Salpeter equation and probability current in the relativistic Hamiltonian quantum mechanics

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The probability current for a quantum spinless relativistic particle is introduced based on the Hamiltonian dynamics approach utilizing the Salpeter equation as an alternative for the Klein-Gordon equation. The correctness of the presented formalism is illustrated by examples of exact solutions to the Salpeter equation including the new ones introduced in this work.

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I. INTRODUCTION

The problem that arose in the very early days of quantum mechanics was finding the relativistic counterpart of the Schrödinger equation. The most popular choices of the relativistic quantum mechanical equations for spinless massive particle are the Klein-Gordon equation [1–4] and less familiar relativistic Schrödinger equation, which is usually referred to as the spinless Salpeter equation [5–23]. The latter can be regarded as the "square root" of the Klein-Gordon equation and is based on the approach that is sometimes referred to as the relativistic Hamiltonian dynamics [24,25].

The advantage of the Klein-Gordon equation is that it is manifestly covariant. Its well-known flaw related to the fact that this equation is of second order in the time derivative is the problem with the probabilistic interpretation, namely, the time component of the probability four-current, which in the nonrelativistic case should coincide with the probability density, can be negative. As a matter of fact, as the second-order equation the Klein-Gordon equation can be cast into a system of first-order equations, which can be represented in matrix-form equation. An example is the Kemmer equation [26,27] and the two-dimensional system discussed by Feshbach and Villars [28]. Nevertheless, the reduction of this kind does not solve problems with the Klein-Gordon equation and generate new ones. Furthermore, the appearance of both signs of energies for the solutions to the Klein-Gordon equation leads to the occurrence of the "*Zitterbewegung*" and the Klein paradox. The *Zitterbewegung* or trembling motion, i.e., rapidly oscillatory motion whose amplitude and period are of order $\hbar/mc$ and $\hbar/mc^2$, respectively, is the result of the interference between positive and negative energy states. The Klein paradox relies on possibility of transmission to states with negative kinetic energy in the electrostatic steplike potentials. To circumvent the problem one usually suggests the creation of the particle-antiparticle pairs; that is, the need is indicated for second quantization as with the Pauli-Weisskopf approach [29], where the probability current is reinterpreted as a charge current. Finally, the scalar product in the space of solutions of the Klein-Gordon equation, although Lorentz covariant, is not positive definite, which causes serious problems. Therefore, severe difficulties arise with the physical interpretation of the Klein-Gordon equation, and it is widely believed that based on this equation one cannot construct consistent one-particle relativistic quantum mechanics.

The disadvantage of the Salpeter equation is that it is not manifestly covariant. Another problem that is sometimes indicated is the nonlocality of the relativistic Hamiltonian, which is the pseudodifferential operator. However, as was pointed out by Lämmerzahl [22], the nonlocality of the Salpeter equation does not disturb the light cone structure. The macrocausality of the second quantized version of this theory was also reported therein. In addition, it has been demonstrated by Foldy [6] that the space $L^2(\mathbb{R}^3,d^3\boldsymbol{x})$ of solutions to the Salpeter equation is invariant under the Lorentz group transformations, which is a direct consequence of a unitary relationship between $L^2(\mathbb{R}^3,d^3\boldsymbol{x})$ and the space of the unitary irreducible representation of the inhomogeneous Lorentz group [25]. On the other hand, it is clear that nonlocal pseudodifferential equations such as the Salpeter equation are much more complicated than the local differential ones such as the Klein-Gordon equation. In spite of difficulties, the great advantage of the Salpeter equation is that it possesses solutions of positive energies only, so we have no problems with paradoxes mentioned above occurring in the case of the Klein-Gordon equation. We also point out that agreement of predictions of the spinless Salpeter equation with the experimental spectrum of mesonic atoms is as good as in the case of the Klein-Gordon equation [30]. Moreover, the possibility of probabilistic interpretation in the quantum case as well as clear classical physical content of the Salpeter equation was the motivation for its wide usage in the phenomenological description of the quark-antiquark-gluon system as a hadron model [31,32].

In this work, after discussion of the relationship of the Salpeter equation with the corresponding integro-differential equation, we introduce and examine the probability current derived from the spinless Salpeter equation. We show that such current has all good properties of its nonrelativistic counterpart. In particular, our analysis shows that the nonlocality of the Salpeter equation does not disturb causal propagation of particles. The theory is illustrated by concrete examples of exact solutions of the Salpeter equation.

