{\mathrm{in}}^\ddagger$ (it is, however, still a vector operator on the joint system-bath Hilbert space). It is because of this assumption, and the limit $\tau \to 0$, that we refer to Result 4 as a quantum Langevin equation, since a Langevin equation typically describes Markovian ($\tau \to 0$) evolution for a "system" vector operator ($\lfloor\hat{\mathbf{s}},d\hat{\mathbf{B}}_{\mathrm{in}}\rceil = \lfloor\hat{\mathbf{s}},d\hat{\mathbf{B}}_{\mathrm{in}}^\ddagger\rceil = 0$). Since neither of these assumptions are made in Result 3, we refer to it simply as a Heisenberg equation. The master equation describes unconditional evolution for the system state, which means that ρ in Result 1 represents an observer's state of knowledge of the system without reference to the measurement output $\mathbf{y}$ [depicted in Fig. 3(b)]. When the observer uses the information in $\mathbf{y}$, his/her state of knowledge (i.e., ρ) about the system then has to be updated according to Result 5. This is known as conditional evolution as the observer is conditioning his/her knowledge on $\mathbf{y}$. Because Result 5 is "like" a master equation except that it contains a Wiener increment $d\mathbf{w}$ we refer to it as a stochastic master equation. Averaging over the realizations of Result 5 (denoted by $\mathrm{E}[d\rho_{\mathrm{c}}]$ in the figure) reproduces the master equation. The stochastic master equation can also be related to the quantum Langevin equation by an identity involving the measurement output in the Heisenberg picture $\hat{\mathbf{y}}_1$. Note that we are writing $\rho_{\mathbf{y}_t}$ to mean the state conditioned on $\mathbf{y}(t)$ (the Schrödinger-picture equivalent of $\hat{\mathbf{y}}_1$ at time t). The identity between Results 5 and 4 should be contrasted with that connecting Results 1 and 4.

where $\hat{\mathbf{c}}$ and $\hat{\mathbf{b}}_{\mathrm{in}}$ are each an L-component vector operator and the Hermitian conjugate of an N vector operator $\hat{\mathbf{A}}$ is defined by

$$\hat{\mathbf{A}}^\dagger = (\hat{A}_1^\dagger, \hat{A}_2^\dagger, \ldots, \hat{A}_N^\dagger). \tag{3}$$

Note that our measurement is performed on the bath, so within the standard quantum theory of indirect measurements [49] the environment acts as our measuring apparatus and (2) effects a measurement interaction. The field $\hat{\mathbf{b}}_{\mathrm{in}}(t)$ represents quantum noise and $d\hat{\mathbf{B}}_{\mathrm{in}}(t) \equiv \hat{\mathbf{b}}_{\mathrm{in}}(t)\,dt$ is a quantum Wiener increment [50]. That is it has zero mean

$$\langle d\hat{\mathbf{B}}_{\mathrm{in}}(t)\rangle = \mathbf{0} \tag{4}$$

and satisfies the (quantum) Itô rule

$$d\hat{\mathbf{B}}_{\mathrm{in}}(t)\,d\hat{\mathbf{B}}_{\mathrm{in}}^\dagger(t) = \hbar\,\hat{\mathrm{I}}_L\,dt, \tag{5}$$

with all other second or higher moments negligible. We are denoting an $L \times L$ identity matrix operator (see Ref. [34]) by $\hat{\mathrm{I}}_L$. The dynamics due to $\hat{H}_0$ is usually well known and we can simplify matters by first transforming to a frame rotating at a frequency set by $\hat{H}_0$ and subsequently define all time evolution with respect to this frame. Unless required we will generally omit the time dependence due to $\hat{H}_0$ and define our Schrödinger and Heisenberg pictures with respect to the rotating frame defined by $\hat{H}_0$ [5].

For simplicity we group $\hat{H}_1$ with $\hat{H}_{\mathrm{m}}$ to define the time-evolution operator due to "measurement" by the Hudson-Parthasarathy equation [51]

$$\hbar\,d\hat{U}_{\mathrm{m}}(t,t_0) = \big(-i\hat{H}_1\,dt - \tfrac{1}{2}\hat{\mathbf{c}}^\dagger\hat{\mathbf{c}}\,dt + d\hat{\mathbf{B}}_{\mathrm{in}}^\dagger\hat{\mathbf{c}} - \hat{\mathbf{c}}^\dagger d\hat{\mathbf{B}}_{\mathrm{in}}\big) \times \hat{U}_{\mathrm{m}}(t,t_0), \tag{6}$$

where $\hat{U}_{\mathrm{m}}(t_0,t_0) = \hat{1}$.

As a consequence of the singluar nature of $\hat{\mathbf{b}}_{\mathrm{in}}$, the unitary evolution specified by (6) gives rise to an output field in the

Heisenberg picture

$$d\hat{\mathbf{B}}_{\rm out}(t) \equiv \hat{U}_{\rm m}^{\dagger}(t+dt,t)\,d\hat{\mathbf{B}}_{\rm in}(t)\,\hat{U}_{\rm m}(t+dt,t), \\ = \hat{\mathbf{c}}(t)dt + d\hat{\mathbf{B}}_{\rm in}(t). \tag{7}$$

Note that $\hat{\mathbf{b}}_{\rm in}$ and $\hat{\mathbf{b}}_{\rm out}$ are different parts of the same quantum field, namely before and after interaction with the system [52, 53]. As such, the input and output fields will commute only with an arbitrary system operator $\hat{\mathbf{s}}$ at different times,

$$\lfloor d\hat{\mathbf{B}}_{\rm in}(t),\hat{\mathbf{s}}(t')\rceil = 0 \quad \forall\, t' \leqslant t, \tag{8}$$

$$\lfloor d\hat{\mathbf{B}}_{\rm out}(t),\hat{\mathbf{s}}(t')\rceil = 0 \quad \forall\, t' > t. \tag{9}$$

Here $\lfloor \hat{\mathbf{A}},\hat{\mathbf{B}}\rceil$ is the matrix-operator bracket for two vector operators $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, defined as [34]

$$\lfloor \hat{\mathbf{A}},\hat{\mathbf{B}}\rceil = \hat{\mathbf{A}}\hat{\mathbf{B}}^{\top} - (\hat{\mathbf{B}}\hat{\mathbf{A}}^{\top})^{\top}. \tag{10}$$

An arbitrary vector operator $\hat{\mathbf{s}}$ will evolve, due to the measurement interaction, according to the Heisenberg equation derived from (6)

$$\hbar\,[d\hat{\mathbf{s}}]_{\rm m} = \big(i\,[\hat{H}_1,\hat{\mathbf{s}}] + \mathcal{J}[\hat{\mathbf{c}}^{\ddagger}]\hat{\mathbf{s}} - \tfrac{1}{2}\{\hat{\mathbf{c}}^{\dagger}\hat{\mathbf{c}},\hat{\mathbf{s}}\}\big)dt \\ + [\hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in} - d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}},\hat{\mathbf{s}}], \tag{11}$$

where $\{\hat{A},\hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}$. We have defined

$$\mathcal{J}[\hat{\mathbf{A}}^{\ddagger}]\,\hat{\mathbf{B}} = (\hat{\mathbf{A}}^{\ddagger}\hat{\mathbf{B}}^{\top})^{\top}\hat{\mathbf{A}}, \tag{12}$$

where $\hat{\mathbf{A}}^{\ddagger} \equiv (\hat{\mathbf{A}}^{\top})^{\dagger}$ (see Ref. [34]). To illustrate the use of the vector-operator algebra introduced in Ref. [34] we have derived (11) in Appendix B. It is then easy to show that transforming this to the Schrödinger picture gives the master equation due to measurement

$$\hbar\,[d\rho]_{\rm m} \equiv \mathcal{L}_{\rm m}\,\rho\,dt = -i\,[\hat{H}_1,\rho]dt + \mathcal{D}[\hat{\mathbf{c}}]\rho\,dt, \tag{13}$$

where

$$\mathcal{D}[\hat{\mathbf{c}}]\rho = \hat{\mathbf{c}}^{\top}\rho\,\hat{\mathbf{c}}^{\ddagger} - \tfrac{1}{2}\{\hat{\mathbf{c}}^{\dagger}\hat{\mathbf{c}},\rho\}. \tag{14}$$

B. Quantum measurements

The output field $\hat{\mathbf{b}}_{\rm out}$ is then measured and the detector produces a current $\mathbf{y}$. In the Heisenberg picture the current is represented by a vector operator, which in general will be some function of the output field $\hat{\mathbf{b}}_{\rm out}$

$$\hat{\mathbf{y}}_1 = g(\hat{\mathbf{b}}_{\rm out},\hat{\boldsymbol{\xi}}), \tag{15}$$

where $\hat{\boldsymbol{\xi}}$ is measurement noise.

For the remainder of this paper we concentrate on the class of diffusive measurements. It was shown previously that the output of such a measurement can be represented by an $R \times 1$ vector operator [5,34]

$$\hbar\,\hat{\mathbf{y}}_1\,dt = \mathsf{M}^{\dagger}d\hat{\mathbf{B}}_{\rm out} + \mathsf{M}^{\top}d\hat{\mathbf{B}}_{\rm out}^{\ddagger} + \hbar\,d\hat{\boldsymbol{v}}_{\rm in}. \tag{16}$$

The subscript for the current here does not mean that it is related to $\hat{H}_0$ and $\hat{H}_1$ in (1); instead, it is to remind us that the current is defined in terms of output field $d\hat{\mathbf{B}}_{\rm out}$. This will be useful when we consider feedback in Sec. III B when the current will be defined in terms of the input field. Note that corresponding to each component of $\hat{\mathbf{c}}$ (or each dissipative channel) we need at most two quadrature measurements so $R \leqslant 2L$. The matrix M is $L \times R$ defined by

$$\mathsf{M}\mathsf{M}^{\dagger}/\hbar \in \mathfrak{H}, \tag{17}$$

where $\mathfrak{H} = \{{\rm diag}(\boldsymbol{\eta})\,|\,\forall\,k,\,\eta_k \in [0,1]\}$. The noise $d\hat{\boldsymbol{v}}_{\rm in}$ in (16) is a $R \times 1$ Hermitian vector operator with zero mean and correlations given by

$$(\hbar\,d\hat{\boldsymbol{v}}_{\rm in})\,(\hbar\,d\hat{\boldsymbol{v}}_{\rm in})^{\top} = \hbar\,\mathsf{Z}\hat{\mathrm{I}}_R\,dt, \tag{18}$$

where

$$\mathsf{Z} = \hbar\,\mathrm{I}_R - \mathsf{M}^{\dagger}\mathsf{M}. \tag{19}$$

We can express $d\hat{\boldsymbol{v}}_{\rm in}$ in terms of independent quantum Wiener increments

$$\hbar\,d\hat{\boldsymbol{v}}_{\rm in} = \sqrt{\mathsf{Z}}\,d\hat{\mathbf{U}}_{\rm in} + \sqrt{\mathsf{Z}}^{*}\,d\hat{\mathbf{U}}_{\rm in}^{\ddagger}. \tag{20}$$

The increments $d\hat{\mathbf{U}}_{\rm in}$ are completely uncorrelated with the system so they satisfy

$$\lfloor d\hat{\mathbf{U}}_{\rm in}(t),\hat{\mathbf{s}}(t')\rceil = 0 \quad \forall\, t,t'. \tag{21}$$

We remind the reader that this is not what is usually referred to as the measurement noise $d\hat{\mathbf{v}}_{\rm m}$.

C. Adding feedback

We can describe feedback on the system by adding another Hamiltonian $\hat{H}_{\rm fb}$ to (1):

$$\hat{H} = \hat{H}_0 + \hat{H}_1 + \hat{H}_{\rm m} + \hat{H}_{\rm fb}. \tag{22}$$

In general, $\hat{H}_{\rm fb}$ will describe the coupling of the input $\hat{\mathbf{u}}$, which may be a functional of the current $\hat{\mathbf{y}}_1$, to the system. Markovian feedback can be defined as the coupling of the measured current $\hat{\mathbf{y}}_1$ (in which case $\hat{\mathbf{u}} = \hat{\mathbf{y}}_1$) to a Hermitian system vector operator $\hat{\mathbf{f}}$ (see Appendix A). As a result of working in the idealized limit where $\hat{\mathbf{y}}_1$ contains white noise it is only sensible to consider $\hat{\mathbf{f}}$ being coupled linearly to $\hat{\mathbf{y}}_1$, i.e.,

$$\hat{H}_{\rm fb} = \hbar\,\hat{\mathbf{f}}^{\top}\,\hat{\mathbf{y}}_1, \tag{23}$$

where $\hat{\mathbf{f}}$ and $\hat{\mathbf{y}}_1$ are at the same time. Coupling $\hat{\mathbf{f}}$ to any nonlinear function of $\hat{\mathbf{y}}_1$ would generate time evolution which is indescribable by (quantum) stochastic calculus.

The careful reader will notice a number of issues with the Hamiltonian (23). First, $\hat{\mathbf{y}}_1$ does not commute with $\hat{\mathbf{f}}$ at the same time, so $\hat{H}_{\rm fb}$ as it stands is not even Hermitian. Second, it does not strictly exist because although $\hat{\mathbf{y}}_1\,dt$ exists as a stochastic increment, $\hat{\mathbf{y}}_1$ does not.

The first problem can be solved in two ways as was recognized in Ref. [26]. The first is to realize that in actuality there must be a finite time delay in the feedback loop. Thus, strictly we have

$$\hat{H}_{\rm fb} = \hbar\,\hat{\mathbf{f}}^{\top}\,\hat{\mathbf{y}}_1(t-\tau), \tag{24}$$

and $\hat{\mathbf{y}}_1(t-\tau)$ commutes with all system operators at times later than $t-\tau$ and so acts as a complex number for $\tau \neq 0$. The limit $\tau \to 0^+$ can be taken at the end of all calculations. We will derive a Markovian ($\tau \to 0^+$) master equation with feedback using this method in Sec. III A. The second approach is to treat the feedback as an instantaneous process at the outset by ensuring that the measurement acts before the feedback. We follow this approach in Sec. III B.

The second issue is more serious, and for general (not necessarily linear) quantum systems care must be taken in determining the evolution generated by Eq. (24). Our definition of Markovian feedback is directly in terms of the feedback Hamiltonian. Placing the definition on the Hamiltonian is sensible and appeals to physicists since the Hamiltonian is the generator of time evolution. In Appendix A we define feedback in a manner that draws on the traditional control systems approach. In this language one can differentiate between system dynamics that is linear and nonlinear and the results of this paper can be seen to apply to the more general (nonlinear) regime.

III. UNCONDITIONAL DYNAMICS

A. Diffusion-mediated feedback starting with nonzero feedback delay

1. Feedback master equation

Since we have already introduced the most general form of a master equation in the absence of feedback (13), we will only derive $\mathcal{L}_{\rm fb}$ in

$$\hbar\,\dot{\rho} = (\mathcal{L}_{\rm m} + \mathcal{L}_{\rm fb})\rho. \tag{25}$$

We will start in the Heisenberg picture in which case the Heisenberg equation of motion corresponding to $\dot{\rho}$ is

$$d\hat{\mathbf{s}} = [d\hat{\mathbf{s}}]_{\rm m} + [d\hat{\mathbf{s}}]_{\rm fb}, \tag{26}$$

where $[d\hat{\mathbf{s}}]_{\rm m}$ is given by Eq. (11). The feedback contribution $[d\hat{\mathbf{s}}]_{\rm fb}$ can be obtained from

$$[d\hat{\mathbf{s}}]_{\rm fb} = \hat{U}^{\dagger}_{\rm fb}(t+dt,t)\,\hat{\mathbf{s}}\,\hat{U}_{\rm fb}(t+dt,t) - \hat{\mathbf{s}}. \tag{27}$$

The unitary operator here is given by

$$\hat{U}_{\rm fb}(t+dt,t) = \ e^{-i\hat{H}_{\rm fb}dt/\hbar}. \tag{28}$$

It is perhaps not entirely obvious that deriving $\mathcal{L}_{\rm fb}$ (from either the Schrödinger or Heisenberg picture) and adding it to $\mathcal{L}_{\rm m}$ should result in the correct master equation since $[d\hat{\mathbf{s}}]_{\rm m}$ and $[d\hat{\mathbf{s}}]_{\rm fb}$ are defined with different time-evolution operators. This is justified in Appendix C. Expanding (28) to order dt,

$$\hat{U}_{\rm fb}(t+dt,t) = \hat{1} - i\,\hat{\mathbf{f}}^{\top}\hat{\mathbf{y}}_1(t-\tau)dt - \tfrac{1}{2}\,\hat{\mathbf{f}}^{\top}\hat{\mathbf{f}}\,dt. \tag{29}$$

We have used the Itô rule to obtain the last term in Eq. (29). Substituting Eq. (29) into (27), retaining only terms of order dt, and multiplying by $\hbar$ we obtain

$$\begin{aligned}\hbar\,[d\hat{\mathbf{s}}]_{\rm fb} = \ & -i\hbar\,\hat{\mathbf{s}}\,[\hat{\mathbf{f}}^{\top}\hat{\mathbf{y}}_1(t-\tau)dt\,] + i\hbar\,[\hat{\mathbf{f}}^{\top}\hat{\mathbf{y}}_1(t-\tau)dt\,]\,\hat{\mathbf{s}} \\ & + \hbar\,\mathcal{D}[\hat{\mathbf{f}}]\,\hat{\mathbf{s}}\,dt.\end{aligned} \tag{30}$$

The bath is assumed to be in the vacuum state so the initial joint system-bath state is

$$\rho_{\rm SB} = \rho \otimes |\mathbf{0}\rangle\langle\mathbf{0}|, \tag{31}$$

where ρ is the system state and $|\mathbf{0}\rangle\langle\mathbf{0}|$ the bath state. Remember that we are in the Heisenberg picture so $\rho_{\rm SB}$ does not evolve. To derive a master equation for ρ we will take the ensemble average of (30) with respect to $\rho_{\rm SB}$ and this immediately eliminates the vacuum noise contained in $\hat{\mathbf{y}}_1$ since the vacuum inputs are completely independent of the system. This also suggests that we should normally order the terms containing $d\hat{\mathbf{B}}_{\rm out}$ (since then $d\hat{\mathbf{B}}_{\rm in}$ will annihilate the vacuum to give zero when averaged). Considering the first term of (30) for the moment, we obtain, on substituting in (16)

$$\begin{aligned}-i\hbar\,\langle\hat{\mathbf{s}}[\hat{\mathbf{f}}^{\top}\hat{\mathbf{y}}_1(t-\tau)\,dt]\rangle = \ & -i\,\langle\,\hat{\mathbf{s}}\,[\,\hat{\mathbf{f}}^{\top}\mathsf{M}^{\dagger}d\hat{\mathbf{B}}_{\rm out}(t-\tau)] \\ & + \hat{\mathbf{s}}\,[\,\hat{\mathbf{f}}^{\top}\mathsf{M}^{\top}d\hat{\mathbf{B}}^{\ddagger}_{\rm out}(t-\tau)]\rangle.\end{aligned} \tag{32}$$

The first term here is already in normal order while the second term can be written as

$$\begin{aligned}\hat{\mathbf{s}}\,[\,\hat{\mathbf{f}}^{\top}\mathsf{M}^{\top}d\hat{\mathbf{B}}^{\ddagger}_{\rm out}(t-\tau)] &= [\hat{\mathbf{s}}\,d\hat{\mathbf{B}}^{\dagger}_{\rm out}(t-\tau)]\,\mathsf{M}\,\hat{\mathbf{f}}^{\top} \\ &= [d\hat{\mathbf{B}}^{\ddagger}_{\rm out}(t-\tau)\,\hat{\mathbf{s}}^{\top}]^{\top}\mathsf{M}\,\hat{\mathbf{f}},\end{aligned} \tag{33}$$

where we have noted the matrix-operator bracket (9) in (33). Using these orderings and (7), the average of (32) is simply

$$\begin{aligned}-i\hbar\,\langle\hat{\mathbf{s}}\,[\,\hat{\mathbf{f}}^{\top}\hat{\mathbf{y}}_1(t-\tau)dt\,]\,\rangle = -i\,\langle\,\hat{\mathbf{s}}\,[\,\hat{\mathbf{f}}^{\top}\mathsf{M}^{\dagger}\hat{\mathbf{c}}(t-\tau)] \\ + [\,\hat{\mathbf{c}}^{\ddagger}(t-\tau)\,\hat{\mathbf{s}}^{\top}]^{\top}\mathsf{M}\,\hat{\mathbf{f}}\rangle.\end{aligned} \tag{34}$$