II. THE SALPETER EQUATION

This section is devoted to the discussion of the basic facts about the Salpeter equation. The Hamiltonian of a relativistic classical particle subject to the potential $V(\boldsymbol{x})$ is

$$H = \sqrt{c^2\boldsymbol{p}^2 + m^2c^4} + V(\boldsymbol{x}), \tag{2.1}$$

where m is the mass of the particle, c is the speed of light, and $\boldsymbol{x}$ is the three-position. In relativistic quantum mechanics the system defined by the Hamiltonian (2.1) is described by the Salpeter equation of the form

$$i\hbar\frac{\partial\phi(\boldsymbol{x},t)}{\partial t} = [\sqrt{m^2c^4 - \hbar^2c^2\Delta} + V(\boldsymbol{x})]\phi(\boldsymbol{x},t), \quad (2.2)$$

where $\Delta \equiv \nabla^2$. Equation (2.2) is obtained by the quantization procedure utilizing the Newton-Wigner localization scheme [6]. In this scheme we have the standard quantization rule $\hat{\boldsymbol{x}} \to \boldsymbol{x}$, and $\hat{\boldsymbol{p}} \to -i\hbar\nabla$. As a consequence, the space of solutions of the Salpeter equation is the Hilbert space $L^2(\mathbb{R}^3,d^3\boldsymbol{x})$ with the scalar product

$$\langle\phi|\psi\rangle = \int d^3\boldsymbol{x}\,\phi^*(\boldsymbol{x})\psi(\boldsymbol{x}). \quad (2.3)$$

Therefore, according to the quantum-mechanical spirit, we should identify $|\phi(\boldsymbol{x},t)|^2$ with the probability density $\rho(\boldsymbol{x},t)$ satisfying the normalization condition:

$$\int d^3\boldsymbol{x}\,\rho(\boldsymbol{x},t) = 1. \quad (2.4)$$

Motivated by usage of limiting procedures in the nonrelativistic and ultrarelativistic case as well as some dimensional considerations, we keep in this section and the following one the physical constants $\hbar$ and c. The natural units $\hbar = 1$ and $c = 1$ are utilized in Sec. IV. Performing the Fourier transformation

$$\phi(\boldsymbol{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}\hbar^3}\int d^3\boldsymbol{p}\,e^{i\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{\hbar}}\tilde{\phi}(\boldsymbol{p},t) \quad (2.5)$$

(in the following, whenever it is clear from the context, we omit designation of a region of integration) we obtain from (2.2) the following equation:

$$i\hbar\frac{\partial\tilde{\phi}(\boldsymbol{p},t)}{\partial t} = [\sqrt{m^2c^4 + \boldsymbol{p}^2c^2} + V(i\hbar\nabla_{\boldsymbol{p}})]\tilde{\phi}(\boldsymbol{p},t). \quad (2.6)$$

It is clear that (2.6) is the partial differential equation of finite order only for $V(\boldsymbol{x})$ polynomial in $\boldsymbol{x}$. Note that $\sqrt{m^2c^4 + \boldsymbol{p}^2c^2}$ is the so-called symbol of the pseudodifferential operator $\sqrt{m^2c^4 - \hbar^2c^2\Delta}$, that is,

$$\sqrt{m^2c^4 - \hbar^2c^2\Delta}\,\phi(\boldsymbol{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}\hbar^3}\int d^3\boldsymbol{p}\,\sqrt{m^2c^4 + \boldsymbol{p}^2c^2}\,e^{i\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{\hbar}}\tilde{\phi}(\boldsymbol{p},t). \quad (2.7)$$

On taking the inverse Fourier transformation

$$\tilde{\phi}(\boldsymbol{p},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\boldsymbol{y}\,e^{-i\frac{\boldsymbol{p}\cdot\boldsymbol{y}}{\hbar}}\phi(\boldsymbol{y},t), \quad (2.8)$$

and making use of the identity [33]

$$\int_{-\infty}^{\infty}\sqrt{x^2+a^2}\,e^{ipx}\,dx = -\frac{2a}{|p|}K_1(a|p|), \quad (2.9)$$

where $K_\nu(z)$ is the modified Bessel function (Macdonald function), as well as the differentiation formula satisfied by the Bessel functions such that

$$K_1'(z) = \frac{1}{z}K_1(z) - K_2(z), \quad (2.10)$$

we get from (2.7) the following formula for the action of the pseudodifferential operator $\sqrt{m^2c^4 - \hbar^2c^2\Delta}$ in the coordinate representation:

$$\sqrt{m^2c^4 - \hbar^2c^2\Delta}\,\phi(\boldsymbol{x},t) = \int d^3\boldsymbol{y}\,K(\boldsymbol{x}-\boldsymbol{y})\phi(\boldsymbol{y},t), \quad (2.11)$$

where the function $K(\boldsymbol{x}-\boldsymbol{y})$ is

$$K(\boldsymbol{x}-\boldsymbol{y}) = -\frac{2m^2c^3}{(2\pi)^2\hbar}\frac{K_2(\frac{mc}{\hbar}|\boldsymbol{x}-\boldsymbol{y}|)}{|\boldsymbol{x}-\boldsymbol{y}|^2}, \quad (2.12)$$

and $|\boldsymbol{a}|$ designates the norm of the vector $\boldsymbol{a}$. Thus, it turns out that the pseudodifferential operator $\sqrt{m^2c^4 - \hbar^2c^2\Delta}$ can be defined as the integral operator with the kernel (2.12). Consequently, the Salpeter equation (2.2) takes the form of the integro-differential equation

$$i\hbar\frac{\partial\phi(\boldsymbol{x},t)}{\partial t} = \int d^3\boldsymbol{y}\,K(\boldsymbol{x}-\boldsymbol{y})\phi(\boldsymbol{y},t) + V(\boldsymbol{x})\phi(\boldsymbol{x},t). \quad (2.13)$$

We also remark that the nonlocality of the Salpeter equation is related only to the kinetic energy term described by the integral operator from the right-hand side of Eq. (2.13) and does not depend on the potential. Therefore, the nonlocality is not connected with potential forces acting on a quantum particle. Consider now the massless limit $m = 0$, when the Salpeter equation is

$$i\hbar\frac{\partial\phi(\boldsymbol{x},t)}{\partial t} = [\hbar c\sqrt{-\Delta} + V(\boldsymbol{x})]\phi(\boldsymbol{x},t). \quad (2.14)$$

Of course, the corresponding Fourier transform $\tilde{\phi}(\boldsymbol{p},t)$ fulfills the massless limit of Eq. (2.6), i.e.,

$$i\hbar\frac{\partial\tilde{\phi}(\boldsymbol{p},t)}{\partial t} = [c|\boldsymbol{p}| + V(i\hbar\nabla_{\boldsymbol{p}})]\tilde{\phi}(\boldsymbol{p},t). \quad (2.15)$$

Taking the limit $m \to 0$ of Eq. (2.12) and using the asymptotic formula

$$K_2(z) = \frac{2}{z^2}, \quad z \to 0, \quad (2.16)$$

we find for $m = 0$

$$K(\boldsymbol{x}-\boldsymbol{y}) = -\frac{2c\hbar}{\pi^2}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^4} \quad (m = 0). \quad (2.17)$$

Finally, consider the simplest case of a relativistic massless particle moving in a line, when the Salpeter equation is

$$i\hbar\frac{\partial\phi(x,t)}{\partial t} = \left[\sqrt{m^2c^4 - \hbar^2c^2\frac{\partial^2}{\partial x^2}} + V(x)\right]\phi(x,t). \quad (2.18)$$

On performing the Fourier transform

$$\phi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}dp\,e^{\frac{ipx}{\hbar}}\tilde{\phi}(p,t), \quad (2.19)$$

we obtain the counterpart of Eq. (2.6),

$$i\hbar\frac{\partial\tilde{\phi}(p,t)}{\partial t} = \left[\sqrt{m^2c^4 + p^2c^2} + V\left(i\hbar\frac{\partial}{\partial p}\right)\right]\tilde{\phi}(p,t). \quad (2.20)$$

Clearly the one-dimensional version of (2.7) is

$$\sqrt{m^2c^4 - \hbar^2 c^2 \frac{\partial^2}{\partial x^2}}\,\phi(x,t) = \frac{1}{\sqrt{2\pi}\hbar}\int_{-\infty}^{\infty} dp\,\sqrt{m^2c^4 + p^2c^2}\,e^{\frac{ipx}{\hbar}}\tilde{\phi}(p,t). \quad (2.21)$$

Taking the inverse Fourier transformation

$$\tilde{\phi}(p,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\, e^{\frac{-ipy}{\hbar}}\phi(y,t), \quad (2.22)$$

and using (2.9) we find

$$\sqrt{m^2c^4 - \hbar^2 c^2 \frac{\partial^2}{\partial x^2}}\,\phi(x,t) = \int_{-\infty}^{\infty} dy\, K(x-y)\phi(y,t), \quad (2.23)$$

where the kernel $K(x-y)$ is given by

$$K(x-y) = -\frac{mc^2}{\pi}\frac{1}{|x-y|}K_1\left(\frac{mc}{\hbar}|x-y|\right). \quad (2.24)$$

It follows immediately from (2.24) and the asymptotic formula

$$K_1(z) = \frac{1}{z}, \quad z \to 0 \quad (2.25)$$

that the kernel in the massless case is of the form

$$K(x-y) = -\frac{c\hbar}{\pi}\frac{1}{(x-y)^2} \quad (m=0). \quad (2.26)$$

The relation (2.26) is also a direct consequence of (2.21) with $m = 0$, Eq. (2.22), and the identity [33]