Now taking the Markovian ($\tau \to 0$) limit and writing the average as a trace we get

$$\begin{aligned}-i\hbar\,\langle\hat{\mathbf{s}}\,[\,\hat{\mathbf{f}}^{\top}\hat{\mathbf{y}}_1\,dt]\rangle &= -i\,{\rm Tr}\{\hat{\mathbf{s}}\,\hat{\mathbf{f}}^{\top}\mathsf{M}^{\dagger}\hat{\mathbf{c}}\,\rho_{\rm SB} + (\hat{\mathbf{c}}^{\ddagger}\,\hat{\mathbf{s}}^{\top})^{\top}\mathsf{M}\,\hat{\mathbf{f}}\,\rho_{\rm SB}\}\,dt \\ &= -i\,{\rm Tr}\{\hat{\mathbf{s}}\,\hat{\mathbf{f}}^{\top}\mathsf{M}^{\dagger}\hat{\mathbf{c}}\,\rho_{\rm SB} + \hat{\mathbf{s}}(\mathsf{M}\,\hat{\mathbf{f}}\,\rho_{\rm SB})^{\top}\hat{\mathbf{c}}^{\ddagger}\}\,dt \\ &= -i\,{\rm Tr}\{\hat{\mathbf{s}}\,\hat{\mathbf{f}}^{\top}(\mathsf{M}^{\dagger}\hat{\mathbf{c}}\,\rho_{\rm SB} + \rho_{\rm SB}\,\mathsf{M}^{\top}\,\hat{\mathbf{c}}^{\ddagger})\}\,dt.\end{aligned} \tag{35}$$

To obtain the Markovian limit of the average of the second term in (30) we can perform a similar calculation as above or, alternatively, note that

$$\begin{aligned}&\lim_{\tau\to 0^+} i\hbar\,\langle[\hat{\mathbf{y}}_1^{\top}(t-\tau)dt\,\hat{\mathbf{f}}]\hat{\mathbf{s}}\rangle \\ &\quad = \left\{\lim_{\tau\to 0^+} -i\hbar\,\langle\,\hat{\mathbf{s}}^{\dagger}\,[\,\hat{\mathbf{f}}^{\top}\hat{\mathbf{y}}_1(t-\tau)\,dt\,]\,\rangle\right\}^{\dagger} \\ &\quad = i\,{\rm Tr}\big\{\hat{\mathbf{s}}\,(\rho_{\rm SB}\,\hat{\mathbf{c}}^{\dagger}\,\mathsf{M}\,\hat{\mathbf{f}} + \hat{\mathbf{c}}^{\top}\,\mathsf{M}^{*}\rho_{\rm SB}\hat{\mathbf{f}})\big\}\,dt.\end{aligned} \tag{36}$$

The last line is obtained by letting $\hat{\mathbf{s}} \to \hat{\mathbf{s}}^{\dagger}$ in (35) and using the cyclic property of trace to permute $\hat{\mathbf{s}}$ to the left. The average of $\mathcal{D}[\hat{\mathbf{f}}]\hat{\mathbf{s}}\,dt$ in (30) can simply be expressed as

$$\hbar\,\langle\mathcal{D}[\hat{\mathbf{f}}]\,\hat{\mathbf{s}}\rangle\,dt = {\rm Tr}\{\hat{\mathbf{s}}\,\hbar\,\mathcal{D}[\hat{\mathbf{f}}]\,\rho_{\rm SB}\}\,dt. \tag{37}$$

Adding (35), (36), and (37), we arrive at

$$\begin{aligned}\hbar\,\langle\,[d\hat{\mathbf{s}}]_{\rm fb}\rangle &= {\rm Tr}\{\hat{\mathbf{s}}[\hbar\,\mathcal{D}[\hat{\mathbf{f}}]\,\rho_{\rm SB} - i\,\hat{\mathbf{f}}^{\top}(\mathsf{M}^{\dagger}\hat{\mathbf{c}}\,\rho_{\rm SB} + \rho_{\rm SB}\,\mathsf{M}^{\top}\,\hat{\mathbf{c}}^{\ddagger}) \\ &\quad + i(\rho_{\rm SB}\,\hat{\mathbf{c}}^{\dagger}\,\mathsf{M} + \hat{\mathbf{c}}^{\top}\,\mathsf{M}^{*}\,\rho_{\rm SB})\hat{\mathbf{f}}\,]\}\,dt \\ &= {\rm Tr}\{\hat{\mathbf{s}}(-i\,\lceil\,\hat{\mathbf{f}},\mathsf{M}^{\dagger}\hat{\mathbf{c}}\,\rho_{\rm SB} + \rho_{\rm SB}\,\mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}\rfloor \\ &\quad + \hbar\,\mathcal{D}[\hat{\mathbf{f}}]\,\rho_{\rm SB})\}\,dt.\end{aligned} \tag{38}$$

In the last equality we have made use of the scalar-operator bracket, defined by [34]

$$\lceil\hat{\mathbf{A}},\hat{\mathbf{B}}\rfloor = \hat{\mathbf{A}}^{\top}\hat{\mathbf{B}} - \hat{\mathbf{B}}^{\top}\hat{\mathbf{A}}. \tag{39}$$

Remember that we are only working out the time evolution due to feedback so the feedback contribution to the full master equation is defined by

$$\hbar\,\langle\,[d\hat{\mathbf{s}}(t)]_{\rm fb}\rangle = {\rm Tr}_{\rm S}\{\hat{\mathbf{s}}(0)\,\hbar\,[d\rho(t)]_{\rm fb}\}, \tag{40}$$

where $\rho(t)$ here is defined by the partial trace over the bath $\rho(t) = {\rm Tr}_{\rm B}\{\rho_{\rm SB}(t)\}$. We thus obtain, in the Schrödinger picture, where operators are understood to be time independent and ρ time dependent,

$$\hbar\,[d\rho\,]_{\rm fb} \equiv \mathcal{L}_{\rm fb}\,\rho\,dt = (\hbar\,\mathcal{D}[\hat{\mathbf{f}}]\rho\, - i\lceil \hat{\mathbf{f}},\mathsf{M}^{\dagger}\hat{\mathbf{c}}\,\rho + \rho\,\mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}\rfloor)dt. \quad (41)$$

Adding this to the measurement master equation defined by (13) we obtain the diffusion-mediated Markovian feedback master equation

Result 1.

$$\hbar\,\dot{\rho} \equiv \mathcal{L}_{\rm mfb}\,\rho = -i[\hat{H}_1,\rho] + \mathcal{D}[\hat{\mathbf{c}}]\rho + \hbar\,\mathcal{D}[\hat{\mathbf{f}}]\rho - i\lceil \hat{\mathbf{f}},\mathsf{M}^{\dagger}\hat{\mathbf{c}}\,\rho + \rho\,\mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}\rfloor. \quad (42)$$

Note that (42) is valid for nonlinear systems (see Appendix A) as no assumptions about $\hat{H}_1$, $\hat{\mathbf{c}}$, and $\hat{\mathbf{f}}$ were made in our derivation. That $\mathcal{L}_{\rm m} + \mathcal{L}_{\rm fb}$ is again of the Lindblad form is a rather lengthy exercise so we have proved it in Appendix E. The result may be written as

Result 2.

$$\hbar\,\dot{\rho} = -i\big[\hat{H}_1 + \tfrac{1}{2}(\hat{\mathbf{f}}^{\top}\mathsf{M}^{\dagger}\hat{\mathbf{c}} + \hat{\mathbf{c}}^{\dagger}\mathsf{M}\,\hat{\mathbf{f}}),\rho\big] + \mathcal{D}[\hat{\mathbf{c}} - i\mathsf{M}\hat{\mathbf{f}}\,]\rho + \mathcal{D}[\sqrt{\hbar\,\mathrm{I}_R - \mathsf{M}^{\dagger}\mathsf{M}}\;\hat{\mathbf{f}}\,]\rho, \quad (43)$$

where $\sqrt{\hbar\,\mathrm{I}_R - \mathsf{M}^{\dagger}\mathsf{M}}$ may be replaced by any matrix square root of $\hbar\,\mathrm{I}_R - \mathsf{M}^{\dagger}\mathsf{M}$.[2] We remark that while the Lindblad form is an important part of the theory, (43) is not necessarily more useful than (42).

We illustrate the process described by (42) in Fig. 3, which we now explain. In the context of continuous quantum measurements the master equation, which describes unconditional evolution, is often contrasted with the stochastic master equation, which describes conditional evolution [and which we will consider in Sec. IV; see (78) and Fig. 4]. Conditional evolution refers to the accounting of a c-number measurement record in the observer's inference of the system state. The current is thus represented as a c number y, as opposed to an operator (or q number) $\hat{y}$. Unconditional evolution then refers to the ignorance of y (a c number) in making inferences about the system state. This is why we have labeled the measurement output in Fig. 3 as y and not $\hat{y}$. On the other hand, we derived (42) from working in the Heisenberg picture and then transforming to the Schrödinger picture in the end. In the Heisenberg picture it is necessary to distinguish between an input field $\hat{\mathbf{b}}_{\rm in}$ and an output field $\hat{\mathbf{b}}_{\rm out}$ for the bath. Later, in Secs. III B and IV, we will also work in the Heisenberg picture, but there we will meet a new output field, $\hat{\mathbf{b}}_{\rm outt}$ (shown in Fig. 4). In order to make a note of this difference we have labeled the bath fields as $\hat{\mathbf{b}}_{\rm in}$ and $\hat{\mathbf{b}}_{\rm out}$ in Fig. 3.[3]

2. Feedback Heisenberg equation

Equation (42), or (43), describes feedback in the Schrödinger picture but they are not the only equations

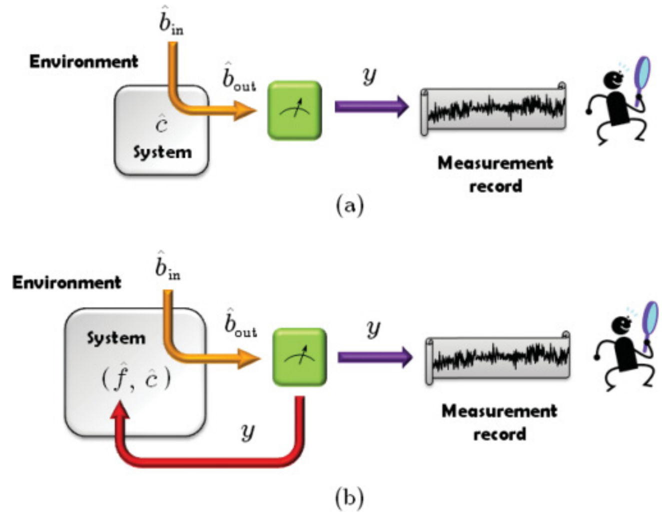

FIG. 3. (Color online) For simplicity we have drawn only one decay channel ($L = 1$) and a SISO feedback loop. (a) The process described by (13), which is equivalent to setting $\hat{f} = 0$ in (42). Equation (13) is in the Schrödinger picture, where the measurement output is treated as a c number y, as opposed to an operator which would be the case when describing measurements in the Heisenberg picture as we have been doing. Unconditional evolution of ρ refers to the observer's ignorance of y when describing how the system state changes in time, depicted here by the observer looking away from the noisy signal y (compared with Fig. 4). (b) The process described by (42) and (43).

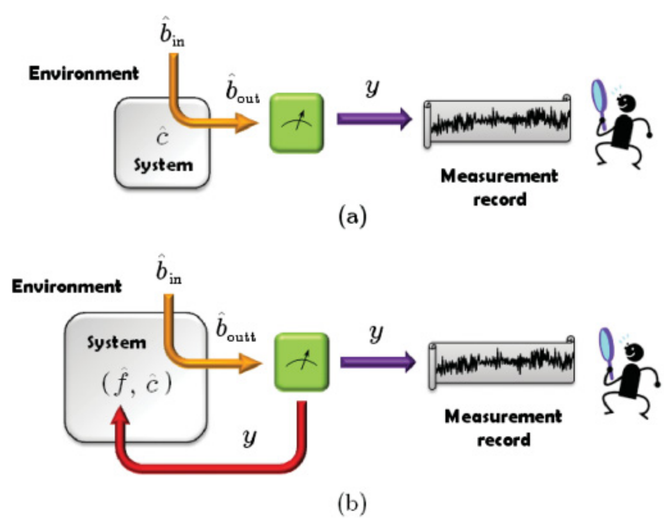

FIG. 4. (Color online) For simplicity we have shown only the SISO case with $L = 1$. (a) The situation described by the works of Refs. [32,33], in which the observer uses his/her knowledge of the measurement record y (treated as a number in the Schrödinger picture) to infer the state of the system but in the absence of feedback. (b) The situation described by the stochastic master Eq. (78): The observer infers the state of the system from the measurement record in the presence of feedback. The inclusion of feedback (i.e., $\hat{f}$) changes the interaction between $\hat{b}_{\rm in}$ and the system. That is, $\hat{b}_{\rm in}$ now "sees" both $\hat{c}$ and $\hat{f}$, which is why we have written them as a pair in the figure. The result of this is to produce an output field $\hat{b}_{\rm outt}$ which evolves according to (55), which differs from $\hat{b}_{\rm out}$ (hence the extra "t" in the subscript). The actual system-bath input-output relation with respect to (55) is worked out in Sec. IV B.

[2]In general the square root of a matrix A is any matrix B such that $B^{\dagger}B = A$. When $A \geqslant 0$, there exists a unique B such that $B \geqslant 0$ and $B^2 = A$, called the positive square root of A. The positive square root of A is denoted by $\sqrt{A}$.

[3]If one wanted a diagram that fully reflects the mentality of the Schrödinger picture then one that is similar to Fig. 1 of Ref. [34] would be appropriate.

of motion capable of capturing the feedback process. An alternative theory of feedback exists in the Heisenberg picture where feedback is described by an equation of motion for an arbitrary vector operator $\hat{\mathbf{s}}$. Such an equation follows unitary evolution and has the interpretation that measurements (namely the collapse of ρ as occurs by using a measurement operator) never happens. Thus it also describes "feedback without measurement" [26]. This mentality is reflected in Fig. 1, whereby the output current is labeled as a vector operator $\hat{\mathbf{y}}_1$ as a opposed to a vector $\mathbf{y}$. This is in contrast to the Schrödinger-picture master equation derived above where the current is represented by a c number.

As before, the calculation can be simplified by first deriving the change in $\hat{\mathbf{s}}$ due to feedback only and then adding it to the measurement contribution. This can be obtained from (30) by substituting in the expression for $\hat{\mathbf{y}}_1$ and then normally ordering $d\hat{\mathbf{B}}_{\rm out}$. The final result, including the measurement contribution is

Result 3.

$$\begin{aligned}\hbar\, d\hat{\mathbf{s}} = &\big(i\,[\hat{H}_1,\hat{\mathbf{s}}] + \mathcal{J}[\hat{\mathbf{c}}^\ddagger]\hat{\mathbf{s}} - \tfrac{1}{2}\{\hat{\mathbf{c}}^\dagger\hat{\mathbf{c}},\hat{\mathbf{s}}\} + \hbar\,\mathcal{D}[\hat{\mathbf{f}}]\hat{\mathbf{s}}\big)dt\\ &+ [\hat{\mathbf{c}}^\dagger d\hat{\mathbf{B}}_{\rm in} - d\hat{\mathbf{B}}_{\rm in}^\dagger\hat{\mathbf{c}},\hat{\mathbf{s}}]\\ &- i\,\lfloor\hat{\mathbf{s}},\mathsf{M}^*\,\hat{\mathbf{f}}\rceil[\hat{\mathbf{c}}(t-\tau)dt + d\hat{\mathbf{B}}_{\rm in}(t-\tau)]\\ &+ i\{[\hat{\mathbf{c}}^\dagger(t-\tau)dt + d\hat{\mathbf{B}}_{\rm in}^\dagger(t-\tau)]\lfloor\mathsf{M}\,\hat{\mathbf{f}},\hat{\mathbf{s}}\rceil\}^\top\\ &- i\,\lfloor\hat{\mathbf{s}},\sqrt{\mathsf{Z}}^*\,\hat{\mathbf{f}}\rceil d\hat{\mathbf{U}}_{\rm in}(t-\tau) + i\{d\hat{\mathbf{U}}_{\rm in}^\dagger(t-\tau)\lfloor\sqrt{\mathsf{Z}}\,\hat{\mathbf{f}},\hat{\mathbf{s}}\rceil\}^\top.\end{aligned} \tag{44}$$

The matrix $\sqrt{\mathsf{Z}}$ is the positive square root of (19) and $d\hat{\mathbf{U}}_{\rm in}$ is an independent Wiener increment [recall (20) and (21)]. This is the diffusion-mediated MIMO generalization of the homodyne-mediated Heisenberg equation (4.16) of Ref. [26]. One can check that (44) is a valid Itô equation, i.e.,

$$d\,(\hat{\mathbf{s}}\hat{\alpha}) = (d\hat{\mathbf{s}})\hat{\alpha} + \hat{\mathbf{s}}\,(d\hat{\alpha}) + (d\hat{\mathbf{s}})(d\hat{\alpha}), \tag{45}$$

for any operator $\hat{\alpha}$. Note that we can take the Markovian limit of (44) by setting $\tau = 0$ in $d\hat{\mathbf{B}}_{\rm in}(t-\tau)$ since $\hat{\mathbf{b}}_{\rm in}(t)$ is continuous in time, although nowhere differentiable. The resulting equation with $\tau = 0$ in (44) is then the Heisenberg-picure equivalent of (42) in the sense that

$$d\,\langle\hat{\mathbf{s}}\rangle = {\rm Tr}[\rho_{\rm SB}(0)\,d\hat{\mathbf{s}}(t)] = {\rm Tr}_S[d\rho(t)\,\hat{\mathbf{s}}(0)]. \tag{46}$$

B. Diffusion-mediated feedback starting with zero feedback delay

When we allow the feedback delay to be zero we are letting the time at which $\hat{\mathbf{f}}$ interacts with the bath converge to the same point in time as the interaction between $\hat{\mathbf{c}}$ and the bath. This eliminates the concept of $\hat{\mathbf{b}}_{\rm out}$. Consequently, the feedback interaction should be defined by

$$\hat{H}_{\rm fb} = \hbar\,\hat{\mathbf{f}}^\top\hat{\mathbf{y}}_0, \tag{47}$$

where $\hat{\mathbf{y}}_0$ is

$$\hbar\,\hat{\mathbf{y}}_0\,dt = \mathsf{M}^\dagger d\hat{\mathbf{B}}_{\rm in} + \mathsf{M}^\top d\hat{\mathbf{B}}_{\rm in}^\ddagger + \hbar\,d\hat{\boldsymbol{v}}_{\rm in}. \tag{48}$$

By working in the limit of zero feedback delay we are also allowing the measurement and feedback interactions to occur in the same infinitesimal time interval $[t,t+dt)$,

$$\hat{U}_{\rm mfb}(t+dt,t) = \hat{U}_{\rm fb}(t+dt,t)\,\hat{U}_{\rm m}(t+dt,t), \tag{49}$$

where

$$\hat{U}_{\rm fb}(t+dt,t) = \exp(-i\,\hat{H}_{\rm fb}dt/\hbar) = \exp(-i\,\hat{\mathbf{f}}^\top\hat{\mathbf{y}}_0\,dt). \tag{50}$$

Since $\hat{H}_{\rm fb}$ and $\hat{H}_{\rm m}$ do not commute the order of $\hat{U}_{\rm fb}$ and $\hat{U}_{\rm m}$ matters and the correct order is defined by the order in which the two processes happen in reality. This order should correspond to the order in which the unitaries act on a state, as shown in (49). The Schrödinger picture provides an intuitive way [in contrast to (51)] to remember the order in which we compose $\hat{U}_{\rm m}$ and $\hat{U}_{\rm fb}$ to give $\hat{U}_{\rm mfb}$. When we evolve a vector operator $\hat{\mathbf{s}}$ in the Heisenberg picture from t to $t+dt$ under measurement and feedback, the order is then given by (with the unitary operators understood to act over an infinitesimal interval from t to $t+dt$)

$$\hat{\mathbf{s}}(t+dt) = \hat{U}_{\rm mfb}^\dagger\,\hat{\mathbf{s}}\,\hat{U}_{\rm mfb} = \hat{U}_{\rm m}^\dagger\,\hat{U}_{\rm fb}^\dagger\,\hat{\mathbf{s}}\,\hat{U}_{\rm fb}\,\hat{U}_{\rm m}. \tag{51}$$

There is, of course, nothing odd about letting $\hat{U}_{\rm fb}$ act on $\hat{\mathbf{s}}$ first in (51); it is simply a consequence of the definition of the Heisenberg picture. If one insists on having $\hat{U}_{\rm m}$ act on $\hat{\mathbf{s}}$ first, even in the Heisenberg picture, then we can rewrite (51) as

$$\hat{\mathbf{s}}(t+dt) = \hat{U}_{\rm fb1}^\dagger\hat{U}_{\rm m}^\dagger\,\hat{\mathbf{s}}\,\hat{U}_{\rm m}\hat{U}_{\rm fb1}, \tag{52}$$

where we have defined

$$\hat{U}_{\rm fb1} = \hat{U}_{\rm m}^\dagger\,\hat{U}_{\rm fb}\,\hat{U}_{\rm m} = \exp(-i\,\hat{H}_{\rm fb1}\,dt/\hbar). \tag{53}$$

The Hamiltonian $\hat{H}_{\rm fb1}$ is given by

$$\hat{H}_{\rm fb1}\,dt = \hbar\,\hat{\mathbf{f}}^\top(t+dt)\,\hat{\mathbf{y}}_1\,dt, \tag{54}$$

and $\hat{\mathbf{y}}_1$ is, as before, given by (16). Note that (54) has no ordering ambiguity [recall the discussion surrounding (24)] on its right-hand side since the current $\hat{\mathbf{y}}_1$ appears at an (infinitesimally) earlier time than $\hat{\mathbf{f}}(t+dt)$. In what follows we will take the former approach, i.e., with a vector operator in the Heisenberg picture defined by (51) and a feedback Hamiltonian given by (47) and (48).