$$\int_{-\infty}^{\infty} |x| e^{ipx}\, dx = -\frac{2}{p^2}. \quad (2.27)$$

As far as we are aware, the first example of the exact solution to the Salpeter equation, referring to the massless free particle moving in a line, was considered by Rosenstein and Horwitz [11], whereas the solution for the massive free particle was discussed by Rosenstein and Usher [12]. The WKB approximation technique to the Salpeter equation was developed in Refs. [8] and [9]. In a series of papers by Hall, Lucha, and Schöberl (see, for instance, Ref. [15]), the energy bounds for the Salpeter equation were analyzed. To our best knowledge, the first example of the nontrivial exact solution to the Salpeter equation in the case of a particle in $\mathbb{R}^3$ was our solution to the Salpeter equation for a relativistic massless harmonic oscillator [23]. More precisely, we derived the exact stationary wave functions and corresponding exact spectrum of the energy expressed by means of zeros of the Airy function. The correctness of the quantization based on the massless Salpeter equation was confirmed by the good behavior of the related probability density and expectation values of observables. Recently, the new exact solution to the Salpeter equation has been reported referring to the free massive particle on a line, with the Gaussian initial wave function [34]. This solution is very complicated, and it has the form of infinite power series expansion with coefficients expressed by means of integrals of special functions. In Sec. IV we introduce other new examples of exact solutions to the Salpeter equation. In particular, we derive the nontrivial solution of this equation for a massless particle in a linear potential.

III. PROBABILITY CURRENT

In this section we introduce the probability current for a quantum spinless relativistic particle and discuss its basic properties. We first discuss the probability density and the probability current for the Klein-Gordon equation such that

$$\left[i\hbar\frac{\partial}{\partial t} - V(\boldsymbol{x})\right]^2 \psi(\boldsymbol{x},t) = (m^2c^4 - \hbar^2c^2\Delta)\psi(\boldsymbol{x},t), \quad (3.1)$$

where the potential $V(\boldsymbol{x})$ is introduced in the equation by means of the vector minimal coupling scheme. We point out that the denomination "square root" of the Klein-Gordon equation, mentioned in the introduction, is appropriate for the Salpeter equation only in the case of a free particle. Indeed, by squaring (2.2) we obtain

$$\left[i\hbar\frac{\partial}{\partial t} - V(\boldsymbol{x})\right]^2 \phi(\boldsymbol{x},t) = \{m^2c^4 - \hbar^2c^2\Delta + [\sqrt{m^2c^4 - \hbar^2c^2\Delta}, V(\boldsymbol{x})]\}\phi(\boldsymbol{x},t). \quad (3.2)$$

We recall that in the case of the Klein-Gordon equation the probability density is given by (compare Ref. [35])

$$\rho_{KG} = \frac{i\hbar}{2mc^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t} + \frac{2i}{\hbar}V|\psi|^2\right). \quad (3.3)$$

Since the Klein-Gordon equation is second order in a time derivative, therefore the initial values of ψ and $\frac{\partial\psi}{\partial t}$ can be arbitrary. We conclude that ρ_{KG} can be either positive or negative, and the problem arises with the probabilistic interpretation of the Klein-Gordon equation. The expression for ρ_{KG} reduces to the nonrelativistic form in the nonrelativistic limit $c \to \infty$. However, to show this one should first assume the validity of the Salpeter equation (2.2) or equivalently restrict to positive energy solutions of the Klein-Gordon equation only. The formula for the probability current for a Klein-Gordon particle is identical with the nonrelativistic one; that is, we have

$$\boldsymbol{j}_{KG} = -\frac{i\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*). \quad (3.4)$$

We stress that neither the probability density (3.3) nor the probability current (3.4) has the correct limit $m \to 0$. Furthermore, it can be checked that the continuity equation implied by the massless Klein-Gordon equation obtained from (3.1) by setting $m = 0$ requires the existence of some universal constant with the dimension of length. This is yet another disadvantage of the Klein-Gordon equation.

We now return to the Salpeter equation (2.2). Proceeding analogously as in the case of the nonrelativistic Schrödinger equation we find

$$\frac{\partial|\phi|^2}{\partial t} + \frac{i}{\hbar}(\phi^*\sqrt{m^2c^4 - \hbar^2c^2\Delta}\,\phi - \phi\sqrt{m^2c^4 - \hbar^2c^2\Delta}\,\phi^*) = 0. \quad (3.5)$$

Now, the probability density $\rho(\boldsymbol{x},t)$ expressed in terms of the Fourier transform $\tilde{\phi}(\boldsymbol{p},t)$ can be written in the form

$$\rho(\boldsymbol{x},t)=|\phi(\boldsymbol{x},t)|^2=\frac{1}{(2\pi)^3\hbar^6}\int d^3\boldsymbol{p}\,d^3\boldsymbol{k}\,e^{\mathrm{i}\frac{(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{x}}{\hbar}}\tilde{\phi}^*(\boldsymbol{p},t)\tilde{\phi}(\boldsymbol{k},t). \tag{3.6}$$

On using (3.5), (3.6), and the continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\boldsymbol{j}=0, \tag{3.7}$$

where $\rho(\boldsymbol{x},t)$ is the probability density and $\boldsymbol{j}(\boldsymbol{x},t)$ is the probability current, we arrive at the following formula for the probability current:

$$\boldsymbol{j}(\boldsymbol{x},t)=\frac{c}{(2\pi)^3\hbar^6}\int d^3\boldsymbol{p}\,d^3\boldsymbol{k}\,\frac{\boldsymbol{p}+\boldsymbol{k}}{\sqrt{m^2c^2+\boldsymbol{p}^2}+\sqrt{m^2c^2+\boldsymbol{k}^2}}\times e^{\mathrm{i}\frac{(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{x}}{\hbar}}\tilde{\phi}^*(\boldsymbol{p},t)\tilde{\phi}(\boldsymbol{k},t). \tag{3.8}$$

It should be noted that the probability current (3.8) has the correct nonrelativistic limit $c\to\infty$. Namely, making use of (2.5), we easily find

$$\lim_{c\to\infty}\boldsymbol{j}=-\frac{\mathrm{i}\hbar}{2m}(\phi^*\nabla\phi-\phi\nabla\phi^*). \tag{3.9}$$

Furthermore, it follows immediately from (3.8) that

$$\int\boldsymbol{j}(\boldsymbol{x},t)\,d^3\boldsymbol{x}=\langle\phi|\hat{\boldsymbol{v}}\phi\rangle, \tag{3.10}$$

where $\hat{\boldsymbol{v}}$ is the operator of the relativistic velocity

$$\hat{\boldsymbol{v}}=\frac{c\hat{\boldsymbol{p}}}{\hat{p}_0}, \tag{3.11}$$

where $\hat{\boldsymbol{p}}=-\mathrm{i}\hbar\nabla$ and $\hat{p}_0=E/c=\sqrt{m^2c^2+\hat{\boldsymbol{p}}^2}=\sqrt{m^2c^2-\hbar^2\Delta}$. The formula (3.10) is the relativistic counterpart of the well-known nonrelativistic expression describing the connection of the integral of the probability current and average velocity in the given state. We stress that the relation (3.10) is not valid in the case of the Klein-Gordon equation. Bearing in mind the critique of the Salpeter equation based on its nonlocality, it is also worthwhile to point out that the length of the average velocity (3.11) related to the discussed probability current via Eq. (3.10), does not exceed the speed of light. Indeed, we have

$$\langle\phi|\hat{\boldsymbol{v}}\phi\rangle^2\leqslant\langle\phi|\hat{\boldsymbol{v}}^2\phi\rangle=c^2\langle\phi|\frac{\hat{\boldsymbol{p}}^2}{\hat{\boldsymbol{p}}^2+m^2c^2}\phi\rangle\leqslant c^2, \tag{3.12}$$

where the inequality is saturated at $m=0$. Yet another good property of the probability current (3.8) is the existence of the massless limit. In fact, putting $m=0$ in (3.8) we immediately get

$$\boldsymbol{j}(\boldsymbol{x},t)=\frac{c}{(2\pi)^3\hbar^6}\int d^3\boldsymbol{p}\,d^3\boldsymbol{k}\,\frac{\boldsymbol{p}+\boldsymbol{k}}{|\boldsymbol{p}|+|\boldsymbol{k}|}\times e^{\mathrm{i}\frac{(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{x}}{\hbar}}\tilde{\phi}^*(\boldsymbol{p},t)\tilde{\phi}(\boldsymbol{k},t)\qquad(m=0). \tag{3.13}$$

The probability current can be expressed in terms of the solution $\phi(\boldsymbol{x},t)$ to the Salpeter equation. Indeed, substituting in (3.8) the Fourier transform (2.8) and using (2.9), (2.10), and the identity [33]

$$\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}px}}{a^2-x^2}\,dx=\frac{\pi}{a}\sin a|p|, \tag{3.14}$$

we find

$$\boldsymbol{j}(\boldsymbol{x},t)=\int d^3\boldsymbol{y}\,d^3\boldsymbol{z}\,\boldsymbol{K}(\boldsymbol{x};\boldsymbol{y},\boldsymbol{z})\phi^*(\boldsymbol{y},t)\phi(\boldsymbol{z},t), \tag{3.15}$$

where

$$\boldsymbol{K}(\boldsymbol{x};\boldsymbol{y},\boldsymbol{z})=-\frac{\mathrm{i}m^2c^3}{(2\pi)^3\hbar^2}(\nabla_{\boldsymbol{y}}-\nabla_{\boldsymbol{z}})\times\left\{\frac{1}{|\boldsymbol{y}-\boldsymbol{x}||\boldsymbol{x}-\boldsymbol{z}|}\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|+|\boldsymbol{x}-\boldsymbol{z}|}\times K_2\left[\frac{mc}{\hbar}(|\boldsymbol{y}-\boldsymbol{x}|+|\boldsymbol{x}-\boldsymbol{z}|)\right]\right\}. \tag{3.16}$$