The Hudson-Parthasarathy equation for $\hat{U}_{\rm mfb}(t,t_0)$ is

$$\begin{aligned}\hbar\,d\hat{U}_{\rm mfb}(t,t_0) = &[(-i\hat{H}_1 - \hat{\mathbf{c}}^\dagger\hat{\mathbf{c}}/2 - \hbar\,\hat{\mathbf{f}}^\top\hat{\mathbf{f}}/2 - i\,\hat{\mathbf{f}}^\top\mathsf{M}^\dagger\hat{\mathbf{c}})\,dt\\ &+ d\hat{\mathbf{B}}_{\rm in}^\dagger(\hat{\mathbf{c}} - i\,\mathsf{M}\hat{\mathbf{f}}) - (\hat{\mathbf{c}}^\dagger + i\,\hat{\mathbf{f}}^\top\mathsf{M}^\dagger)d\hat{\mathbf{B}}_{\rm in}\\ &- i(\hat{\mathbf{f}}^\top\sqrt{\mathsf{Z}}\,d\hat{\mathbf{U}}_{\rm in} + d\hat{\mathbf{U}}_{\rm in}^\dagger\sqrt{\mathsf{Z}}\,\hat{\mathbf{f}})]\hat{U}_{\rm mfb}(t,t_0),\end{aligned} \tag{55}$$

with the initial condition $\hat{U}_{\rm mfb}(t_0,t_0) = \hat{1}$. From this we can derive yet another Heisenberg equation, a quantum Langevin equation (see the caption in Fig. 2 for the difference between a Heisenberg equation and a Langevin equation). If the above argument [(47)–(51)] is correct, then one would expect the quantum Langevin equation obtained from (55) to be equivalent to (44) in the limit of $\tau = 0$. This is not trivial and we show in Appendix D that this second approach [using (55)] does indeed give us a quantum Langevin equation which corresponds to setting $\tau = 0$ in (44). The final result is the Markovian quantum Langevin equation

Result 4.

$$\begin{aligned}\hbar\,d\hat{\mathbf{s}} = &\,i\,[\hat{H}_1,\hat{\mathbf{s}}]\,dt + \hbar\,\mathcal{D}[\hat{\mathbf{f}}]\hat{\mathbf{s}}\,dt - \lfloor\hat{\mathbf{s}},\hat{\mathbf{c}}^\ddagger\rceil\big(\tfrac{1}{2}\,\hat{\mathbf{c}}\,dt + d\hat{\mathbf{B}}_{\rm in}\big)\\ &- \big\{\big(\tfrac{1}{2}\hat{\mathbf{c}}^\dagger dt + d\hat{\mathbf{B}}_{\rm in}^\dagger\big)\lfloor\hat{\mathbf{c}},\hat{\mathbf{s}}\rceil\big\}^\top + i\{(\hat{\mathbf{c}}^\dagger dt + d\hat{\mathbf{B}}_{\rm in}^\dagger)\lfloor\mathsf{M}\,\hat{\mathbf{f}},\hat{\mathbf{s}}\rceil\}^\top\\ &- i\,\lfloor\hat{\mathbf{s}},\mathsf{M}^*\,\hat{\mathbf{f}}\rceil(\hat{\mathbf{c}}\,dt + d\hat{\mathbf{B}}_{\rm in}) - i\,\lfloor\hat{\mathbf{s}},\sqrt{\mathsf{Z}}^*\,\hat{\mathbf{f}}\rceil d\hat{\mathbf{U}}_{\rm in}\\ &+ i\{d\hat{\mathbf{U}}_{\rm in}^\dagger\lfloor\sqrt{\mathsf{Z}}\,\hat{\mathbf{f}},\hat{\mathbf{s}}\rceil\}^\top.\end{aligned} \tag{56}$$

This is again a valid Itô equation in the sense of (45) and we have also placed the bath fields on the exterior so terms containing $d\hat{\mathbf{B}}_{\rm in}$ or $d\hat{\mathbf{B}}_{\rm in}^{\dagger}$ vanish when averaged against a vacuum bath state.

Here we have to be careful that (56) is not quite the same as the equation which results from setting $\tau = 0$ in (44). Setting $\hat{\mathbf{s}} = d\hat{\mathbf{B}}_{\rm in}$ in (56) will not give the correct output bath field $d\hat{\mathbf{B}}_{\rm out}$, whereas (44) will. The reason is that we have explicitly used the commutability of $d\hat{\mathbf{B}}_{\rm in}$ and $d\hat{\mathbf{B}}_{\rm in}^{\ddagger}$ with $\hat{\mathbf{s}}$ to rewrite the first line of (44) in the form shown in (56). This assumption is then violated when we let $\hat{\mathbf{s}} = d\hat{\mathbf{B}}_{\rm in}$ since $d\hat{\mathbf{B}}_{\rm in}^{\ddagger}$ does not commute with $d\hat{\mathbf{B}}_{\rm in}$ (see Appendix B). The form of (56) is motivated by the form of (4.21) in Ref. [26] or (5.162) of Ref. [5].

To derive a master equation with $\tau = 0$ from the start we may move into the Schrödinger picture from (55) or simply take the average of (56). Here we show how to obtain the master equation from (56). Since (56) is normally ordered in the bath vector operators, its average with respect to (31) is simply

$$\hbar\,\langle d\hat{\mathbf{s}}\rangle = \Big\langle i[\hat{H}_1,\hat{\mathbf{s}}] - \tfrac{1}{2}\lfloor\hat{\mathbf{s}},\hat{\mathbf{c}}^{\ddagger}\rceil\hat{\mathbf{c}} - \tfrac{1}{2}(\hat{\mathbf{c}}^{\dagger}\lfloor\hat{\mathbf{c}},\hat{\mathbf{s}}\rceil)^{\top} + \hbar\,\mathcal{D}[\hat{\mathbf{f}}]\hat{\mathbf{s}} + i\,(\hat{\mathbf{c}}^{\dagger}\lfloor \mathsf{M}\hat{\mathbf{f}},\hat{\mathbf{s}}\rceil)^{\top} - i\,\lfloor\hat{\mathbf{s}},\mathsf{M}^{*}\hat{\mathbf{f}}\rceil\Big\rangle\,dt. \quad (57)$$

It is easy to show that

$$-\tfrac{1}{2}\lfloor\hat{\mathbf{s}},\hat{\mathbf{c}}^{\ddagger}\rceil\hat{\mathbf{c}} - \tfrac{1}{2}(\hat{\mathbf{c}}^{\dagger}\lfloor\hat{\mathbf{c}},\hat{\mathbf{s}}\rceil)^{\top} = (\hat{\mathbf{c}}^{\ddagger}\hat{\mathbf{s}}^{\top})^{\top}\hat{\mathbf{c}} - \tfrac{1}{2}\{\hat{\mathbf{c}}^{\dagger}\hat{\mathbf{c}},\hat{\mathbf{s}}\} = \mathcal{J}[\hat{\mathbf{c}}^{\ddagger}]\hat{\mathbf{s}} - \tfrac{1}{2}\{\hat{\mathbf{c}}^{\dagger}\hat{\mathbf{c}},\hat{\mathbf{s}}\}. \quad (58)$$

So the first line of (57) is just the average of (11) for which the contribution to $\dot{\rho}$ is well known, given by (13). The first term on the second line is given by (37) while

$$i\,\langle\hat{\mathbf{c}}^{\dagger}\lfloor\mathsf{M}\hat{\mathbf{f}},\hat{\mathbf{s}}\rceil\rangle^{\top} = i\,{\rm Tr}\{\hat{\mathbf{s}}(\rho_{\rm SB}\,\hat{\mathbf{c}}^{\dagger}\mathsf{M}\,\hat{\mathbf{f}} - \hat{\mathbf{f}}^{\top}\rho_{\rm SB}\,\mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger})\}, \quad (59)$$

$$-i\,\langle\lfloor\hat{\mathbf{s}},\mathsf{M}^{*}\,\hat{\mathbf{f}}\rceil\,\hat{\mathbf{c}}\rangle = -i\,{\rm Tr}\{\hat{\mathbf{s}}(\hat{\mathbf{f}}^{\top}\mathsf{M}^{\dagger}\hat{\mathbf{c}}\,\rho_{\rm SB} - \hat{\mathbf{c}}^{\top}\mathsf{M}^{*}\rho_{\rm SB}\,\hat{\mathbf{f}})\}. \quad (60)$$

Adding (37), (59), and (60), we see that the second line of (57) is in fact (38). From these it should be clear that a master equation in exactly the form of (42) results, as expected. If we were not interested in the quantum Langevin equation, then we would derive the master equation directly from (55), and this is, in fact, quicker.

IV. CONDITIONAL DYNAMICS

To better understand applications of feedback one would like to know the controlled system dynamics as the monitoring and feedback occurs in real time. It is well-known that continuously measured systems can be described by a nonlinear stochastic differential equation for the system state [54,55]. Here we will derive a general diffusion-mediated stochastic feedback master equation in the Heisenberg picture. This is an extension of the diffusive stochastic master equation found in Refs. [32,33] to include feedback but using a different parametrization of the measurement. We illustrate the two cases (with and without feedback) in Fig. 4.

A. Diffusion-mediated feedback stochastic master equation

Previously, we found the most general diffusive stochastic master equation with measurements alone to be given by

$$d\rho_{\rm c} = \mathcal{L}_{\rm m}\,\rho_{\rm c}\,dt + \mathcal{H}[d\mathbf{w}^{\top}\mathsf{M}^{\dagger}\hat{\mathbf{c}}]\,\rho_{\rm c}, \quad (61)$$

where $\mathcal{L}_m$ is given by (13) and $d\mathbf{w}$ is an $R \times 1$ (vector) Wiener increment defined by ${\rm E}[d\mathbf{w}(t)] = \mathbf{0}$ and

$$d\mathbf{w}(t)\,d\mathbf{w}^{\top}(t) = {\rm I}_R\,dt, \quad (62)$$

$$d\mathbf{w}(t)\,d\mathbf{w}^{\top}(t') = 0 \quad \forall\, t \neq t'. \quad (63)$$

The superoperator $\mathcal{H}[\hat{A}]$, for any $\hat{A}$, is defined to be

$$\mathcal{H}[\hat{A}]\rho = \hat{A}\rho + \rho\hat{A}^{\dagger} - {\rm Tr}[\hat{A}\rho + \rho\hat{A}^{\dagger}]\rho. \quad (64)$$

To generalize (61) to account for feedback we first note that our foregoing derivation of the master equation prescribes us with the rule

$$\mathcal{L}_{\rm m} \;\longrightarrow\; \mathcal{L}_{\rm mfb} = \mathcal{L}_{\rm m} + \mathcal{L}_{\rm fb} \quad (65)$$

for the unconditioned evolution. But how does the conditional dynamics change? That is, how can the nonlinear term in (61) be altered to include feedback?

A derivation of the stochastic master equation in the Heisenberg picture would be possible if we can establish a relation about the time evolution in the Schrödinger and Heisenberg pictures that involves the conditioning. For unconditional evolution such a relation is given by (46), which made the derivation of the master equation possible in the Heisenberg picture. We can in fact find an analogous relation that incorporates the conditioning of ρ on the measured current. By considering the evolution over an infinitesimal time interval such an equation is given by

$$\hbar^2\,\langle(\hat{\mathbf{y}}_1 - \langle\hat{\mathbf{y}}_1\rangle)\,\hat{\mathbf{s}}^{\top}(t+dt)\rangle\,dt = \hbar^2\,{\rm E}\,\{d\mathbf{w}\,{\rm Tr}_{\rm S}[\hat{\mathbf{s}}^{\top}\rho_{\mathbf{y}_t}(t+dt)]\}, \quad (66)$$

where we have multiplied each side by $\hbar^2$ for convenience. This identity can be derived using quantum measurement theory. For simplicity we are assuming the state to be given at time t (i.e., deterministic). The state on the right-hand side of (66) is conditioned on the vector-valued current

$$\hbar\,\mathbf{y}\,dt = \langle\mathsf{M}^{\dagger}\hat{\mathbf{c}} + \mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}\rangle\,dt + \hbar\,d\mathbf{w}, \quad (67)$$

at only one time, t, where $d\mathbf{w}$ is a vector Wiener increment. To use (66) we note from quantum measurement theory that any diffusive unravelling will be of the form

$$d\rho_{\rm c} = \mathcal{L}\,\rho_{\rm c}\,dt + \mathcal{H}[d\mathbf{w}^{\top}\hat{\boldsymbol{\alpha}}\,]\,\rho_{\rm c}, \quad (68)$$

for some $\hat{\boldsymbol{\alpha}}$ and $\mathcal{L}$. We therefore make this ansatz in (66) with $\hat{\boldsymbol{\alpha}}$ to be determined by the left-hand side, which is in the Heisenberg picture.

Using (68) and the fact that $\rho_{\rm c}(t) \equiv \rho(t)$ is known, the right-hand side of (66) simply reduces to

$$\hbar^2\,{\rm E}\big\{d\mathbf{w}\,{\rm Tr}_{\rm S}\big[\hat{\mathbf{s}}^{\top}\rho_{\mathbf{y}_t}(t+dt)\big]\big\} = \hbar\,{\rm Tr}\{d\mathbf{w}\,\hat{\mathbf{s}}^{\top}\mathcal{H}[d\mathbf{w}^{\top}\hat{\boldsymbol{\alpha}}\,]\,\rho\} = \hbar\,{\rm Tr}\{[\hat{\mathbf{s}}\,(\hat{\boldsymbol{\alpha}}^{\top}\rho + \rho\hat{\boldsymbol{\alpha}}^{\dagger} - \langle\hat{\boldsymbol{\alpha}} + \hat{\boldsymbol{\alpha}}^{\dagger}\rangle\rho)]^{\top}\}\,dt. \quad (69)$$

The left-hand side of (66) is

$$\hbar^2\,\langle(\hat{\mathbf{y}}_1-\langle\hat{\mathbf{y}}_1\rangle)\hbar\,\hat{\mathbf{s}}^\top(t+dt)\rangle\,dt = \langle(\hbar\,\hat{\mathbf{y}}_1 dt)\hbar\,\hat{\mathbf{s}}^\top(t+dt)\rangle - \langle\hbar\,\hat{\mathbf{y}}_1 dt\rangle\langle\hbar\,\hat{\mathbf{s}}^\top(t+dt)\rangle. \tag{70}$$

On substituting in (16), the first term in (70) is

$$\begin{aligned}\langle(\hbar\,\hat{\mathbf{y}}_1 dt)\hbar\,\hat{\mathbf{s}}^\top(t+dt)\rangle = {}& \langle \mathsf{M}^\dagger d\hat{\mathbf{B}}_{\text{out}}^\top\hbar\,\hat{\mathbf{s}}^\top(t+dt)\rangle\\ &+\langle \mathsf{M}^\top d\hat{\mathbf{B}}_{\text{out}}^\ddagger\hbar\,\hat{\mathbf{s}}^\top(t+dt)\rangle\\ &+\langle(\hbar\,d\hat{\boldsymbol{\upsilon}}_{\text{in}})\hbar\,\hat{\mathbf{s}}^\top(t+dt)\rangle.\end{aligned} \tag{71}$$

By examining (56) it is not difficult to see that

$$\langle \mathsf{M}^\dagger d\hat{\mathbf{B}}_{\text{out}}^\top\hbar\,\hat{\mathbf{s}}^\top(t+dt)\rangle = \hbar\,\langle\,[\hat{\mathbf{s}}\,(\mathsf{M}^\dagger\hat{\mathbf{c}})^\top]^\top\rangle\,dt + i\,\hbar\,\langle\lfloor \mathsf{M}^\dagger\mathsf{M}\hat{\mathbf{f}},\hat{\mathbf{s}}\rceil\rangle\,dt, \tag{72}$$

$$\langle \mathsf{M}^\top d\hat{\mathbf{B}}_{\text{out}}^\ddagger\hbar\,\hat{\mathbf{s}}^\top(t+dt)\rangle = \hbar\,\langle \mathsf{M}^\top\hat{\mathbf{c}}^\ddagger\hat{\mathbf{s}}^\top\rangle\,dt, \tag{73}$$

$$\langle(\hbar\,d\hat{\boldsymbol{\upsilon}}_{\text{in}})\hbar\,\hat{\mathbf{s}}^\top(t+dt)\rangle = i\,\hbar^2\,\langle\lfloor\hat{\mathbf{f}},\hat{\mathbf{s}}\rceil\rangle\,dt - i\,\hbar\,\langle\lfloor \mathsf{M}^\dagger\mathsf{M}\hat{\mathbf{f}},\hat{\mathbf{s}}\rceil\rangle\,dt, \tag{74}$$

$$\langle\hbar\,\hat{\mathbf{y}}_1 dt\rangle\langle\hbar\,\hat{\mathbf{s}}^\top(t+dt)\rangle = \hbar\langle \mathsf{M}^\dagger\hat{\mathbf{c}}\rangle\langle\hat{\mathbf{s}}^\top\rangle dt + \hbar\langle \mathsf{M}^\top\hat{\mathbf{c}}^\ddagger\rangle\langle\hat{\mathbf{s}}^\top\rangle dt. \tag{75}$$

Writing (72)-(75) as a trace we get

$$\begin{aligned}&\hbar^2\,\langle(\hat{\mathbf{y}}_1-\langle\hat{\mathbf{y}}_1\rangle)[\hbar\,\hat{\mathbf{s}}^\top(t+dt)]\rangle\,dt\\ &= \hbar\,\mathrm{Tr}\{[\hat{\mathbf{s}}\{(\mathsf{M}^\dagger\hat{\mathbf{c}}-i\hbar\,\hat{\mathbf{f}})^\top\rho+\rho\,(\mathsf{M}^\dagger\hat{\mathbf{c}}-i\hbar\,\hat{\mathbf{f}})^\dagger\\ &\quad-\langle(\mathsf{M}^\dagger\hat{\mathbf{c}})^\top+(\mathsf{M}^\dagger\hat{\mathbf{c}})^\dagger\rangle\rho\}]^\top\}\,dt.\end{aligned} \tag{76}$$

Equating (76) to (69) and solving for $\hat{\boldsymbol{\alpha}}$ gives

$$\hat{\boldsymbol{\alpha}} = \mathsf{M}^\dagger\hat{\mathbf{c}} - i\,\hbar\,\hat{\mathbf{f}}. \tag{77}$$

Invoking (65) and (77), we arrive at the diffusion-mediated stochastic feedback master equation

Result 5.

$$\begin{aligned}\hbar\,d\rho_{\mathrm{c}} = {}& -i[\hat{H}_1,\rho_{\mathrm{c}}]+\mathcal{D}[\hat{\mathbf{c}}]\rho_{\mathrm{c}}+\hbar\,\mathcal{D}[\hat{\mathbf{f}}]\rho_{\mathrm{c}}\\ &-i\,\lceil\hat{\mathbf{f}},\mathsf{M}^\dagger\hat{\mathbf{c}}\,\rho_{\mathrm{c}}+\rho_{\mathrm{c}}\,\mathsf{M}^\top\hat{\mathbf{c}}^\ddagger\rfloor\\ &+\mathcal{H}[d\mathbf{w}^\top(\mathsf{M}^\dagger\hat{\mathbf{c}}-i\hbar\,\hat{\mathbf{f}})]\,\rho_{\mathrm{c}}.\end{aligned} \tag{78}$$

Comparing (78) to (61) we can summarize the changes necessary to include feedback in the stochastic master equation (61) by the two transformations

$$\mathcal{L}_{\mathrm{m}} \longrightarrow \mathcal{L}_{\mathrm{mfb}}, \tag{79}$$

$$\mathsf{M}^\dagger\hat{\mathbf{c}} \longrightarrow \mathsf{M}^\dagger\hat{\mathbf{c}} - i\hbar\,\hat{\mathbf{f}}. \tag{80}$$

We can understand why $\hat{\mathbf{f}}$ must appear in the nonlinear term by considering the case when $\mathsf{M}=0$. In this case the feedback master equation is simply

$$\hbar\,\dot{\rho} = -i\,[\hat{H}_1,\rho]+\mathcal{D}[\hat{\mathbf{c}}]\rho+\hbar\,\mathcal{D}[\hat{\mathbf{f}}]\rho, \tag{81}$$

and the current fed back is pure noise

$$\mathbf{y}\,dt = d\mathbf{w}. \tag{82}$$

Equation (81) is the unconditional evolution for the measurement defined by (82). If we now condition the state on the pure-noise output then the stochastic master equation which unravels (81) is

$$\begin{aligned}\hbar\,d\rho_{\mathrm{c}} = {}& -i\,[\hat{H}_1,\rho_{\mathrm{c}}]+\mathcal{D}[\hat{\mathbf{c}}]\rho_{\mathrm{c}}+\hbar\,\mathcal{D}[\hat{\mathbf{f}}]\rho_{\mathrm{c}}\\ &+\hbar\,\mathcal{H}[-i\,d\mathbf{w}^\top\hat{\mathbf{f}}\,]\rho_{\mathrm{c}}.\end{aligned} \tag{83}$$

This can be seen by noting that (82) can also be written as

$$\mathbf{y}\,dt = \langle -i\,\hat{\mathbf{f}}+(-i\,\hat{\mathbf{f}})^\ddagger\rangle + d\mathbf{w}, \tag{84}$$

which gives rise to the nonlinear term in (83). When $\mathsf{M}\neq 0$ we get the general case of (78).