Hence, making use of the theorem on the gradient [36]

$$\int_V d^3\boldsymbol{x}\,\nabla\varphi(\boldsymbol{x})=\int_S d\boldsymbol{S}\,\varphi(\boldsymbol{x}), \tag{3.17}$$

where S is the oriented boundary of the volume V, and taking into account that the Bessel functions $K_\nu(z)$ approach zero as $|z|\to\infty$, we arrive at the relation

$$\boldsymbol{j}(\boldsymbol{x},t)=\int d^3\boldsymbol{y}\,d^3\boldsymbol{z}\,K(|\boldsymbol{y}-\boldsymbol{x}|,|\boldsymbol{x}-\boldsymbol{z}|)\times[\phi^*(\boldsymbol{y},t)\nabla_{\boldsymbol{z}}\phi(\boldsymbol{z},t)-\phi(\boldsymbol{z},t)\nabla_{\boldsymbol{y}}\phi^*(\boldsymbol{y},t)], \tag{3.18}$$

where

$$K(|\boldsymbol{u}|,|\boldsymbol{w}|)=-\frac{\mathrm{i}m^2c^3}{(2\pi)^3\hbar^2}\frac{1}{|\boldsymbol{u}||\boldsymbol{w}|}\frac{1}{|\boldsymbol{u}|+|\boldsymbol{w}|}\times K_2\left[\frac{mc}{\hbar}(|\boldsymbol{u}|+|\boldsymbol{w}|)\right]. \tag{3.19}$$

The formula (3.18) is remarkable. In fact, it means that the relativistic probability current has the form resembling the "smeared" nonrelativistic one.

We now discuss the case $m=0$. On taking the massless limit $m\to 0$ and making use of (2.16) we easily get from (3.16) the following formula:

$$\boldsymbol{K}(\boldsymbol{x};\boldsymbol{y},\boldsymbol{z})=-\frac{2\mathrm{i}c}{(2\pi)^3}(\nabla_{\boldsymbol{y}}-\nabla_{\boldsymbol{z}})\times\left[\frac{1}{|\boldsymbol{y}-\boldsymbol{x}||\boldsymbol{x}-\boldsymbol{z}|}\frac{1}{(|\boldsymbol{y}-\boldsymbol{x}|+|\boldsymbol{x}-\boldsymbol{z}|)^3}\right]\qquad(m=0). \tag{3.20}$$

The relation (3.20) can be also easily derived from (3.13), (2.8), (3.14), and the identity [33]

$$\int_{-\infty}^{\infty}\varepsilon(x)\sin ax\,e^{\mathrm{i}px}\,dx=\frac{2a}{a^2-p^2}, \tag{3.21}$$

where $\varepsilon(x)$ is the sign function. Proceeding analogously as with (3.15) we get

$$K(|\boldsymbol{u}|,|\boldsymbol{w}|)=-\frac{2\mathrm{i}c}{(2\pi)^3}\frac{1}{|\boldsymbol{u}||\boldsymbol{w}|}\frac{1}{(|\boldsymbol{u}|+|\boldsymbol{w}|)^3}\qquad(m=0). \tag{3.22}$$

On the other hand, the formula (3.22) is an immediate consequence of (3.19) and the asymptotic formula (2.16).

Now, whenever the initial wave function is real, then the solutions to the Salpeter equation (2.2) satisfy $\phi^*(\boldsymbol{x},t) = \phi(\boldsymbol{x},-t)$. Of course, it is a well-known property of the Schrödinger equation. An immediate consequence of this relation and (3.15) is $\boldsymbol{j}(\boldsymbol{x},-t) = -\boldsymbol{j}(\boldsymbol{x},t)$, that is, $\boldsymbol{j}(\boldsymbol{x},t)$ is an odd function of time. Furthermore, it follows easily from (3.16) that if the wave packet fulfils $\phi(-\boldsymbol{x},t) = \phi(\boldsymbol{x},t)$ or $\phi(-\boldsymbol{x},t) = -\phi(\boldsymbol{x},t)$, then $\boldsymbol{j}(-\boldsymbol{x},t) = -\boldsymbol{j}(\boldsymbol{x},t)$.

We now return to the Salpeter equation (2.2). Using the continuity equation (3.7) and the formal power series expansion of the square root from the right-hand side of (2.2) we get the following formula for the probability current:

$$\boldsymbol{j} = -\frac{\mathrm{i}mc^2}{\hbar}\sum_{n=1}^{\infty}\frac{(2n-3)!!}{(2n)!!}\left(\frac{\hbar}{mc}\right)^{2n} \times \sum_{k=0}^{2n-1}(-1)^k\nabla^k\phi^*\nabla^{2n-k-1}\phi. \tag{3.23}$$

We point out that (3.8) can be formally obtained from (3.23) and (2.5). Nevertheless, the formula for the probability current (3.23) is mathematically less sound and technically less convenient than the integral representation (3.8). Furthermore, in opposition to (3.8), the current given by (3.23) has no correct limit for $m \to 0$.

Finally, let us restrict to the simplest case of a relativistic massless particle on a line. Equations (3.6), (3.8), and (3.13) take then the following form:

$$\rho(x,t) = |\phi(x,t)|^2 = \frac{1}{2\pi\hbar^2}\int dp\,dk\,e^{\mathrm{i}\frac{(k-p)x}{\hbar}}\tilde{\phi}^*(p,t)\tilde{\phi}(k,t), \tag{3.24}$$

$$j(x,t) = \frac{c}{2\pi\hbar^2}\int dp\,dk\,\frac{p+k}{\sqrt{m^2c^2+p^2}+\sqrt{m^2c^2+k^2}} \times e^{\mathrm{i}\frac{(k-p)x}{\hbar}}\tilde{\phi}^*(p,t)\tilde{\phi}(k,t), \tag{3.25}$$

$$j(x,t) = \frac{c}{2\pi\hbar^2}\int dp\,dk\,\frac{p+k}{|p|+|k|}e^{\mathrm{i}\frac{(k-p)x}{\hbar}}\tilde{\phi}^*(p,t)\tilde{\phi}(k,t) \quad (m=0), \tag{3.26}$$

respectively; here we have used the Fourier transformation (2.19). Let us focus our attention on Eq. (3.25) . Taking into account (2.22), (2.9), and the identity [33]

$$\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}px}}{x+a}dx = i\pi\varepsilon(p)e^{-\mathrm{i}ap}, \tag{3.27}$$

we get from (3.25)

$$j(x,t) = \int dy\,dz\,K(x;y,z)\phi^*(y,t)\phi(z,t), \tag{3.28}$$

where

$$K(x;y,z) = -\frac{imc^2}{2\pi\hbar}\frac{\varepsilon(x-y)-\varepsilon(x-z)}{|y-z|}K_1\left(\frac{mc}{\hbar}|y-z|\right). \tag{3.29}$$

Hence, taking the limit $m \to 0$ and using (2.25), we find the following formula for the function $K(x;y,z)$ for the probability current in the massless case (3.26)

$$K(x;y,z) = -\frac{\mathrm{i}c}{2\pi}\frac{\varepsilon(x-y)-\varepsilon(x-z)}{(y-z)^2} \qquad (m=0). \tag{3.30}$$

We now discuss the case of the massless particle on a line in a more detail. Let $\phi_\pm(x,t)$ be solution of the Salpeter equation

$$\mathrm{i}\frac{\partial\phi(x,t)}{\partial t} = c\sqrt{-\frac{\partial^2}{\partial x^2}}\,\phi(x,t) \tag{3.31}$$

for a free massless particle with positive (negative) momentum moving in the line to the right (left). Then we can identify

$$\tilde{\phi}_\pm(p,t) = \theta(\pm p)\tilde{\phi}(p,t), \tag{3.32}$$

where $\theta(p)$ is the Heaviside step function. By means of the Fourier transformation (2.19) for $m=0$, we conclude that ϕ_+ and ϕ_- depends on $x-ct$ and $x+ct$, respectively, that is,

$$\phi_+(x,t) = \varphi(x-ct) \tag{3.33}$$

and

$$\phi_-(x,t) = \psi(x+ct). \tag{3.34}$$

Let furthermore $\rho_\pm(x,t) = |\phi_\pm(x,t)|^2$ and $j_\pm(x,t)$ designate the probability density and probability current, respectively, corresponding to $\phi_\pm(x,t)$. An immediate consequence of (3.24) and (3.26) is

$$j_\pm = \pm c\rho_\pm. \tag{3.35}$$

We point out that the relation (3.35) is a natural massless relativistic counterpart of the well-known nonrelativistic formula for the probability current [see (4.29)] expressed by means of the velocity and the probability density. The continuity equation in the one-dimensional case such that

$$\frac{\partial\rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \tag{3.36}$$

is satisfied identically. Of course, the general solution $\phi(x,t)$ to the Salpeter equation (3.31) is

$$\phi(x,t) = \frac{1}{\sqrt{2}}[\phi_+(x,t)+\phi_-(x,t)] = \frac{1}{\sqrt{2}}[\varphi(x-ct)+\psi(x+ct)]. \tag{3.37}$$

On the other hand, (3.37) is the well-known general solution of the one-dimensional Klein-Gordon equation for a massless free particle of the form

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)\phi(x,t) = 0. \tag{3.38}$$

Notice, finally, that for massless particle moving to the right or left, the expectation value $\langle\hat{x}(t)\rangle$ of the position operator coincides with the classical trajectory of a massless particle on a line. Indeed, we have

$$\langle\hat{x}(t)\rangle = \langle\phi_\pm|\hat{x}\phi_\pm\rangle = \int_{-\infty}^{\infty}dx\,x\rho(x\mp ct) = \pm ct + \langle\hat{x}(0)\rangle. \tag{3.39}$$

IV. EXACT SOLUTIONS

In this section we introduce the exact solutions to the Salpeter equation and discuss the corresponding probability density and probability current. We begin with the one-dimensional cases.