B. Output correlation function

When we include feedback in our theory the controlled dynamics can be accounted for by transforming the input fields according to $\hat{U}_{\mathrm{mfb}}$ as opposed to $\hat{U}_{\mathrm{m}}$. That is, instead of (7) we now have the new output field

$$\begin{aligned}d\hat{\mathbf{B}}_{\mathrm{outt}}(t) \equiv {}& \hat{U}_{\mathrm{mfb}}^\dagger(t+dt,t)\,d\hat{\mathbf{B}}_{\mathrm{in}}(t)\,\hat{U}_{\mathrm{mfb}}(t+dt,t)\\ = {}& [\hat{\mathbf{c}}(t)-i\,\mathsf{M}\hat{\mathbf{f}}(t)]\,dt + d\hat{\mathbf{B}}_{\mathrm{in}},\end{aligned} \tag{85}$$

which can be derived from (55). The use of the subscript "outt" is deliberate, to be read as "out twice." This is to remind us that $d\hat{\mathbf{B}}_{\mathrm{outt}}$ is the output field obtained from using $\hat{U}_{\mathrm{mfb}}$, which is a composition of two unitaries.[4] The input field $d\hat{\mathbf{B}}_{\mathrm{in}}$ would still have the same matrix-operator bracket with an arbitrary system vector operator $\hat{\mathbf{s}}$ as given by (8), but $d\hat{\mathbf{B}}_{\mathrm{out}}$ in (9) should be replaced by $d\hat{\mathbf{B}}_{\mathrm{outt}}$. Thus we now have

$$\lfloor d\hat{\mathbf{B}}_{\mathrm{in}}(t),\hat{\mathbf{s}}(t')\rceil = 0 \quad \forall\, t'\leqslant t, \tag{86}$$

$$\lfloor d\hat{\mathbf{B}}_{\mathrm{outt}}(t),\hat{\mathbf{s}}(t')\rceil = 0 \quad \forall\, t'> t. \tag{87}$$

We should not forget to change the input field $d\hat{\mathbf{U}}_{\mathrm{in}}$ as well since it will now evolve under the dynamics of feedback. Recall that $d\hat{\mathbf{U}}_{\mathrm{in}}$ was introduced in (20), where it appeared as a vacuum noise in the current that did not interact with the system. When we add feedback this noise is redirected onto the system so it is no longer correct to assume that it is independent of the system as was the case in (20). We thus have an additional input-output relation, which can also be derived from (55),

$$\begin{aligned}d\hat{\mathbf{U}}_{\mathrm{outt}}(t) \equiv {}& \hat{U}_{\mathrm{mfb}}^\dagger(t+dt,t)\,d\hat{\mathbf{U}}_{\mathrm{in}}(t)\,\hat{U}_{\mathrm{mfb}}(t+dt,t)\\ = {}& d\hat{\mathbf{U}}_{\mathrm{in}}(t) - i\,\sqrt{\mathsf{Z}}\;\hat{\mathbf{f}}(t)\,dt.\end{aligned} \tag{88}$$

Similarly to (86) and (87),

$$\lfloor d\hat{\mathbf{U}}_{\mathrm{in}}(t),\hat{\mathbf{s}}(t')\rceil = 0 \quad \forall\, t'\leqslant t, \tag{89}$$

$$\lfloor d\hat{\mathbf{U}}_{\mathrm{outt}}(t),\hat{\mathbf{s}}(t')\rceil = 0 \quad \forall\, t'> t. \tag{90}$$

Relations (85) and (88) in turn define a new vector-operator-valued current

$$\hbar\,\hat{\mathbf{y}}_2\,dt = \mathsf{M}^\dagger d\hat{\mathbf{B}}_{\mathrm{outt}} + \mathsf{M}^\top d\hat{\mathbf{B}}_{\mathrm{outt}}^\ddagger + \hbar\,d\hat{\boldsymbol{\upsilon}}_{\mathrm{outt}}, \tag{91}$$

where the subscript on the current should remind us that it is defined in terms of $d\hat{\mathbf{B}}_{\mathrm{outt}}$, or the number of times the letter "t" appears on the right-hand side.

$$\begin{aligned}\hbar\,d\hat{\boldsymbol{\upsilon}}_{\mathrm{outt}} \equiv {}& \sqrt{\mathsf{Z}}\;d\hat{\mathbf{U}}_{\mathrm{outt}} + \sqrt{\mathsf{Z}}^*\,d\hat{\mathbf{U}}_{\mathrm{outt}}^\ddagger\\ = {}& \hbar\,d\hat{\boldsymbol{\upsilon}}_{\mathrm{in}} - i\,\mathsf{Z}\,\hat{\mathbf{f}}\,dt + i\,\mathsf{Z}^*\,\hat{\mathbf{f}}\,dt.\end{aligned} \tag{92}$$

[4] Note, however, that $\hat{U}_{\mathrm{mfb}}^\dagger\hat{\mathbf{b}}_{\mathrm{in}}\hat{U}_{\mathrm{mfb}} \neq \hat{U}_{\mathrm{fb}}^\dagger\hat{\mathbf{b}}_{\mathrm{out}}\hat{U}_{\mathrm{fb}}$.

From (85), (91), and (92), we can see that

$$\hat{\mathbf{y}}_2 = \hat{\mathbf{y}}_1. \tag{93}$$

This is the current evolved over an infinitesimal interval from t to $t + dt$ under both measurement and feedback is in fact the same as the current evolved in the same time interval but with measurement alone. Equation (93) can also be seen from the form of the Hamiltonian (47), which gives

$$[\hat{H}_{\rm fb}, \hat{\mathbf{y}}_0] = 0. \tag{94}$$

Substituting $\hat{U}_{\rm mfb}$ into the definition of $\hat{\mathbf{y}}_2$ we obtain

$$\begin{aligned}\hat{\mathbf{y}}_2(t) &= \hat{U}^\dagger_{\rm mfb}(t+dt,t)\,\hat{\mathbf{y}}_0(t)\,\hat{U}_{\rm mfb}(t+dt,t)\\ &= \hat{U}^\dagger_{\rm m}(t+dt,t)\,\hat{\mathbf{y}}_0(t)\,\hat{U}_{\rm m}(t+dt,t) = \hat{\mathbf{y}}_1(t).\end{aligned} \tag{95}$$

Using (91) we can calculate how the current at time t is correlated to the current at a later time $t+\tau$ during which feedback is applied. The time separation τ is assumed to be non-negative. We then obtain

$$\begin{aligned}&\hbar^2\langle\hat{\mathbf{y}}_2(t)\,\hat{\mathbf{y}}_2^\top(t+\tau)\rangle\\ &\quad= \langle \mathsf{M}^\dagger\,\hat{\mathbf{b}}_{\rm outt}(t)\,\hat{\mathbf{b}}^\top_{\rm outt}(t+\tau)\,\mathsf{M}^*\rangle + \langle \mathsf{M}^\dagger\,\hat{\mathbf{b}}_{\rm outt}(t)\,\hat{\mathbf{b}}^\dagger_{\rm outt}(t+\tau)\,\mathsf{M}\rangle\\ &\qquad+ \langle \mathsf{M}^\dagger\,\hat{\mathbf{b}}_{\rm outt}(t)\hbar\,\hat{\boldsymbol{\zeta}}^\top_{\rm outt}(t+\tau)\rangle + \langle \mathsf{M}^\top\,\hat{\mathbf{b}}^\ddagger_{\rm outt}(t)\,\hat{\mathbf{b}}^\top_{\rm outt}(t+\tau)\mathsf{M}^*\rangle\\ &\qquad+ \langle \mathsf{M}^\top\,\hat{\mathbf{b}}^\ddagger_{\rm outt}(t)\,\hat{\mathbf{b}}^\dagger_{\rm outt}(t+\tau)\mathsf{M}\rangle + \langle \mathsf{M}^\top\,\hat{\mathbf{b}}^\ddagger_{\rm outt}(t)\,\hbar\,\hat{\boldsymbol{\zeta}}^\top_{\rm outt}(t+\tau)\rangle\\ &\qquad+ \langle\hbar\,\hat{\boldsymbol{\zeta}}_{\rm outt}(t)\,\hat{\mathbf{b}}^\top_{\rm outt}(t+\tau)\,\mathsf{M}^*\rangle + \langle\hbar\,\hat{\boldsymbol{\zeta}}_{\rm outt}(t)\,\hat{\mathbf{b}}^\dagger_{\rm outt}(t+\tau)\,\mathsf{M}\rangle\\ &\qquad+ \langle\hbar\,\hat{\boldsymbol{\zeta}}_{\rm outt}(t)\,\hbar\,\hat{\boldsymbol{\zeta}}^\top_{\rm outt}(t+\tau)\rangle.\end{aligned} \tag{96}$$

Note that here we have introduced quantum stochastic processes $\hat{\boldsymbol{\zeta}}_{\rm outt}$ and $\hat{\boldsymbol{\mu}}_{\rm outt}$, defined in terms of the increments by

$$d\hat{\boldsymbol{\upsilon}}_{\rm outt} = \hat{\boldsymbol{\zeta}}_{\rm outt}\,dt = (\sqrt{\mathsf{Z}}\,\hat{\boldsymbol{\mu}}_{\rm outt} + \sqrt{\mathsf{Z}}^*\hat{\boldsymbol{\mu}}^\ddagger_{\rm outt})dt, \tag{97}$$

where

$$d\hat{\mathbf{U}}_{\rm outt} = \hat{\boldsymbol{\mu}}_{\rm outt}\,dt = (\hat{\boldsymbol{\mu}}_{\rm in} - i\sqrt{\mathsf{Z}}\,\hat{\mathbf{f}})dt. \tag{98}$$

As with earlier calculations, the assumption of a vacuum bath state suggests that we should substitute (97) into (96) and then normal and time order each term before the average is taken. The output field vector operators satisfy the familiar free-field matrix-operator brackets

$$\lfloor\hat{\mathbf{b}}_{\rm outt}(t),\hat{\mathbf{b}}^\ddagger_{\rm outt}(t')\rceil = \hbar\,\hat{\mathrm{I}}_L\,\delta(t-t')\quad\forall\, t,t', \tag{99}$$

and

$$\lfloor\hat{\mathbf{b}}_{\rm outt}(t),\hat{\mathbf{b}}_{\rm outt}(t')\rceil = \lfloor\hat{\mathbf{b}}^\ddagger_{\rm outt}(t),\hat{\mathbf{b}}^\ddagger_{\rm outt}(t')\rceil = 0 \quad\forall\, t,t'. \tag{100}$$

The same is true for $\hat{\boldsymbol{\mu}}_{\rm outt}$ since it is also a free field, but remember that $\hat{\boldsymbol{\mu}}_{\rm outt}$ is $R\times 1$ so

$$\lfloor\hat{\boldsymbol{\mu}}_{\rm outt}(t),\hat{\boldsymbol{\mu}}^\ddagger_{\rm outt}(t')\rceil = \hbar\,\hat{\mathrm{I}}_R\,\delta(t-t'). \tag{101}$$

We also have, and it is not difficult to see, that

$$\lfloor\hat{\mathbf{b}}_{\rm outt}(t),\hat{\boldsymbol{\mu}}_{\rm outt}(t')\rceil = \lfloor\hat{\mathbf{b}}_{\rm outt}(t),\hat{\boldsymbol{\mu}}^\ddagger_{\rm outt}(t')\rceil = 0 \quad\forall\, t,t'. \tag{102}$$

We summarize the result of each term in Appendix F. Using the results therein we arrive at

$$\begin{aligned}\hbar^2\langle\hat{\mathbf{y}}_2(t)\,\hat{\mathbf{y}}_2^\top(t+\tau)\rangle &= \langle[\mathsf{M}^\dagger\hat{\mathbf{c}}(t+\tau) + \mathsf{M}^\top\hat{\mathbf{c}}^\ddagger(t+\tau)][\hat{\mathbf{c}}^\top(t)\mathsf{M}^*\\ &\quad - i\hbar\,\hat{\mathbf{f}}^\top(t)]\rangle^\top + \langle[\mathsf{M}^\top\hat{\mathbf{c}}^\ddagger(t) + i\hbar\,\hat{\mathbf{f}}(t)]\\ &\quad\times[\hat{\mathbf{c}}^\top(t+\tau)\mathsf{M}^* + \hat{\mathbf{c}}^\dagger(t+\tau)\mathsf{M}]\rangle\\ &\quad+\hbar^2\,\mathrm{I}_R\,\delta(\tau).\end{aligned} \tag{103}$$

Applying vector-operator quantum regression formulas to (103) we finally arrive at

Result 6.

$$\begin{aligned}&\hbar^2\langle\hat{\mathbf{y}}_2(t)\,\hat{\mathbf{y}}_2^\top(t+\tau)\rangle\\ &\quad= (\mathrm{Tr}\{(\mathsf{M}^\dagger\hat{\mathbf{c}} + \mathsf{M}^\top\hat{\mathbf{c}}^\ddagger)e^{\mathcal{L}_{\rm mfb}}[(\hat{\mathbf{c}}^\top\mathsf{M}^* - i\hbar\,\hat{\mathbf{f}}^\top)\rho(t)\\ &\qquad+\rho(t)(\hat{\mathbf{c}}^\dagger\mathsf{M} + i\hbar\,\hat{\mathbf{f}}^\top)]\})^\top + \hbar^2\,\mathrm{I}_R\,\delta(\tau)\,,\end{aligned} \tag{104}$$

where the time dependence has been placed in the system state and the vector operators are time independent. Note that (104) could have obtained by using the transformations (79) and (80) in the measurement-only correlation function

$$\begin{aligned}&\hbar^2\,\langle\hat{\mathbf{y}}_1(t)\,\hat{\mathbf{y}}_1^\top(t+\tau)\rangle\\ &\quad= (\mathrm{Tr}\{(\mathsf{M}^\dagger\hat{\mathbf{c}} + \mathsf{M}^\top\hat{\mathbf{c}}^\ddagger)e^{\mathcal{L}_{\rm m}\tau}[\hat{\mathbf{c}}^\top\mathsf{M}^*\,\rho(t) + \rho(t)\,\hat{\mathbf{c}}^\dagger\mathsf{M}]\})^\top\\ &\qquad+\hbar^2\,\mathrm{I}_R\,\delta(\tau),\end{aligned} \tag{105}$$

as one might have guessed.

V. SIMPLE CASES

Here we illustrate how the above theory can be used by considering Markovian feedback mediated by homodyne and heterodyne detection. For simplicity we take $L=1$. In the case of homodyne-mediated feedback we then obtain a SISO theory, whereas for heterodyne-mediated feedback we get a one-input–two-output theory. In the case of homodyne-mediated feedback we recover results previously derived in Refs. [26] and [56]. We will allow for nonunit detection efficiency in both cases and write η in place of H.

A. Consistency with previous results: Homodyne-mediated feedback

Consider the SISO limit defined by a quadrature measurement of the form

$$\hbar\,\langle\hat{y}_1\rangle\,dt \propto \langle d\hat{B}_{\rm out} + d\hat{B}^\dagger_{\rm out}\rangle. \tag{106}$$

The condition that M, now a scalar, must satisfy is simply

$$|\mathsf{M}|^2 = \hbar\,\eta. \tag{107}$$

The measurement defined by (106) can be achieved by choosing $\mathsf{M} = \sqrt{\hbar\,\eta}$. This gives $\mathsf{Z} = \hbar\,\bar{\eta}$, where we have defined $\bar{\eta} = 1-\eta$ for convenience. We find

$$\hbar\,\hat{y}_1\,dt = \sqrt{\hbar\,\eta}\,(\hat{c}+\hat{c}^\dagger)dt + \hbar\,d\hat{\upsilon}_{\rm m}, \tag{108}$$

where the measurement noise is

$$\hbar\,d\hat{\upsilon}_{\rm m} = 2\sqrt{\hbar\,\eta}\;\mathrm{Re}[d\hat{B}_{\rm in}] + 2\sqrt{\hbar\,\bar{\eta}}\;\mathrm{Re}[d\hat{U}_{\rm in}]. \tag{109}$$

It is clear that

$$(\hbar\, d\hat{v}_{\rm m})^2 = 4\hbar\,\eta\,({\rm Re}[d\hat{B}_{\rm in}])^2 + 4\hbar\,\bar{\eta}\,({\rm Re}[d\hat{U}_{\rm in}])^2 = \hbar^2 dt. \tag{110}$$

From (78) the stochastic feedback master equation is then

$$\begin{aligned}\hbar\, d\rho_{\rm c} = (&-i\,[\hat{H}_1,\rho_{\rm c}] + \mathcal{D}[\hat{c}]\rho_{\rm c} + \hbar\,\mathcal{D}[\hat{f}]\rho_{\rm c}\\ &- i\sqrt{\hbar\,\eta}\,[\hat{f},\hat{c}\,\rho_{\rm c} + \rho_{\rm c}\,\hat{c}^\dagger])\,dt\\ &+ dw\,\mathcal{H}[\sqrt{\hbar\,\eta}\,\hat{c} - i\hbar\,\hat{f}\,]\rho_{\rm c}.\end{aligned} \tag{111}$$

This is consistent with the stochastic master equation found in Ref. [56] for $\hbar = 1$ and when the current is suitably rescaled. It also reproduces the master equation in Ref. [26] when $\hbar = \eta = 1$. The Lindblad form of the unconditioned evolution can be found directly from (43),

$$\begin{aligned}\hbar\,\dot{\rho} \equiv \mathcal{L}_{\rm hom}\,\rho = &-i\left[\hat{H}_1 + \tfrac{1}{2}\sqrt{\hbar\,\eta}\,(\hat{f}\,\hat{c} + \hat{c}^\dagger\,\hat{f}),\rho\right]\\ &+ \mathcal{D}[\hat{c} - i\sqrt{\hbar\,\eta}\,\hat{f}\,]\rho + \hbar\,\bar{\eta}\mathcal{D}[\hat{f}\,]\rho.\end{aligned} \tag{112}$$

Again, this is consistent with the Lindblad form obtained in Ref. [56] (for $\hbar = 1$) and Ref. [26] (for $\hbar = \eta = 1$), but in these works the Lindblad form was obtained by algebraic manipulation of (111). The two-time correlation function of (108) is, from (104),

$$\begin{aligned}\hbar^2\langle\hat{y}_2(t)\,\hat{y}_2(t{+}\tau)\rangle = \sqrt{\hbar\,\eta}\;{\rm Tr}\{(\hat{c} + \hat{c}^\dagger)e^{\mathcal{L}_{\rm hom}\tau}[(\sqrt{\hbar\,\eta}\,\hat{c} - i\hbar\,\hat{f})\rho\\ + {\rm H.c.}]\} + \hbar^2\,\delta(\tau)\end{aligned} \tag{113}$$

and reproduces (4.10) of Ref. [26] when $\hbar = \eta = 1$.

We can also find an equation of motion for $\hat{\mathbf{s}}$ from either (44) or (56). There is no restriction on the number of components that $\hat{\mathbf{s}}$ is allowed. For simplicity we take it to be a scalar operator. Taking the Markovian limit of (44) the homodyne feedback Heisenberg equation is

$$\begin{aligned}\hbar\, d\hat{s} = &\left(i\,[\hat{H}_1,\hat{s}\,] + \mathcal{J}[\hat{c}^\dagger]\,\hat{s} - \tfrac{1}{2}\{\hat{c}^\dagger\hat{c},\hat{s}\} + \hbar\,\mathcal{D}[\hat{f}\,]\,\hat{s}\right)dt\\ &+ [\,\hat{c}^\dagger d\hat{B}_{\rm in} - d\hat{B}^\dagger_{\rm in}\hat{c},\hat{s}\,] - i\sqrt{\hbar\,\eta}\,[\hat{s},\hat{f}\,](\hat{c}\,dt + d\hat{B}_{\rm in})\\ &+ i\sqrt{\hbar\,\eta}\,(\hat{c}^\dagger\,dt + d\hat{B}^\dagger_{\rm in})[\hat{f},\hat{s}\,] - i\sqrt{\hbar\,\bar{\eta}}\,[\hat{s},\hat{f}\,]d\hat{U}_{\rm in}\\ &+ i\sqrt{\hbar\,\bar{\eta}}\;d\hat{U}^\dagger_{\rm in}[\hat{f},\hat{s}\,].\end{aligned} \tag{114}$$

As before, when $\hbar = \eta = 1$ this correctly reproduces (4.16) of Ref. [26]. Note the extra noise terms $d\hat{U}_{\rm in}$ and $d\hat{U}^\dagger_{\rm in}$ in (114) which do not appear in (4.16) of Ref. [26], since there, the Heisenberg equation was derived in the limit of $\eta = 1$.

B. Heterodyne-mediated feedback

A heterodyne detection is equivalent to two homodyne measurements of orthogonal quadratures each with half the detection efficiency so this requires $R = 2$. Consider the heterodyne current defined by

$$\hbar\,\langle\hat{\mathbf{y}}_1\rangle\,dt \;\propto\; \sqrt{\frac{\eta}{2}}\begin{pmatrix}\langle d\hat{B}_{\rm out} + d\hat{B}^\dagger_{\rm out}\rangle\\ -i\,\langle d\hat{B}_{\rm out} - d\hat{B}^\dagger_{\rm out}\rangle\end{pmatrix}. \tag{115}$$

This can be effected by

$$\mathsf{M} = \sqrt{\frac{\hbar\,\eta}{2}}\,(1,\,i), \tag{116}$$

which satisfies (17). The stochastic master equation from (78) is thus

$$\begin{aligned}\hbar\, d\rho_{\rm c} = \Bigg(&-i\,[\hat{H}_1,\rho_{\rm c}] + \mathcal{D}[\hat{c}]\rho_{\rm c} + \hbar\,\mathcal{D}[\hat{f}_1]\rho_{\rm c}\\ &+ \hbar\,\mathcal{D}[\hat{f}_2]\rho_{\rm c} - i\sqrt{\frac{\hbar\,\eta}{2}}\,[\hat{f}_1,\hat{c}\rho_{\rm c} + \rho_{\rm c}\hat{c}^\dagger]\\ &- i\sqrt{\frac{\hbar\,\eta}{2}}\,[\hat{f}_2,\, -i(\hat{c}\rho_{\rm c} - \rho_{\rm c}\hat{c}^\dagger)]\Bigg)dt\\ &+ dw_1\,\mathcal{H}[\sqrt{\hbar\,\eta/2}\;\hat{c} - i\hbar\,\hat{f}_1]\rho_{\rm c}\\ &+ dw_2\,\mathcal{H}[-i\sqrt{\hbar\,\eta/2}\;\hat{c} - i\hbar\,\hat{f}_2]\rho_{\rm c}.\end{aligned} \tag{117}$$

Setting $\hbar = \eta = 1$ this is consistent with a special case of the heterodyne feedback master equation in Ref. [57] [see (5.19)–(5.24) with $N = M = 0$]. The Lindblad form of the master equation unravelled by (117) can be obtained by noting that (116) leads to

$$\mathsf{Z} = \hbar\begin{pmatrix}1 - \eta/2 & -i\,\eta/2\\ i\,\eta/2 & 1 - \eta/2\end{pmatrix}, \tag{118}$$

which has the positive square root

$$\sqrt{\mathsf{Z}} = \frac{\sqrt{\hbar}}{2}\begin{pmatrix}1 + \sqrt{\bar{\eta}} & -i\,(1 - \sqrt{\bar{\eta}}\,)\\ i(1 - \sqrt{\bar{\eta}}\,) & 1 + \sqrt{\bar{\eta}}\end{pmatrix}. \tag{119}$$

By introducing

$$\hat{F} = \hat{f}_1 + i\,\hat{f}_2, \tag{120}$$

the unconditioned evolution unravelled by (117) has a Lindblad form which can be written compactly as

$$\begin{aligned}\hbar\,\dot{\rho} \equiv \mathcal{L}_{\rm het}\,\rho = &-i[\hat{H}_1,\rho] - i\sqrt{\frac{\hbar\eta}{8}}\;[\hat{F}^\dagger\hat{c} + \hat{c}^\dagger\hat{F},\rho]\\ &+ \frac{\hbar}{4}\,\mathcal{D}[\hat{F}^\dagger + \sqrt{\bar{\eta}}\,\hat{F}]\rho + \frac{\hbar}{4}\,\mathcal{D}[\hat{F}^\dagger - \sqrt{\bar{\eta}}\,\hat{F}]\rho\\ &+ \mathcal{D}[\hat{c} - i\sqrt{\hbar\eta/2}\,\hat{F}]\rho.\end{aligned} \tag{121}$$

We find the heterodyne current has correlations given by

$$\begin{aligned}&\hbar^2\left\langle\hat{\mathbf{y}}_2(t)\,\hat{\mathbf{y}}_2^\top(t+\tau)\right\rangle\\ &= \sqrt{\frac{\hbar\,\eta}{2}}\begin{pmatrix}{\rm Tr}\big\{(\hat{c}+\hat{c}^\dagger)e^{\mathcal{L}_{\rm het}\tau}\big[\big(\sqrt{\tfrac{\hbar\eta}{2}}\,\hat{c} - i\hbar\,\hat{f}_1\big)\rho + {\rm H.c.}\big]\big\} & {\rm Tr}\big\{-i(\hat{c}-\hat{c}^\dagger)e^{\mathcal{L}_{\rm het}\tau}\big[\big(\sqrt{\tfrac{\hbar\eta}{2}}\,\hat{c} - i\hbar\,\hat{f}_1\big)\rho + {\rm H.c.}\big]\big\}\\ {\rm Tr}\big\{(\hat{c}+\hat{c}^\dagger)e^{\mathcal{L}_{\rm het}\tau}\big[\big(-i\sqrt{\tfrac{\hbar\eta}{2}}\,\hat{c} - i\hbar\,\hat{f}_2\big)\rho + {\rm H.c.}\big]\big\} & {\rm Tr}\big\{-i(\hat{c}-\hat{c}^\dagger)e^{\mathcal{L}_{\rm het}\tau}\big[\big(-i\sqrt{\tfrac{\hbar\eta}{2}}\,\hat{c} - i\hbar\,\hat{f}_2\big)\rho + {\rm H.c.}\big]\big\}\end{pmatrix}\\ &\quad + \hbar^2\,{\rm I}_2\,\delta(\tau).\end{aligned} \tag{122}$$

As with the homodyne case we can derive a heterodyne Heisenberg equation assuming $\hat{\mathbf{s}}$ to be a scalar operator by taking the Markovian limit from (44). The result is

$$\hbar\, d\hat{s} = \big(i\,[\hat{H}_1,\hat{s}\,] + \mathcal{J}[\hat{c}^\dagger]\,\hat{s} - \frac{1}{2}\{\hat{c}^\dagger\hat{c},\hat{s}\} + \hbar\,\mathcal{D}[\hat{f}\,]\,\hat{s}\,\big)dt + [\hat{c}^\dagger d\hat{B}_{\rm in} - d\hat{B}_{\rm in}^\dagger\hat{c},\hat{s}] - i\sqrt{\frac{\hbar\,\eta}{2}}\,[\hat{s},\hat{F}^\dagger](\hat{c}\,dt + d\hat{B}_{\rm in}) + i\sqrt{\frac{\hbar\,\eta}{2}}\,(\hat{c}^\dagger\,dt + d\hat{B}_{\rm in}^\dagger)[\hat{F},\hat{s}] - i[\hat{s},\hat{F} + \sqrt{\bar{\eta}}\,F^\dagger]d\hat{U}_{\rm in1} - i[\hat{s}, -i(\hat{F} + \sqrt{\bar{\eta}}\,\hat{F}^\dagger)]d\hat{U}_{\rm in2} + i\,d\hat{U}_{\rm in1}^\dagger[\sqrt{\bar{\eta}}\,\hat{F} + \hat{F}^\dagger,\hat{s}] + i\,d\hat{U}_{\rm in2}^\dagger[-i(\sqrt{\bar{\eta}}\,\hat{F} - \hat{F}^\dagger),\hat{s}]. \tag{123}$$

VI. DISCUSSION

We have constructed a theory of Markovian quantum feedback control for nonlinear systems with an arbitrary number of decay channels, inputs, and outputs and mediated by arbitrary diffusive measurements. We have derived the time evolution of the system state both with and without conditioning for a vacuum bath input. When the evolution is unconditioned, one may find an equivalent formulation in terms of Heisenberg equations of motion and we have derived these equations, too. We also derived the two-time correlation function for the measured current including feedback.

We have performed our derivations using the Heisenberg picture, where the entire feedback loop is described by unitary evolution. Most notably, we established relation (66), which can be viewed as the analog of (46) but for conditional evolution. This is what allowed us to derive the stochastic master equation from the quantum Langevin equation.

It is interesting to note that the two-time correlation function of the measured current is an expression about measurements at two separated times. It therefore lends itself as a different way of deriving the stochastic master equation. In this approach one would calculate the correlation function in the Schrödinger picture by making the ansatz (68) and equating the end result to (104). Solving for $\hat{\boldsymbol{\alpha}}$ should result in (77). If one was only interested in the stochastic master equation then this second method is, however, much less direct than the first approach, as the calculation of the correlation function in the Heisenberg picture is a very lengthy process. Alternatively, one could derive a stochastic master equation first and then use it to derive the autocorrelation of the current on which the state is conditioned in the Schrödinger picture. However, our approach to obtaining the autocorrelation of the current and the stochastic master equation has not been to derive one result from the other but rather each result independently.

The interpretation of the Heisenberg picture approach was recognized in Ref. [26] and also discussed in detail in Ref. [5]. In essence this is a no-measurement (or more precisely no-collapse) model where the observer is never aware of the measurement record from the monitoring. Consequently, we have refrained from using terms such as "unravellings" or "conditional" unless in explicit reference to results in the Schrödinger picture.

Finally, we note that it would be possible to generalize the results of this paper even further by allowing the bath to be nonvacuum. In such a theory we would have to allow $d\hat{\mathbf{B}}_{\rm in}$ to have a nonzero mean and correlated more generally as opposed just (5).

ACKNOWLEDGMENTS

This research was conducted by the Australian Research Council Centre of Excellence for Quantum Computation and Communication Technology (project no. CE110001027).

APPENDIX A : CONNECTION TO CONTROL SYSTEMS ENGINEERING

The standard engineering approach to feedback control is to start with a stochastic differential equation of a vector $\mathbf{x}$. A model of the system in the time domain is known as a state-space model and the vector $\mathbf{x}$ a state. The state contains variables such that if these variables are known at time t then all other system variables at time t may be calculated from it [58].

The state-space model can be translated to quantum dynamics most easily via the Heisenberg picture. In the Heisenberg picture we define an $N \times 1$ Hermitian vector operator $\hat{\mathbf{x}}$, from which an arbitrary system operator $\hat{s}$ can be defined. We will be considering continuous Markov processes in which case,

$$d\hat{\mathbf{x}} = \alpha(\hat{\mathbf{x}},\hat{\mathbf{u}},t)\,dt + \beta(\hat{\mathbf{x}},t)\,d\hat{\mathbf{v}}_{\rm p}, \tag{A1}$$

where $\hat{\mathbf{u}}$ is the input (potentially arising from feedback) and $d\hat{\mathbf{v}}_{\rm p}$ is an $M \times 1$ quantum Wiener increment defined by

$$\langle d\hat{\mathbf{v}}_{\rm p}(t)\rangle = \mathbf{0}, \tag{A2}$$

and the Itô rules

$$d\hat{\mathbf{v}}_{\rm p}(t)\,d\hat{\mathbf{v}}_{\rm p}^\top(t') = 0 \quad \forall\, t \neq t', \tag{A3}$$

$$d\hat{\mathbf{v}}_{\rm p}(t)\,d\hat{\mathbf{v}}_{\rm p}^\top(t) = \hat{\mathrm{I}}_M\,dt. \tag{A4}$$

Note that α is a vector-operator-valued function while β maps to a $N \times M$ matrix (which, in general, may matrix-operator valued).

We will assume the system to be monitored via R channels and that the measurement noise is diffusive. Let us denote the measurement results by $\hat{\mathbf{y}}_1$, which can be written in the general form

$$\hat{\mathbf{y}}_1\,dt = g(\hat{\mathbf{x}},\hat{\mathbf{u}},t)\,dt + d\hat{\mathbf{v}}_{\rm m}. \tag{A5}$$

The noise term $d\hat{\mathbf{v}}_{\rm m}$ is another Wiener increment and is what defines the measurement to be diffusive. It is often assumed that $d\hat{\mathbf{v}}_{\rm p}$ is uncorrelated with $d\hat{\mathbf{v}}_{\rm m}$. One could, of course, drop this assumption and allow the two noises to be correlated if necessary [5]. It is conventional (and we will follow this convention) to call $\hat{\mathbf{y}}_1$ the output.

Equation (A1) is generated by a Hamiltonian which one often writes in the general form

$$\hat{H} = \hat{H}_1 + \hat{H}_{\rm m} + \hat{H}_{\rm fb}. \tag{A6}$$

Here $\hat{H}_{\rm m}$ is still defined by (2) but the feedback Hamiltonian is kept general, of the form,

$$\hat{H}_{\rm fb} = \hat{\mathbf{f}}^{\top}\hat{\mathbf{u}}, \tag{A7}$$

where $\hat{\mathbf{u}}$ and $\hat{\mathbf{f}}$ are Hermitian and $\lceil\hat{\mathbf{f}},\hat{\mathbf{u}}\rfloor = 0$ to ensure the Hermiticity of $\hat{H}_{\rm fb}$. The input is then used to influence some system observable $\hat{\mathbf{f}}$. Note that when $\hat{\mathbf{u}}$ is a feedback input it will be a functional of the output $\hat{\mathbf{y}}_1$, which is a bath vector operator so the condition $\lceil\hat{\mathbf{f}},\hat{\mathbf{u}}\rfloor = 0$ will be guaranteed.

When the input is chosen to be linear in the output

$$\hat{\mathbf{u}}(t) = L\,\hat{\mathbf{y}}_1(t-\tau), \tag{A8}$$

where τ is the feedback delay, the feedback is said to be proportional or Markovian (provided $\tau \to 0^+$). Note that to obtain Markovian system evolution the matrix L needs to be independent of time. We will absorb L into the definition of $\hat{\mathbf{f}}$ and just define Markovian feedback by

$$\hat{\mathbf{u}}(t) = \hat{\mathbf{y}}_1(t-\tau), \tag{A9}$$

and $\hat{\mathbf{f}}$ a $R \times 1$ vector operator. This will keep our calculation simpler, without the need to write out L explicitly. Taking the input and output to be of the same dimension is no less general than if they were of different dimensions as we can always pad zeros in $\hat{\mathbf{u}}$ (or $\hat{\mathbf{y}}_1$, since $\hat{\mathbf{u}}$ is just the time-delayed version of $\hat{\mathbf{y}}_1$) if there is no feedback in some of the input channels. It is also not sensible to allow $\hat{\mathbf{u}}$ (and therefore $\hat{\mathbf{y}}_1$) to have more than R components since then the inner product $\hat{\mathbf{f}}^{\top}\hat{\mathbf{u}}$ is undefined.

APPENDIX B :DERIVATION OF EQ. (11)

From (6) we have

$$\hbar\,\hat{U}_{\rm m}(t+dt,t) = \hbar\,\hat{1} - i\hat{H}_1\,dt - \tfrac{1}{2}\hat{\mathbf{c}}^{\dagger}\hat{\mathbf{c}}\,dt + d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}} - \hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in}. \tag{B1}$$

By definition,

$$\hbar^2[d\hat{\mathbf{s}}]_{\rm m} = [\hbar\,\hat{U}_{\rm m}^{\dagger}(t+dt,t)]\hat{\mathbf{s}}[\hbar\,\hat{U}_{\rm m}(t+dt,t)] - \hat{\mathbf{s}}, \tag{B2}$$

where we have suppressed the time argument for vector operators at time t for ease of writing. Substituting (B1) into (B2), and expanding and collecting like terms, we obtain

$$\begin{aligned}\hbar^2[d\hat{\mathbf{s}}]_{\rm m} = {} & i\hbar\,[\hat{H}_1,\hat{\mathbf{s}}]dt + \hbar\,[\hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in} - d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}},\hat{\mathbf{s}}] \\ & + (\hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in})\hat{\mathbf{s}}(d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}}) - \frac{\hbar}{2}\{\hat{\mathbf{c}}^{\dagger}\hat{\mathbf{c}},\hat{\mathbf{s}}\}\,dt.\end{aligned} \tag{B3}$$

To simplify the first term in the second line we will derive a slightly more general relation that will also be useful for deriving (56) (Result 4) in Appendix D. For two arbitrary vector operators $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ we may write

$$\begin{aligned}(\hat{\boldsymbol{\alpha}}^{\top}d\hat{\mathbf{B}}_{\rm in})\,\hat{\mathbf{s}}\,(d\hat{\mathbf{B}}_{\rm in}^{\dagger}\,\hat{\boldsymbol{\beta}}) = {} & (\hat{\boldsymbol{\alpha}}^{\top}d\hat{\mathbf{B}}_{\rm in})(\hat{\mathbf{s}}\,d\hat{\mathbf{B}}_{\rm in}^{\dagger})\hat{\boldsymbol{\beta}} \\ = {} & (\hat{\boldsymbol{\alpha}}^{\top}d\hat{\mathbf{B}}_{\rm in})(d\hat{\mathbf{B}}_{\rm in}^{\ddagger}\hat{\mathbf{s}}^{\top})^{\top}\hat{\boldsymbol{\beta}},\end{aligned} \tag{B4}$$

where in the first line above we have simply reassociated the terms while the second line follows from using $\lfloor\hat{\mathbf{s}},d\hat{\mathbf{B}}_{\rm in}^{\ddagger}\rceil = 0$.

Using $\hat{c}\,\hat{\mathsf{A}}^{\top} = (\hat{c}\,\hat{\mathsf{A}})^{\top}$ for any scalar operator $\hat{c}$ and matrix operator $\hat{\mathsf{A}}$ we have

$$\begin{aligned}(\hat{\boldsymbol{\alpha}}^{\top}d\hat{\mathbf{B}}_{\rm in})(d\hat{\mathbf{B}}_{\rm in}^{\ddagger}\,\hat{\mathbf{s}}^{\top})^{\top}\hat{\boldsymbol{\beta}} = {} & [(\hat{\boldsymbol{\alpha}}^{\top}d\hat{\mathbf{B}}_{\rm in})d\hat{\mathbf{B}}_{\rm in}^{\ddagger}\hat{\mathbf{s}}]^{\top}\hat{\boldsymbol{\beta}} \\ = {} & [(\hat{\boldsymbol{\alpha}}^{\top}d\hat{\mathbf{B}}_{\rm in}d\hat{\mathbf{B}}_{\rm in}^{\dagger})^{\top}\hat{\mathbf{s}}^{\top}]^{\top}\hat{\boldsymbol{\beta}} \\ = {} & \hbar\,(\hat{\boldsymbol{\alpha}}\,\hat{\mathbf{s}}^{\top})^{\top}\hat{\boldsymbol{\beta}}\,dt,\end{aligned} \tag{B5}$$

where we have used the quantum Itô rule (5) to obtain the last line; that is, we have

$$(\hat{\boldsymbol{\alpha}}^{\top}d\hat{\mathbf{B}}_{\rm in})\,\hat{\mathbf{s}}\,(d\hat{\mathbf{B}}_{\rm in}^{\dagger}\,\hat{\boldsymbol{\beta}}) = \hbar\,(\hat{\boldsymbol{\alpha}}\,\hat{\mathbf{s}}^{\top})^{\top}\hat{\boldsymbol{\beta}}\,dt. \tag{B6}$$

Now letting $\hat{\boldsymbol{\alpha}} = \hat{\mathbf{c}}^{\ddagger}$ and $\hat{\boldsymbol{\beta}} = \hat{\mathbf{c}}$ in (B6) we obtain

$$(\hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in})\hat{\mathbf{s}}(d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}}) = \hbar\,(\hat{\mathbf{c}}^{\ddagger}\,\hat{\mathbf{s}}^{\top})^{\top}\hat{\mathbf{c}}\,dt. \tag{B7}$$

Substituting (B7) into (B3) and canceling one factor of $\hbar$ we arrive at

$$\begin{aligned}\hbar\,[d\hat{\mathbf{s}}]_{\rm m} = {} & \big(i\,[\hat{H}_1,\hat{\mathbf{s}}] + [\hat{\mathbf{c}}^{\ddagger}\hat{\mathbf{s}}^{\top}]^{\top}\hat{\mathbf{c}} - \tfrac{1}{2}\{\hat{\mathbf{c}}^{\dagger}\hat{\mathbf{c}},\hat{\mathbf{s}}\}\big)dt \\ & + [\hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in} - d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}},\hat{\mathbf{s}}].\end{aligned} \tag{B8}$$

When we come to describe feedback, it will be useful to write our Heisenberg equations in a form which resembles the SISO results of Ref. [26] for ease of comparison. To this end we rewrite (B8) by noting that

$$(\hat{\mathbf{c}}^{\ddagger}\hat{\mathbf{s}}^{\top})^{\top}\hat{\mathbf{c}} - \tfrac{1}{2}\{\hat{\mathbf{c}}^{\dagger}\hat{\mathbf{c}},\hat{\mathbf{s}}\} = -\tfrac{1}{2}\lfloor\hat{\mathbf{s}},\hat{\mathbf{c}}^{\ddagger}\rceil\,\hat{\mathbf{c}} - \tfrac{1}{2}(\hat{\mathbf{c}}^{\dagger}\lfloor\hat{\mathbf{c}},\hat{\mathbf{s}}\rceil)^{\top}, \tag{B9}$$

$$[\hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in} - d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}},\hat{\mathbf{s}}] = -\lfloor\hat{\mathbf{s}},\hat{\mathbf{c}}^{\ddagger}\rceil d\hat{\mathbf{B}}_{\rm in} - (d\hat{\mathbf{B}}_{\rm in}^{\dagger}\lfloor\hat{\mathbf{c}},\hat{\mathbf{s}}\rceil)^{\top}. \tag{B10}$$

Equation (B8) is then

$$\begin{aligned}\hbar\,[d\hat{\mathbf{s}}]_{\rm m} = {} & i\,[\hat{H}_1,\hat{\mathbf{s}}]\,dt - \lfloor\hat{\mathbf{s}},\hat{\mathbf{c}}^{\ddagger}\rceil\big(\tfrac{1}{2}\,\hat{\mathbf{c}}\,dt + d\hat{\mathbf{B}}_{\rm in}\big) \\ & - \big\{\big(\tfrac{1}{2}\,\hat{\mathbf{c}}^{\dagger}dt + d\hat{\mathbf{B}}_{\rm in}^{\dagger}\big)\lfloor\hat{\mathbf{c}},\hat{\mathbf{s}}\rceil\big\}^{\top}.\end{aligned} \tag{B11}$$

Note that (B11) does not give the correct input-output relation (7), whereas (B8) does. This can be seen by letting $\hat{\mathbf{s}} = d\hat{\mathbf{B}}_{\rm in}$ in (B8). Doing so, we find the first line of (B8) is negligible (it is a higher-order infinitesimal) while the second line of (B8) is

$$[\hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in} - d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}},d\hat{\mathbf{B}}_{\rm in}] = \hbar\,\hat{\mathbf{c}}\,dt. \tag{B12}$$

This gives the right $d\hat{\mathbf{B}}_{\rm out}$ since the left-hand side of the Heisenberg equation is, by definition, $\hbar(d\hat{\mathbf{B}}_{\rm out} - d\hat{\mathbf{B}}_{\rm in})$. The reason that (B11) does not reproduce the correct $d\hat{\mathbf{B}}_{\rm out}$ is because it rearranges the commutator as shown in (B10) by assuming $\lfloor\hat{\mathbf{s}},d\hat{\mathbf{B}}_{\rm in}\rceil = \lfloor\hat{\mathbf{s}},d\hat{\mathbf{B}}_{\rm in}^{\ddagger}\rceil = 0$. This assumption is no longer true when we let $\hat{\mathbf{s}} = d\hat{\mathbf{B}}_{\rm in}$. In fact, we know that for a system with L decay channels in a vacuum bath,

$$\lfloor d\hat{\mathbf{B}}_{\rm in},d\hat{\mathbf{B}}_{\rm in}^{\ddagger}\rceil = \hbar\,\hat{1}_L dt. \tag{B13}$$

Note that it is only the commutator $[\hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in} - d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}},\hat{\mathbf{s}}]$ that matters in producing the correct form of $d\hat{\mathbf{B}}_{\rm out}$ because the first line of (B8) does not contribute when $\hat{\mathbf{s}} = d\hat{\mathbf{B}}_{\rm in}$.

APPENDIX C: DERIVATION OF EQ. (26)

It would be most natural to derive the inifinitesimal evolution given by (26) with the full unitary operator

$$\hat{U}_{\rm mfb}(t+dt,t) = \exp[(\hat{H}_1 + \hat{H}_{\rm m} + \hat{H}_{\rm fb})\,dt/\hbar], \tag{C1}$$

where $\hat{H}_{\rm m}$ and $\hat{H}_{\rm fb}$ are given by (2) and (24), respectively. Expanding this to order dt,

$$\hat{U}_{\rm mfb}(t+dt,t) = \hat{1} - i\,\frac{dt}{\hbar}\,(\hat{H}_1 + \hat{H}_{\rm m} + \hat{H}_{\rm fb}) - \frac{1}{2\hbar^2}\,(\hat{H}_{\rm m}\,dt + \hat{H}_{\rm fb}\,dt)^2. \quad \text{(C2)}$$

The important step here is to note that cross terms between $\hat{H}_{\rm m}$ and $\hat{H}_{\rm fb}$ do not contribute for a nonzero feedback delay τ:

$$(\hat{H}_{\rm m}\,dt)(\hat{H}_{\rm fb}\,dt) = (d\hat{\mathbf{B}}_{\rm in}^{\dagger}\,\hat{\mathbf{c}} - \hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in})[\hbar\,\hat{\mathbf{f}}^{\top}\hat{\mathbf{y}}_1(t-\tau)\,dt] \quad \text{(C3)}$$
$$= \hbar\,\hat{\mathbf{c}}^{\top}[d\hat{\mathbf{B}}_{\rm in}^{\ddagger}\,\hat{\mathbf{y}}_1^{\top}(t-\tau)\,dt]\hat{\mathbf{f}} - \hbar\,\hat{\mathbf{c}}^{\dagger}[d\hat{\mathbf{B}}_{\rm in}\,\hat{\mathbf{y}}_1^{\top}(t-\tau)\,dt]\hat{\mathbf{f}}. \quad \text{(C4)}$$

Recall that $\hat{\mathbf{y}}_1(t-\tau)\,dt$ is defined in terms of $d\hat{\mathbf{B}}_{\rm out}(t-\tau)$ and $d\hat{\mathbf{B}}_{\rm out}^{\ddagger}(t-\tau)$, which for $\tau > 0$,

$$\lfloor d\hat{\mathbf{B}}_{\rm out}(t-\tau), d\hat{\mathbf{B}}_{\rm in}(t)\rceil = \lfloor d\hat{\mathbf{B}}_{\rm out}(t-\tau), d\hat{\mathbf{B}}_{\rm in}^{\ddagger}(t)\rceil = 0, \quad \text{(C5)}$$

and similarly with $d\hat{\mathbf{B}}_{\rm out}$ replaced by $d\hat{\mathbf{B}}_{\rm out}^{\ddagger}$. Therefore the products $d\hat{\mathbf{B}}_{\rm in}^{\ddagger}\,\hat{\mathbf{y}}_1^{\top}(t-\tau)\,dt$ and $d\hat{\mathbf{B}}_{\rm in}\,\hat{\mathbf{y}}_1^{\top}(t-\tau)$ can always be written as normally ordered functions in the input fields which average to zero for a vacuum bath. Similarly, $(\hat{H}_{\rm fb}\,dt)(\hat{H}_{\rm m}\,dt)$ is also negligible. Letting $\hbar \equiv 1$ for simplicity, we thus obtain

$$\hat{U}_{\rm mfb}^{\dagger}(t+dt,t)\,\hat{\mathbf{s}}\,\hat{U}_{\rm mfb}(t+dt,t) = \hat{\mathbf{s}} - i\,\hat{\mathbf{s}}\,(\hat{H}_1 + \hat{H}_{\rm m} + \hat{H}_{\rm fb})dt - \tfrac{1}{2}\,\hat{\mathbf{s}}\,(\hat{H}_{\rm m}\,dt + \hat{H}_{\rm fb}\,dt)^2 + i\,(\hat{H}_1 + \hat{H}_{\rm m} + \hat{H}_{\rm fb})dt\,\hat{\mathbf{s}} + (\hat{H}_{\rm m}\,dt + \hat{H}_{\rm fb}\,dt)\,\hat{\mathbf{s}} \times (\hat{H}_{\rm m}\,dt + \hat{H}_{\rm fb}\,dt) - \tfrac{1}{2}\,(\hat{H}_{\rm m}\,dt + \hat{H}_{\rm fb}\,dt)^2\,\hat{\mathbf{s}}. \quad \text{(C6)}$$

Expanding and collecting terms proportional to $\hat{H}_1 + \hat{H}_{\rm m}$ as one group and terms proportional to $\hat{H}_{\rm fb}$ as another group we get

$$\hat{U}_{\rm mfb}^{\dagger}(t+dt,t)\,\hat{\mathbf{s}}\,\hat{U}_{\rm mfb}(t+dt,t) = \hat{\mathbf{s}} + [e^{i(\hat{H}_1+\hat{H}_{\rm m})dt}\,\hat{\mathbf{s}}\,e^{-i(\hat{H}_1+\hat{H}_{\rm m})dt} - \hat{\mathbf{s}}] + [e^{i\hat{H}_{\rm fb}dt}\,\hat{\mathbf{s}}\,e^{-i\hat{H}_{\rm fb}dt} - \hat{\mathbf{s}}]. \quad \text{(C7)}$$

We have noted that adding $\hat{H}_1$ to $\hat{H}_{\rm m}$ on the exponent of the exponential only has an effect to order dt. Subtracting $\hat{\mathbf{s}}$ from each side this is simply

$$d\hat{\mathbf{s}} = [d\hat{\mathbf{s}}]_{\rm m} + [d\hat{\mathbf{s}}]_{\rm fb}. \quad \text{(C8)}$$

It should be apparent from the above that the validity of (C8) relies on the procedure of first allowing $\tau \neq 0$ and then letting $\tau \to 0^+$ in the end.

APPENDIX D: DERIVATION OF EQ. (56)

Using the Hudson-Parthasarathy equation (55) we get

$$\hbar^2\hat{U}_{\rm mfb}(t+dt,t) = \hbar^2\hat{1} + i\hbar\,\hat{H}_1\,dt - \hbar\,\hat{\mathbf{c}}^{\dagger}\hat{\mathbf{c}}\,dt/2 + \hbar(\hat{\mathbf{c}}\,d\hat{\mathbf{B}}_{\rm in} - d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}}) + i\hbar(d\hat{\mathbf{B}}_{\rm in}^{\dagger}\mathsf{M} + d\hat{\mathbf{B}}_{\rm in}^{\top}\mathsf{M}^* + \hbar\,d\hat{\boldsymbol{v}}_{\rm in}^{\top})\,\hat{\mathbf{f}} + i\hbar\,\hat{\mathbf{c}}^{\dagger}\mathsf{M}\,\hat{\mathbf{f}}\,dt - \hbar^2\,\hat{\mathbf{f}}^{\top}\hat{\mathbf{f}}\,dt/2. \quad \text{(D1)}$$

Note that for compactness we are using the correlated noise $d\hat{\boldsymbol{v}}_{\rm in}$ in (D1). This is related to $d\hat{\mathbf{U}}_{\rm in}$ by (20). From this we obtain

$$\begin{aligned}
\hbar^4\hat{\mathbf{s}}(t+dt) &= [\hbar^2\hat{U}_{\rm mfb}^{\dagger}(t+dt,t)]\,\hat{\mathbf{s}}(t)\,[\hbar^2\hat{U}_{\rm mfb}^{\dagger}(t+dt,t)] \\
&= \hbar^4\hat{\mathbf{s}} - \underbrace{i\hbar^3\hat{\mathbf{s}}\,\hat{H}_1\,dt}_{\text{term A1}} - \underbrace{\hbar^3\,\hat{\mathbf{c}}^{\dagger}\hat{\mathbf{c}}\,dt/2}_{\text{term B1}} + \underbrace{\hbar^3\hat{\mathbf{s}}(d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}} - \hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in})}_{\text{term C1}} \\
&\quad - \underbrace{i\hbar^3\hat{\mathbf{s}}\,(\mathsf{M}^*\,\hat{\mathbf{f}})^{\top}d\hat{\mathbf{B}}_{\rm in}}_{\text{term D1}} - \underbrace{i\hbar^3\hat{\mathbf{s}}\,(\mathsf{M}\,\hat{\mathbf{f}})^{\top}d\hat{\mathbf{B}}_{\rm in}^{\ddagger}}_{\text{term E1}} \\
&\quad - \underbrace{i\hbar^3\hat{\mathbf{s}}\,\hat{\mathbf{f}}^{\top}(\hbar\,d\hat{\boldsymbol{v}}_{\rm in})}_{\text{term F1}} - \underbrace{i\hbar^3\hat{\mathbf{s}}(\mathsf{M}^*\hat{\mathbf{f}})^{\top}\hat{\mathbf{c}}\,dt}_{\text{term G1}} - \underbrace{\hbar^4\,\hat{\mathbf{s}}\,\hat{\mathbf{f}}^{\top}\hat{\mathbf{f}}\,dt/2}_{\text{term H1}} \\
&\quad + \underbrace{i\hbar^3\,\hat{H}_1\,\hat{\mathbf{s}}\,dt}_{\text{term A2}} - \underbrace{\hbar^3\,\hat{\mathbf{c}}^{\dagger}\hat{\mathbf{c}}\,\hat{\mathbf{s}}\,dt/2}_{\text{term B2}} + \underbrace{\hbar^3(\hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in} - d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}})\hat{\mathbf{s}}}_{\text{term C2}} \\
&\quad + \underbrace{\hbar^2(\hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in})\hat{\mathbf{s}}(d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}})}_{\text{term B3}} - \underbrace{i\hbar^2(\hat{\mathbf{c}}^{\dagger}d\hat{\mathbf{B}}_{\rm in})\hat{\mathbf{s}}(\hat{\mathbf{f}}^{\top}\mathsf{M}^{\top}d\hat{\mathbf{B}}_{\rm in}^{\ddagger})}_{\text{term I1}} \\
&\quad + \underbrace{i\hbar^3 d\hat{\mathbf{B}}_{\rm in}^{\dagger}(\mathsf{M}\,\hat{\mathbf{f}})\,\hat{\mathbf{s}}}_{\text{term E2}} + \underbrace{i\hbar^3 d\hat{\mathbf{B}}_{\rm in}^{\top}\mathsf{M}^*\hat{\mathbf{f}}\,\hat{\mathbf{s}}}_{\text{term D2}} + \underbrace{i\hbar^3(\hbar\,d\hat{\boldsymbol{v}}_{\rm in}^{\top})\hat{\mathbf{f}}\,\hat{\mathbf{s}}}_{\text{term F2}} \\
&\quad + \underbrace{i\hbar^2(d\hat{\mathbf{B}}_{\rm in}^{\top}\mathsf{M}^*\,\hat{\mathbf{f}})\hat{\mathbf{s}}(d\hat{\mathbf{B}}_{\rm in}^{\dagger}\hat{\mathbf{c}})}_{\text{term G2}} + \underbrace{\hbar^2(d\hat{\mathbf{B}}_{\rm in}^{\top}\mathsf{M}^*\,\hat{\mathbf{f}})\hat{\mathbf{s}}(\hat{\mathbf{f}}^{\top}\mathsf{M}^{\top}d\hat{\mathbf{B}}_{\rm in}^{\ddagger})}_{\text{term H2}} \\
&\quad + \underbrace{\hbar^4(d\hat{\boldsymbol{v}}_{\rm in}^{\top}\,\hat{\mathbf{f}})\hat{\mathbf{s}}(\hat{\mathbf{f}}^{\top}\,d\hat{\boldsymbol{v}}_{\rm in})}_{\text{term H3}} + \underbrace{i\hbar^3\hat{\mathbf{c}}^{\dagger}(\mathsf{M}\,\hat{\mathbf{f}})\hat{\mathbf{s}}\,dt}_{\text{term I2}} - \underbrace{\hbar^4\,\hat{\mathbf{f}}^{\top}\hat{\mathbf{f}}\,\hat{\mathbf{s}}\,dt/2.}_{\text{term H4}}
\end{aligned} \quad \text{(D2)}$$

We have labeled each term to indicate that terms designated with the same letter should be grouped together, e.g., terms A1 and A2 should be grouped together, and B1, B2, and B3 should be grouped together. The sum of the A terms, B terms, and C terms gives (B8) (with the appropriate factors of $\hbar$). Using

$$(\hat{\mathbf{A}}\hat{\mathbf{B}}^{\top})^{\top}\hat{\mathbf{C}} = [\hat{\mathbf{A}}^{\top}(\hat{\mathbf{B}}\hat{\mathbf{C}}^{\top})^{\top}]^{\top} \quad \text{(D3)}$$
$$(\hat{\mathbf{A}}^{\top}\hat{\mathbf{B}})\hat{\mathbf{C}} = (\hat{\mathbf{A}}^{\top}\hat{\mathbf{B}}\hat{\mathbf{C}}^{\top})^{\top} \quad \text{(D4)}$$

we can rewrite terms E1, D2, and F2, as follows.

Term E1:

$$-i\hbar^3\,\hat{\mathbf{s}}(\mathsf{M}\,\hat{\mathbf{f}})^{\top}d\hat{\mathbf{B}}_{\rm in}^{\ddagger} = -i\hbar^3\,\hat{\mathbf{s}}\,d\hat{\mathbf{B}}_{\rm in}^{\dagger}(\mathsf{M}\,\hat{\mathbf{f}}) = -i\hbar^3(d\hat{\mathbf{B}}_{\rm in}^{\ddagger}\hat{\mathbf{s}}^{\top})^{\top}(\mathsf{M}\,\hat{\mathbf{f}}) = -i\hbar^3\{d\hat{\mathbf{B}}_{\rm in}^{\dagger}[\hat{\mathbf{s}}\,(\mathsf{M}\,\hat{\mathbf{f}}\,)^{\top}]^{\top}\}^{\top}. \quad \text{(D5)}$$

Term E2:

$$i\hbar^3 d\hat{\mathbf{B}}_{\rm in}^{\dagger}(\mathsf{M}\,\hat{\mathbf{f}})\,\hat{\mathbf{s}} = i\hbar^3[d\hat{\mathbf{B}}_{\rm in}^{\dagger}(\mathsf{M}\,\hat{\mathbf{f}})\hat{\mathbf{s}}^{\top}]^{\top}. \quad \text{(D6)}$$

Term D2:

$$i\hbar^3 d\hat{\mathbf{B}}_{\rm in}^{\top}(\mathsf{M}^*\hat{\mathbf{f}})\,\hat{\mathbf{s}} = i\hbar^3(\mathsf{M}^*\,\hat{\mathbf{f}})^{\top}d\hat{\mathbf{B}}_{\rm in}\,\hat{\mathbf{s}} = i\hbar^3[(\mathsf{M}^*\,\hat{\mathbf{f}})^{\top}d\hat{\mathbf{B}}_{\rm in}\,\hat{\mathbf{s}}^{\top}]^{\top} = i\hbar^3[(\mathsf{M}^*\,\hat{\mathbf{f}})^{\top}\,(\hat{\mathbf{s}}\,d\hat{\mathbf{B}}_{\rm in}^{\top})^{\top}\,]^{\top} = i\hbar^3[(\mathsf{M}^*\hat{\mathbf{f}})\,\hat{\mathbf{s}}^{\top}]^{\top}d\hat{\mathbf{B}}_{\rm in}. \quad \text{(D7)}$$

Term F2:

$$i\hbar^4\,d\hat{\boldsymbol{v}}_{\rm in}^{\top}\,\hat{\mathbf{f}}\,\hat{\mathbf{s}} = i\hbar^4\,(\hat{\mathbf{f}}\,\hat{\mathbf{s}}^{\top})^{\top}d\hat{\boldsymbol{v}}_{\rm in}. \quad \text{(D8)}$$

Term I2:

$$i\hbar^3\,\hat{\mathbf{c}}^{\dagger}(\mathsf{M}\,\hat{\mathbf{f}})\,\hat{\mathbf{s}}\,dt = i\hbar^3[\hat{\mathbf{c}}^{\dagger}(\mathsf{M}\,\hat{\mathbf{f}})\,\hat{\mathbf{s}}^{\top}]^{\top}dt. \quad \text{(D9)}$$

Note that (D8) can be derived similarly to (D7). By moving $d\hat{\mathbf{B}}_{\rm in}$ to the position as shown in (B6) we can rewrite terms I1, G2, H2, and H3 as follows.

Term I1:

$$-i\hbar^2(\hat{\mathbf{c}}^\dagger d\hat{\mathbf{B}}_{\rm in})\hat{\mathbf{s}}(\hat{\mathbf{f}}^\top \mathsf{M}^\top d\hat{\mathbf{B}}_{\rm in}^\ddagger) = -i\hbar^3(\hat{\mathbf{c}}^\ddagger\hat{\mathbf{s}}^\top)^\top \mathsf{M}\hat{\mathbf{f}}\,dt = -i\hbar^3\{\hat{\mathbf{c}}^\dagger[\hat{\mathbf{s}}\,(\mathsf{M}\,\hat{\mathbf{f}})^\top]^\top\}^\top. \quad \text{(D10)}$$

Term G2:

$$i\hbar^2(d\hat{\mathbf{B}}_{\rm in}^\top \mathsf{M}^*\,\hat{\mathbf{f}})\hat{\mathbf{s}}(d\hat{\mathbf{B}}_{\rm in}^\dagger\hat{\mathbf{c}}) = i\hbar^3(\mathsf{M}^*\,\hat{\mathbf{f}}\,\hat{\mathbf{s}}^\top)^\top\hat{\mathbf{c}}\,dt. \quad \text{(D11)}$$

Term H2:

$$\hbar^2(d\hat{\mathbf{B}}_{\rm in}^\top \mathsf{M}^*\,\hat{\mathbf{f}})\hat{\mathbf{s}}(\hat{\mathbf{f}}^\top \mathsf{M}^\top d\hat{\mathbf{B}}_{\rm in}^\ddagger) = \hbar^3(\mathsf{M}^*\hat{\mathbf{f}}\,\hat{\mathbf{s}}^\top)^\top \mathsf{M}\,\hat{\mathbf{f}}\,dt. \quad \text{(D12)}$$

Term H3:

$$\hbar^4(d\hat{\boldsymbol{\upsilon}}_{\rm in}^\top\hat{\mathbf{f}})\hat{\mathbf{s}}(\hat{\mathbf{f}}^\top d\hat{\boldsymbol{\upsilon}}_{\rm in}) = \hbar^3(\mathsf{Z}\,\hat{\mathbf{f}}\,\hat{\mathbf{s}}^\top)^\top\hat{\mathbf{f}}\,dt. \quad \text{(D13)}$$

Note that (D13) can be obtained in similar fashion to (B4) and (B5) except that we have to use (18) instead of (5). The sum of terms H2 and H3 is then

$$\begin{aligned}
&\hbar^3(\mathsf{M}\,\hat{\mathbf{f}}\,\hat{\mathbf{s}}^\top)^\top \mathsf{M}^*\,\hat{\mathbf{f}}\,dt + \hbar^3(\mathsf{Z}\,\hat{\mathbf{f}}\,\hat{\mathbf{s}}^\top)^\top\hat{\mathbf{f}}\,dt\\
&= \hbar^3\{(\mathsf{M}\hat{\mathbf{f}}\,\hat{\mathbf{s}}^\top)^\top \mathsf{M}^*\hat{\mathbf{f}} + [(\hbar\,\mathrm{I}_R - \mathsf{M}^\dagger\mathsf{M})\hat{\mathbf{f}}\,\hat{\mathbf{s}}^\top]^\top\hat{\mathbf{f}}\}\,dt\\
&= \hbar^4(\hat{\mathbf{f}}\hat{\mathbf{s}}^\top)^\top\hat{\mathbf{f}}\,dt + \hbar^3\{(\mathsf{M}\hat{\mathbf{f}}\,\hat{\mathbf{s}}^\top)^\top\mathsf{M}^* - (\mathsf{M}^\dagger\mathsf{M}\,\hat{\mathbf{f}}\,\hat{\mathbf{s}}^\top)^\top\}\,\hat{\mathbf{f}}\,dt\\
&= \hbar^4(\hat{\mathbf{f}}\hat{\mathbf{s}}^\top)^\top\hat{\mathbf{f}}\,dt,
\end{aligned} \quad \text{(D14)}$$

where the last line follows from the fact that the terms proportional to $\hbar^3$ inside the braces cancel. This can be seen by noting that

$$(M\hat{\mathsf{A}})^\top = \hat{\mathsf{A}}^\top M^\top \quad \text{(D15)}$$

for any matrix M and any matrix operator $\hat{\mathsf{A}}$.

Substituting (D5)–(D11) and (D14) into (D2), and reassociating the terms as indicated, we arrive at

$$\begin{aligned}
\hbar\,d\hat{\mathbf{s}} &= \underbrace{i\,[\hat{H}_1,\hat{\mathbf{s}}]dt}_{\text{A terms}} - \underbrace{\lfloor\hat{\mathbf{s}},\hat{\mathbf{c}}^\dagger\rceil\big(\tfrac{1}{2}\,\hat{\mathbf{c}}\,dt + d\hat{\mathbf{B}}_{\rm in}\big)}_{\text{B + C terms}}\\
&\quad - \underbrace{\big\{\big(\tfrac{1}{2}\,\hat{\mathbf{c}}^\dagger dt + d\hat{\mathbf{B}}_{\rm in}^\dagger\big)\lfloor\hat{\mathbf{c}},\hat{\mathbf{s}}\rceil\big\}^\top}_{\text{B + C terms}} + \underbrace{\big[(\hat{\mathbf{f}}\,\hat{\mathbf{s}}^\top)^\top\hat{\mathbf{f}} - \tfrac{1}{2}\{\hat{\mathbf{f}}^\top\hat{\mathbf{f}},\hat{\mathbf{s}}\}\big]dt}_{\text{H terms}}\\
&\quad - \underbrace{i\,\lfloor\hat{\mathbf{s}},\mathsf{M}^*\,\hat{\mathbf{f}}\rceil(\hat{\mathbf{c}}\,dt + d\hat{\mathbf{B}}_{\rm in})}_{\text{D + G terms}} + \underbrace{i\,\big\{\big(\hat{\mathbf{c}}^\dagger dt + d\hat{\mathbf{B}}_{\rm in}^\dagger\big)\lfloor\mathsf{M}\,\hat{\mathbf{f}},\hat{\mathbf{s}}\rceil\big\}^\top}_{\text{E + I terms}}\\
&\quad - \underbrace{i\hbar\lfloor\hat{\mathbf{s}},\hat{\mathbf{f}}\rceil d\hat{\boldsymbol{\upsilon}}_{\rm in}}_{\text{F terms}}.
\end{aligned} \quad \text{(D16)}$$

Note that we have used (B10) and (B9) to write the Heisenberg equation in a form that resembles (4.21) from Ref. [26]. As we noted in Appendix B, this assumes that $d\hat{\mathbf{B}}_{\rm in}$ and $d\hat{\mathbf{B}}_{\rm in}^\ddagger$ commutes with $\hat{\mathbf{s}}$ which allows us to rewrite $[\hat{\mathbf{c}}^\dagger d\hat{\mathbf{B}}_{\rm in} - d\hat{\mathbf{B}}_{\rm in}^\dagger\hat{\mathbf{c}},\hat{\mathbf{s}}]$ as shown in (B10). Substituting the definition (20) of $d\hat{\boldsymbol{\upsilon}}_{\rm in}$ in terms of the independent Wiener increments $d\hat{\mathbf{U}}_{\rm in}$ into (D16) and nomal ordering gives (56). However, to verify that (56) is a valid Itô equation [i.e., (45)], one may prefer to use (D16) as it is more compact.

APPENDIX E :DERIVATION OF EQ. (43)

We wish to derive (43) from (42). For convenience we restate (42) here

$$\mathcal{L}\rho = -i[\hat{H}_1,\rho] + \mathcal{D}[\hat{\mathbf{c}}]\rho + \hbar\,\mathcal{D}[\hat{\mathbf{f}}]\rho - i\,\lceil\hat{\mathbf{f}},\mathsf{M}^\dagger\hat{\mathbf{c}}\,\rho + \rho\,\mathsf{M}^\top\hat{\mathbf{c}}^\ddagger\rfloor. \quad \text{(E1)}$$

Consider, first, the two terms $\mathcal{D}[\hat{\mathbf{c}}]\rho$ and $\lceil\hat{\mathbf{f}},\mathsf{M}^\top\hat{\mathbf{c}}\rho + \rho\,\mathsf{M}^\dagger\hat{\mathbf{c}}^\ddagger\rfloor$ and expanding the scalar-operator bracket, we obtain:

$$\begin{aligned}
\mathcal{D}[\hat{\mathbf{c}}]\rho - i\,\lceil\hat{\mathbf{f}},\mathsf{M}^\dagger\hat{\mathbf{c}}\rho + \rho\,\mathsf{M}^\top\hat{\mathbf{c}}^\ddagger\rfloor &= \hat{\mathbf{c}}^\top\rho\,\hat{\mathbf{c}}^\ddagger - \tfrac{1}{2}\,\rho\,\hat{\mathbf{c}}^\dagger\hat{\mathbf{c}} - \tfrac{1}{2}\,\hat{\mathbf{c}}^\dagger\hat{\mathbf{c}}\,\rho\\
&\quad + i\,\hat{\mathbf{c}}^\top\rho\,\mathsf{M}^*\,\hat{\mathbf{f}} - i\,\hat{\mathbf{f}}^\top\mathsf{M}^\top\rho\,\hat{\mathbf{c}}^\ddagger\\
&\quad + i\rho\,\hat{\mathbf{c}}^\dagger\mathsf{M}\,\hat{\mathbf{f}} - i\,\hat{\mathbf{f}}^\top\mathsf{M}^\dagger\hat{\mathbf{c}}\,\rho.
\end{aligned} \quad \text{(E2)}$$

We can regroup terms as follows:

$$\begin{aligned}
&\mathcal{D}[\hat{\mathbf{c}}]\rho - i\,\lceil\hat{\mathbf{f}},\mathsf{M}^\dagger\hat{\mathbf{c}}\rho + \rho\,\mathsf{M}^\top\hat{\mathbf{c}}^\ddagger\rfloor\\
&= \frac{1}{2}\,\hat{\mathbf{c}}^\top\rho(\hat{\mathbf{c}}^\ddagger + i\,\mathsf{M}^*\,\hat{\mathbf{f}}) + \frac{1}{2}(\hat{\mathbf{c}}^\top - i\,\hat{\mathbf{f}}^\top\mathsf{M}^\top)\rho\,\hat{\mathbf{c}}^\ddagger\\
&\quad - \frac{1}{2}\,\rho\,\hat{\mathbf{c}}^\dagger(\hat{\mathbf{c}} - i\,\mathsf{M}\hat{\mathbf{f}}) - \frac{1}{2}(\hat{\mathbf{c}}^\dagger + i\,\hat{\mathbf{f}}^\top\mathsf{M}^\dagger)\hat{\mathbf{c}}\,\rho\\
&\quad + \frac{i}{2}\,\hat{\mathbf{c}}^\top\rho\,\mathsf{M}^*\,\hat{\mathbf{f}} - \frac{i}{2}\,\hat{\mathbf{f}}^\top\mathsf{M}^\top\rho\,\hat{\mathbf{c}}^\ddagger + \frac{i}{2}\,\rho\,\hat{\mathbf{c}}^\dagger\mathsf{M}\hat{\mathbf{f}} - \frac{i}{2}\,\hat{\mathbf{f}}^\top\mathsf{M}^\dagger\hat{\mathbf{c}}\,\rho.
\end{aligned} \quad \text{(E3)}$$

Guided by the terms with parentheses in (E3) we add and subtract $\mathcal{D}[\mathsf{M}\,\hat{\mathbf{f}}]\rho$ to the last line in (E3). Using the identity

$$\frac{i}{2}\,\rho\,\hat{\mathbf{c}}^\dagger\mathsf{M}\hat{\mathbf{f}} - \frac{i}{2}\,\hat{\mathbf{f}}^\top\mathsf{M}^\dagger\,\hat{\mathbf{c}}\,\rho = \frac{i}{2}\,\hat{\mathbf{c}}^\dagger\mathsf{M}\hat{\mathbf{f}}\rho - \frac{i}{2}\,\rho\,\hat{\mathbf{f}}^\top\mathsf{M}^\dagger\hat{\mathbf{c}} - \frac{i}{2}\,[\hat{\mathbf{f}}^\top\mathsf{M}^\dagger\hat{\mathbf{c}} + \hat{\mathbf{c}}^\dagger\mathsf{M}\,\hat{\mathbf{f}},\rho], \quad \text{(E4)}$$

the last line of (E3) can be written as

$$\begin{aligned}
&\frac{i}{2}\,\hat{\mathbf{c}}^\top\rho\,\mathsf{M}^*\,\hat{\mathbf{f}} - \frac{i}{2}\,\hat{\mathbf{f}}^\top\mathsf{M}^\top\rho\,\hat{\mathbf{c}}^\ddagger + \frac{i}{2}\,\rho\,\hat{\mathbf{c}}^\dagger\mathsf{M}\hat{\mathbf{f}} - \frac{i}{2}\,\hat{\mathbf{f}}^\top\mathsf{M}^\dagger\hat{\mathbf{c}}\,\rho\\
&= -\frac{i}{2}[\hat{\mathbf{f}}^\top\mathsf{M}^\dagger\hat{\mathbf{c}} + \hat{\mathbf{c}}^\dagger\mathsf{M}\,\hat{\mathbf{f}},\rho] + \mathcal{D}[\mathsf{M}\,\hat{\mathbf{f}}]\rho - \mathcal{D}[\mathsf{M}\,\hat{\mathbf{f}}]\rho\\
&\quad + \frac{i}{2}\,\hat{\mathbf{c}}^\top\,\rho\,\mathsf{M}^*\,\hat{\mathbf{f}} - \frac{i}{2}\,\hat{\mathbf{f}}^\top\mathsf{M}^\top\rho\,\hat{\mathbf{c}}^\ddagger + \frac{i}{2}\,\hat{\mathbf{c}}^\dagger\mathsf{M}\hat{\mathbf{f}}\rho - \frac{i}{2}\,\rho\,\hat{\mathbf{f}}^\top\mathsf{M}^\dagger\hat{\mathbf{c}}\\
&= -\frac{i}{2}[\hat{\mathbf{f}}^\top\mathsf{M}^\dagger\hat{\mathbf{c}} + \hat{\mathbf{c}}^\dagger\mathsf{M}\,\hat{\mathbf{f}},\rho] - \mathcal{D}[\mathsf{M}\,\hat{\mathbf{f}}]\rho\\
&\quad + \frac{i}{2}(\hat{\mathbf{c}}^\top - i\,\hat{\mathbf{f}}^\top\mathsf{M}^\top)\rho\,\mathsf{M}^*\,\hat{\mathbf{f}} - \frac{i}{2}\,\hat{\mathbf{f}}^\top\mathsf{M}^\top\rho(\hat{\mathbf{c}}^\ddagger + i\,\mathsf{M}^*\,\hat{\mathbf{f}})\\
&\quad + \frac{i}{2}(\hat{\mathbf{c}}^\dagger + i\,\hat{\mathbf{f}}^\top\mathsf{M}^\dagger)\mathsf{M}\hat{\mathbf{f}}\rho - \frac{i}{2}\,\rho\,\hat{\mathbf{f}}^\top\mathsf{M}^\dagger(\hat{\mathbf{c}} - i\,\mathsf{M}\hat{\mathbf{f}}).
\end{aligned} \quad \text{(E5)}$$

Substituting this back into (E3) and collecting like terms we get

$$\begin{aligned}
&\mathcal{D}[\hat{\mathbf{c}}]\rho - i\,\lceil\hat{\mathbf{f}},\mathsf{M}^\dagger\hat{\mathbf{c}}\rho + \rho\,\mathsf{M}^\top\hat{\mathbf{c}}^\ddagger\rfloor\\
&= -\frac{i}{2}[\hat{\mathbf{f}}^\top\mathsf{M}^\dagger\hat{\mathbf{c}} + \hat{\mathbf{c}}^\dagger\mathsf{M}\,\hat{\mathbf{f}},\rho] + \mathcal{D}[\hat{\mathbf{c}} - i\mathsf{M}\,\hat{\mathbf{f}}]\rho\\
&\quad - \mathcal{D}[\mathsf{M}\,\hat{\mathbf{f}}]\rho.
\end{aligned} \quad \text{(E6)}$$

Substituting this back into $\mathcal{L}\rho$ we arrive at

$$\begin{aligned}
\mathcal{L}\rho &= -i\big[\hat{H}_1 + \tfrac{1}{2}(\hat{\mathbf{f}}^\top\mathsf{M}^\dagger\hat{\mathbf{c}} + \hat{\mathbf{c}}^\dagger\mathsf{M}\,\hat{\mathbf{f}}),\rho\big]\\
&\quad + \mathcal{D}[\hat{\mathbf{c}} - i\mathsf{M}\,\hat{\mathbf{f}}]\rho + \hbar\,\mathcal{D}[\hat{\mathbf{f}}]\rho - \mathcal{D}[\mathsf{M}\,\hat{\mathbf{f}}]\rho.
\end{aligned} \quad \text{(E7)}$$

The final two terms can be written as

$$\begin{aligned}\hbar\mathcal{D}[\hat{\mathbf{f}}]\rho-\mathcal{D}[\mathsf{M}\hat{\mathbf{f}}]\rho &= \hbar\,\hat{\mathbf{f}}^{\top}\rho\,\hat{\mathbf{f}}-\frac{1}{2}\hat{\mathbf{f}}^{\top}\mathsf{M}^{\top}\rho\,\mathsf{M}^{*}\hat{\mathbf{f}}\\ &\quad-\frac{\hbar}{2}\hat{\mathbf{f}}^{\top}\hat{\mathbf{f}}\rho+\frac{1}{2}\hat{\mathbf{f}}^{\top}\mathsf{M}^{\dagger}\mathsf{M}\hat{\mathbf{f}}\rho\\ &\quad-\frac{\hbar}{2}\rho\,\hat{\mathbf{f}}^{\top}\hat{\mathbf{f}}+\frac{1}{2}\rho\,\hat{\mathbf{f}}^{\top}\mathsf{M}^{\dagger}\mathsf{M}\hat{\mathbf{f}}\\ &=\hat{\mathbf{f}}^{\top}\rho\mathsf{Z}^{*}\hat{\mathbf{f}}-\frac{1}{2}\hat{\mathbf{f}}^{\top}\mathsf{Z}\hat{\mathbf{f}}\rho-\frac{1}{2}\rho\,\hat{\mathbf{f}}^{\top}\mathsf{Z}\hat{\mathbf{f}}.\end{aligned}\tag{E8}$$

Recall that $\mathsf{Z}=\hbar\,\mathsf{I}_R-\mathsf{M}^{\dagger}\mathsf{M}$, which was defined under (18). Since $\mathsf{Z}\geqslant 0$, there exists a B such that $\mathsf{Z}=\mathsf{B}^{\dagger}\mathsf{B}$. Therefore, we are free to write

$$\begin{aligned}&\hbar\mathcal{D}[\hat{\mathbf{f}}]\rho-\mathcal{D}[\mathsf{M}\hat{\mathbf{f}}]\rho\\ &\quad=\hat{\mathbf{f}}^{\top}\mathsf{B}^{\top}\rho\,\mathsf{B}^{*}\hat{\mathbf{f}}-\tfrac{1}{2}\hat{\mathbf{f}}^{\top}(\mathsf{B}^{\dagger}\mathsf{B})\hat{\mathbf{f}}\rho-\tfrac{1}{2}\rho\,\hat{\mathbf{f}}^{\top}(\mathsf{B}^{\dagger}\mathsf{B})\hat{\mathbf{f}}\\ &\quad=\mathcal{D}[\mathsf{B}\hat{\mathbf{f}}]\rho.\end{aligned}\tag{E9}$$

The final master equation in Lindblad form is therefore (leaving B as a general matrix square root)

$$\begin{aligned}\hbar\,\dot{\rho}= &-i\big[\hat{H}_1+\tfrac{1}{2}(\hat{\mathbf{f}}^{\top}\mathsf{M}^{\dagger}\hat{\mathbf{c}}+\hat{\mathbf{c}}^{\dagger}\mathsf{M}\hat{\mathbf{f}}),\rho\big]\\ &+\mathcal{D}[\hat{\mathbf{c}}-i\mathsf{M}\hat{\mathbf{f}}]\rho+\mathcal{D}[\mathsf{B}\hat{\mathbf{f}}]\rho.\end{aligned}\tag{E10}$$

APPENDIX F :DERIVATION OF EQ. (103)

For clarity we will label each term in (96):

$$\begin{aligned}\hbar^2\langle\hat{\mathbf{y}}_2(t)\,\hat{\mathbf{y}}_2^{\top}(t+\tau)\rangle = &\underbrace{\langle\mathsf{M}^{\dagger}\,\hat{\mathbf{b}}_{\rm outt}(t)\,\hat{\mathbf{b}}_{\rm outt}^{\top}(t+\tau)\,\mathsf{M}^{*}\rangle}_{\text{term A}}+\underbrace{\langle\mathsf{M}^{\dagger}\,\hat{\mathbf{b}}_{\rm outt}(t)\,\hat{\mathbf{b}}_{\rm outt}^{\dagger}(t+\tau)\,\mathsf{M}\rangle}_{\text{term B}}+\underbrace{\langle\mathsf{M}^{\dagger}\,\hat{\mathbf{b}}_{\rm outt}(t)\,\hbar\,\hat{\boldsymbol{\zeta}}_{\rm outt}^{\top}(t+\tau)\rangle}_{\text{term E}}\\ &+\underbrace{\langle\mathsf{M}^{\top}\,\hat{\mathbf{b}}_{\rm outt}^{\ddagger}(t)\,\hat{\mathbf{b}}_{\rm outt}^{\top}(t+\tau)\,\mathsf{M}^{*}\rangle}_{\text{term C}}+\underbrace{\langle\mathsf{M}^{\top}\,\hat{\mathbf{b}}_{\rm outt}^{\ddagger}(t)\,\hat{\mathbf{b}}_{\rm outt}^{\dagger}(t+\tau)\,\mathsf{M}\rangle}_{\text{term D}}+\underbrace{\langle\mathsf{M}^{\top}\,\hat{\mathbf{b}}_{\rm outt}^{\ddagger}(t)\,\hbar\,\hat{\boldsymbol{\zeta}}_{\rm outt}^{\top}(t+\tau)\rangle}_{\text{term F}}\\ &+\underbrace{\langle\hbar\,\hat{\boldsymbol{\zeta}}_{\rm outt}(t)\,\hat{\mathbf{b}}_{\rm outt}^{\top}(t+\tau)\,\mathsf{M}^{*}\rangle}_{\text{term G}}+\underbrace{\langle\hbar\,\hat{\boldsymbol{\zeta}}_{\rm outt}(t)\,\hat{\mathbf{b}}_{\rm outt}^{\dagger}(t+\tau)\,\mathsf{M}\rangle}_{\text{term H}}+\underbrace{\langle\hbar\,\hat{\boldsymbol{\zeta}}_{\rm outt}(t)\,\hbar\,\hat{\boldsymbol{\zeta}}_{\rm outt}^{\top}(t+\tau)\rangle}_{\text{term I}}.\end{aligned}\tag{F1}$$

For convenience we use "cw" to denote "cancels with." Normal and time ordering of each term leads to

Term A:

$$\begin{aligned}\langle\mathsf{M}^{\dagger}\,\hat{\mathbf{b}}_{\rm outt}(t)\,\hat{\mathbf{b}}_{\rm outt}^{\top}(t+\tau)\,\mathsf{M}^{*}\rangle = &\underbrace{\langle\mathsf{M}^{\dagger}\hat{\mathbf{c}}(t+\tau)\,\hat{\mathbf{c}}^{\top}(t)\,\mathsf{M}^{*}\rangle^{\top}}_{\text{A1}}-\underbrace{\langle i\,\mathsf{M}^{\dagger}\hat{\mathbf{c}}(t+\tau)\,\hat{\mathbf{f}}^{\top}(t)(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle^{\top}}_{\text{A2 (cw G2)}}-\underbrace{\langle i\,\mathsf{M}^{\dagger}\mathsf{M}\,\hat{\mathbf{f}}(t+\tau)\,\hat{\mathbf{c}}^{\top}(t)\mathsf{M}^{*}\rangle^{\top}}_{\text{A3 (cw E1)}}\\ &-\underbrace{\langle\mathsf{M}^{\dagger}\mathsf{M}\,\hat{\mathbf{f}}(t+\tau)\,\hat{\mathbf{f}}^{\top}(t)(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle^{\top}}_{\text{A4 (cw G4)}}.\end{aligned}\tag{F2}$$

Term B:

$$\begin{aligned}\langle\mathsf{M}^{\dagger}\,\hat{\mathbf{b}}_{\rm outt}(t)\,\hat{\mathbf{b}}_{\rm outt}^{\dagger}(t+\tau)\,\mathsf{M}\rangle = &\underbrace{\langle\mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}(t+\tau)\,\hat{\mathbf{c}}^{\top}(t)\,\mathsf{M}^{*}\rangle^{\top}}_{\text{B1}}-\underbrace{\langle i\,\mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}(t+\tau)\,\hat{\mathbf{f}}^{\top}(t)\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle^{\top}}_{\text{B2 (cw H2)}}+\underbrace{\hbar^2\,\mathsf{M}^{\dagger}\mathsf{M}\,\delta(\tau)}_{\text{B5 (cw I1)}}\\ &+\underbrace{\langle i\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\,\hat{\mathbf{f}}(t+\tau)\,\hat{\mathbf{c}}^{\top}(t)\,\mathsf{M}^{*}\rangle^{\top}}_{\text{B3 (cw E3)}}+\underbrace{\langle(\mathsf{M}^{\dagger}\mathsf{M})^{*}\,\hat{\mathbf{f}}(t+\tau)\,\hat{\mathbf{f}}(t)\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle^{\top}}_{\text{B4 (cw E4)}}.\end{aligned}\tag{F3}$$

Term C:

$$\begin{aligned}\langle\mathsf{M}^{\top}\,\hat{\mathbf{b}}_{\rm outt}^{\ddagger}(t)\,\hat{\mathbf{b}}_{\rm outt}^{\top}(t+\tau)\,\mathsf{M}^{*}\rangle = &\underbrace{\langle\mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}(t)\,\hat{\mathbf{c}}^{\top}(t+\tau)\,\mathsf{M}^{*}\rangle}_{\text{C1}}-\underbrace{\langle i\,\mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}(t)\,\hat{\mathbf{f}}^{\top}(t+\tau)\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle}_{\text{C2 (cw F1)}}+\underbrace{\langle i\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\,\hat{\mathbf{f}}(t)\,\hat{\mathbf{c}}^{\top}(t+\tau)\,\mathsf{M}^{*}\rangle}_{\text{C3 (cw G6)}}\\ &+\underbrace{\langle(\mathsf{M}^{\dagger}\mathsf{M})^{*}\,\hat{\mathbf{f}}(t)\,\hat{\mathbf{f}}^{\top}(t+\tau)\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle}_{\text{C4 (cw F2)}}.\end{aligned}\tag{F4}$$

Term D:

$$\begin{aligned}\langle\mathsf{M}^{\top}\,\hat{\mathbf{b}}_{\rm outt}^{\ddagger}(t)\,\hat{\mathbf{b}}_{\rm outt}^{\dagger}(t+\tau)\,\mathsf{M}\rangle = &\underbrace{\langle\mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}(t)\,\hat{\mathbf{c}}^{\dagger}(t+\tau)\,\mathsf{M}\rangle}_{\text{D1}}+\underbrace{\langle i\,\mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}(t)\,\hat{\mathbf{f}}^{\top}(t+\tau)\,\mathsf{M}^{\dagger}\mathsf{M}\rangle}_{\text{D2 (cw F3)}}+\underbrace{\langle i\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\,\hat{\mathbf{f}}(t)\,\hat{\mathbf{c}}^{\dagger}(t+\tau)\,\mathsf{M}\rangle}_{\text{D3 (cw H6)}}\\ &-\underbrace{\langle(\mathsf{M}^{\dagger}\mathsf{M})^{*}\,\hat{\mathbf{f}}(t)\,\hat{\mathbf{f}}^{\top}(t+\tau)\,\mathsf{M}^{\dagger}\mathsf{M}\rangle}_{\text{D4 (cw F4)}}.\end{aligned}\tag{F5}$$

Term E:

$$\begin{aligned}\langle\mathsf{M}^{\dagger}\,\hat{\mathbf{b}}_{\rm outt}(t)\,\hbar\,\hat{\boldsymbol{\zeta}}_{\rm outt}^{\top}(t+\tau)\rangle = &\underbrace{\langle i\,\mathsf{M}^{\dagger}\mathsf{M}\,\hat{\mathbf{f}}(t+\tau)\,\hat{\mathbf{c}}^{\top}(t)\,\mathsf{M}^{*}\rangle^{\top}}_{\text{E1 (cw A3)}}+\underbrace{\langle\mathsf{M}^{\dagger}\mathsf{M}\,\hat{\mathbf{f}}(t+\tau)\,\hat{\mathbf{f}}^{\top}(t)\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle^{\top}}_{\text{E2 (cw I3)}}-\underbrace{\langle i\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\,\hat{\mathbf{f}}(t+\tau)\,\hat{\mathbf{c}}^{\top}(t)\,\mathsf{M}^{*}\rangle^{\top}}_{\text{E3 (cw B3)}}\\ &-\underbrace{\langle(\mathsf{M}^{\dagger}\mathsf{M})^{*}\hat{\mathbf{f}}(t+\tau)\,\hat{\mathbf{f}}^{\top}(t)(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle^{\top}}_{\text{E4 (cw B4)}}.\end{aligned}\tag{F6}$$

Term F:

$$\begin{aligned}\langle\mathsf{M}^{\top}\,\hat{\mathbf{b}}_{\rm outt}^{\ddagger}(t)\,\hbar\,\hat{\boldsymbol{\zeta}}_{\rm outt}^{\top}(t+\tau)\rangle = &\underbrace{\langle i\,\mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}(t)\,\hat{\mathbf{f}}^{\top}(t+\tau)\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle}_{\text{F1 (cw C2)}}-\underbrace{\langle(\mathsf{M}^{\dagger}\mathsf{M})^{*}\,\hat{\mathbf{f}}(t)\,\hat{\mathbf{f}}^{\top}(t+\tau)\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle}_{\text{F2 (cw C4)}}-\underbrace{\langle i\,\mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}(t)\,\hat{\mathbf{f}}^{\top}(t+\tau)\,\mathsf{M}^{\dagger}\mathsf{M}\rangle}_{\text{F3 (cw D2)}}\\ &+\underbrace{\langle(\mathsf{M}^{\dagger}\mathsf{M})^{*}\,\hat{\mathbf{f}}(t)\,\hat{\mathbf{f}}^{\top}(t+\tau)\,\mathsf{M}^{\dagger}\mathsf{M}\rangle}_{\text{F4 (cw D4)}}.\end{aligned}\tag{F7}$$

Term G:

$$\begin{aligned}\langle \hbar\, \hat{\boldsymbol{\xi}}_{\rm outt}(t)\, \hat{\mathbf{b}}_{\rm outt}^{\top}(t+\tau)\, \mathsf{M}^{*}\rangle =\ & \underbrace{\langle -i\hbar\, \mathsf{M}^{\dagger}\hat{\mathbf{c}}(t+\tau)\hat{\mathbf{f}}^{\top}(t)\rangle^{\top}}_{\text{G1}} + \underbrace{\langle i\, \mathsf{M}^{\dagger}\hat{\mathbf{c}}(t+\tau)\,\hat{\mathbf{f}}^{\top}(t)(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle^{\top}}_{\text{G2 (cw A2)}} - \underbrace{\langle \hbar\, \mathsf{M}^{\dagger}\mathsf{M}\, \hat{\mathbf{f}}(t+\tau)\hat{\mathbf{f}}^{\top}(t)\rangle^{\top}}_{\text{G3 (cw I2)}} \\ & + \underbrace{\langle \mathsf{M}^{\dagger}\mathsf{M}\, \hat{\mathbf{f}}(t+\tau)\hat{\mathbf{f}}^{\top}(t)(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle^{\top}}_{\text{G4 (cw A4)}} + \underbrace{\langle i\hbar\, \hat{\mathbf{f}}(t)\, \hat{\mathbf{c}}^{\top}(t+\tau)\, \mathsf{M}^{*}\rangle}_{\text{G5}} - \underbrace{\langle i\, (\mathsf{M}^{\dagger}\mathsf{M})^{*}\, \hat{\mathbf{f}}(t)\, \hat{\mathbf{c}}^{\top}(t+\tau)\, \mathsf{M}^{*}\rangle}_{\text{G6 (cw C3)}} \\ & + \underbrace{\langle \hbar\, \hat{\mathbf{f}}(t)\, \hat{\mathbf{f}}^{\top}(t+\tau)\, (\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle}_{\text{G7 (cw I6)}} - \underbrace{\langle (\mathsf{M}^{\dagger}\mathsf{M})^{*}\, \hat{\mathbf{f}}(t)\, \hat{\mathbf{f}}^{\top}(t+\tau)(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle}_{\text{G8 (cw I7)}}. \end{aligned} \tag{F8}$$

Term H:

$$\begin{aligned}\langle \hbar\, \hat{\boldsymbol{\xi}}_{\rm outt}(t)\, \hat{\mathbf{b}}_{\rm outt}^{\dagger}(t+\tau)\, \mathsf{M}\rangle =\ & \underbrace{\langle -i\hbar\, \mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}(t+\tau)\,\hat{\mathbf{f}}^{\top}(t)\rangle^{\top}}_{\text{H1}} + \underbrace{\langle i\, \mathsf{M}^{\top}\hat{\mathbf{c}}^{\ddagger}(t+\tau)\,\hat{\mathbf{f}}^{\top}(t+\tau)\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle^{\top}}_{\text{H2 (cw B2)}} + \underbrace{\langle \hbar\, (\mathsf{M}^{\dagger}\mathsf{M})^{*}\, \hat{\mathbf{f}}(t+\tau)\,\hat{\mathbf{f}}^{\top}(t)\rangle^{\top}}_{\text{H3 (cw I4)}} \\ & - \underbrace{\langle (\mathsf{M}^{\dagger}\mathsf{M})^{*}\, \hat{\mathbf{f}}(t+\tau)\,\hat{\mathbf{f}}^{\top}(t)\,(\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle^{\top}}_{\text{H4 (cw I5)}} + \underbrace{\langle i\hbar\, \hat{\mathbf{f}}(t)\, \hat{\mathbf{c}}^{\dagger}(t+\tau)\mathsf{M}\rangle}_{\text{H5}} - \underbrace{\langle i\, (\mathsf{M}^{\dagger}\mathsf{M})^{*}\, \hat{\mathbf{f}}(t)\, \hat{\mathbf{c}}^{\dagger}(t+\tau)\, \mathsf{M}\rangle}_{\text{H6 (cw D3)}} \\ & - \underbrace{\langle \hbar\, \hat{\mathbf{f}}(t)\, \hat{\mathbf{f}}^{\top}(t+\tau)\, \mathsf{M}^{\dagger}\mathsf{M}\rangle}_{\text{H7 (cw I8)}} + \underbrace{\langle (\mathsf{M}^{\dagger}\mathsf{M})^{*}\, \hat{\mathbf{f}}(t)\, \hat{\mathbf{f}}^{\top}(t+\tau)\, \mathsf{M}^{\dagger}\mathsf{M}\rangle}_{\text{H8 (cw I9)}}. \end{aligned} \tag{F9}$$

Term I:

$$\begin{aligned}\langle \hbar\, \hat{\boldsymbol{\xi}}_{\rm outt}(t)\, \hbar\, \hat{\boldsymbol{\xi}}_{\rm outt}^{\top}(t+\tau)\rangle =\ & \hbar^{2}\, \mathrm{I}_{R}\, \delta(\tau) - \underbrace{\hbar\, \mathsf{M}^{\dagger}\mathsf{M}\, \delta(\tau)}_{\text{I1 (cw B5)}} + \underbrace{\langle \hbar\, \mathsf{M}^{\dagger}\mathsf{M}\, \hat{\mathbf{f}}(t+\tau)\, \hat{\mathbf{f}}^{\top}(t)\rangle^{\top}}_{\text{I2 (cw G3)}} - \underbrace{\langle \mathsf{M}^{\dagger}\mathsf{M}\, \hat{\mathbf{f}}(t+\tau)\, \hat{\mathbf{f}}^{\top}(t)\, (\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle^{\top}}_{\text{I3 (cw E2)}} \\ & - \underbrace{\langle \hbar(\mathsf{M}^{\dagger}\mathsf{M})^{*}\, \hat{\mathbf{f}}(t+\tau)\, \hat{\mathbf{f}}^{\top}(t)\rangle^{\top}}_{\text{I4 (cw H3)}} + \underbrace{\langle (\mathsf{M}^{\dagger}\mathsf{M})^{*}\hat{\mathbf{f}}(t+\tau)\, \hat{\mathbf{f}}^{\top}(t)\, (\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle^{\top}}_{\text{I5 (cw H4)}} \\ & - \underbrace{\langle \hbar\, \hat{\mathbf{f}}(t)\, \hat{\mathbf{f}}^{\top}(t+\tau)\, (\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle}_{\text{I6 (cw G7)}} + \underbrace{\langle (\mathsf{M}^{\dagger}\mathsf{M})^{*}\, \hat{\mathbf{f}}(t)\, \hat{\mathbf{f}}^{\top}(t+\tau)\, (\mathsf{M}^{\dagger}\mathsf{M})^{*}\rangle}_{\text{I7 (cw G8)}} \\ & + \underbrace{\langle \hbar\, \hat{\mathbf{f}}(t)\, \hat{\mathbf{f}}^{\top}(t+\tau)\, \mathsf{M}^{\dagger}\mathsf{M}\rangle}_{\text{I8 (cw H7)}} + \underbrace{\langle (\mathsf{M}^{\dagger}\mathsf{M})^{*}\, \hat{\mathbf{f}}(t)\, \hat{\mathbf{f}}^{\top}(t+\tau)\, \mathsf{M}^{\dagger}\mathsf{M}\rangle}_{\text{I9 (cw H8)}}. \end{aligned} \tag{F10}$$

The remaining terms are A1, B1, C1, D1, G1, G5, H1, H5, and the $\hbar^{2}\, \mathrm{I}_{R}\, \delta(\tau)$ in term I. Adding these and collecting like terms we arrive at (103).

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