A. Free massless particle on a line

We first study a relativistic free massless particle moving in a line. The corresponding Salpeter equation can be written in the form

$$\mathrm{i}\frac{\partial\phi(x,t)}{\partial t}=\sqrt{-\frac{\partial^2}{\partial x^2}}\,\phi(x,t), \tag{4.1}$$

where we set $c=1$. Consider the evolution of the (normalized) wave packet

$$\phi(x,0)=\sqrt{\frac{2}{\pi}}\frac{a^{\frac{3}{2}}}{x^2+a^2},\quad a>0. \tag{4.2}$$

This package is referred to in Ref. [11] as to the "Lorentzian" one. On performing the Fourier transformation

$$\phi(x,t)=\frac{1}{\sqrt{2\pi}}\int dp\, e^{\mathrm{i}px}\tilde{\phi}(p,t), \tag{4.3}$$

where we set $\hbar=1$, we get from (4.1) the following equation:

$$\mathrm{i}\frac{\partial\tilde{\phi}(p,t)}{\partial t}=|p|\tilde{\phi}(p,t), \tag{4.4}$$

subject to the initial condition

$$\tilde{\phi}(p,0)=\sqrt{a}e^{-a|p|}, \tag{4.5}$$

where use was made of the identity

$$\int_{-\infty}^{\infty}\frac{1}{x^2+a^2}\,e^{\mathrm{i}px}\,dx=\frac{\pi}{a}e^{-a|p|}. \tag{4.6}$$

The solution to the elementary equation (4.4) with the initial condition (4.5) is

$$\tilde{\phi}(p,t)=\sqrt{a}e^{-(a+\mathrm{i}t)|p|}. \tag{4.7}$$

Hence, using (4.3) we obtain the normalized wave function at any time

$$\phi(x,t)=\sqrt{\frac{2a}{\pi}}\frac{a+\mathrm{i}t}{x^2+(a+\mathrm{i}t)^2}. \tag{4.8}$$

The solution (4.8) was originally derived in Ref. [11]; however, the definition of the probability current suggested therein is different from ours. An immediate consequence of (4.8) is the following formula for the probability density

$$\rho(x,t)=|\phi(x,t)|^2=\frac{2a}{\pi}\frac{a^2+t^2}{(x^2-t^2+a^2)^2+4a^2t^2}. \tag{4.9}$$

The time development of the probability density (4.9) is presented in Fig. 1. Furthermore, taking into account (3.26) and (3.36), we find after some calculation

$$j(x,t)=\frac{a}{4\pi t^2}\ln\frac{(x+t)^2+a^2}{(x-t)^2+a^2}-\frac{ax}{\pi t}\frac{x^2-3t^2+a^2}{(x^2-t^2+a^2)^2+4a^2t^2}. \tag{4.10}$$

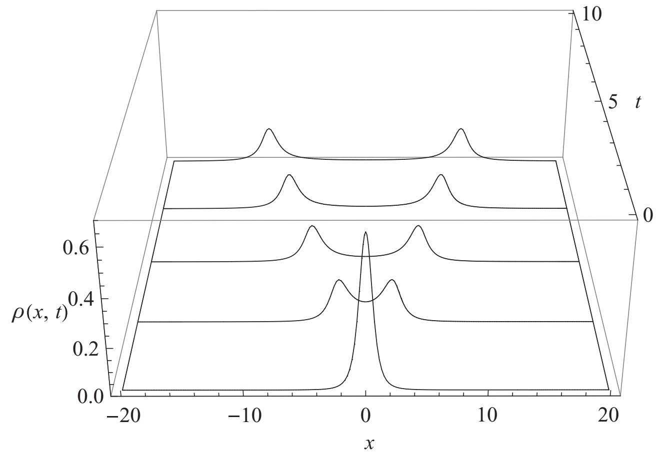

FIG. 1. The time evolution of the probability density (4.9) related to the solution of the Salpeter equation for a free massless particle in one dimension. Because of choice of the natural units ($c=1$ and $\hbar=1$), the units of ρ, j, x, and t used in the figures referring to the one-dimensional case are m^{-1}, m^{-1}, m, and m, respectively. The parameter $a=1$ m.

The time evolution of the probability current (4.10) is shown in Fig. 2. We point out that there is no singularity in (4.10) for $t=0$. Namely, we have $\lim_{t\to 0} j(x,t)=0$.

We now return to (4.7). In view of (3.32) we have

$$\tilde{\phi}_{\pm}(p,t)=\sqrt{2a}\theta(\pm p)e^{-(a+\mathrm{i}t)|p|}. \tag{4.11}$$

Hence, taking into account (4.3) it follows that the wave packet referring to the particle moving to the right and left, respectively, is given by

$$\phi_{\pm}(x,t)=\sqrt{\frac{a}{\pi}}\frac{\pm\mathrm{i}}{x\mp t\pm\mathrm{i}a}. \tag{4.12}$$

Therefore, the corresponding probability density and probability current are

$$\rho_{\pm}(x,t)=|\phi_{\pm}(x,t)|^2=\pm\, j_{\pm}(x,t)=\frac{a}{\pi}\frac{1}{(x\mp t)^2+a^2}. \tag{4.13}$$

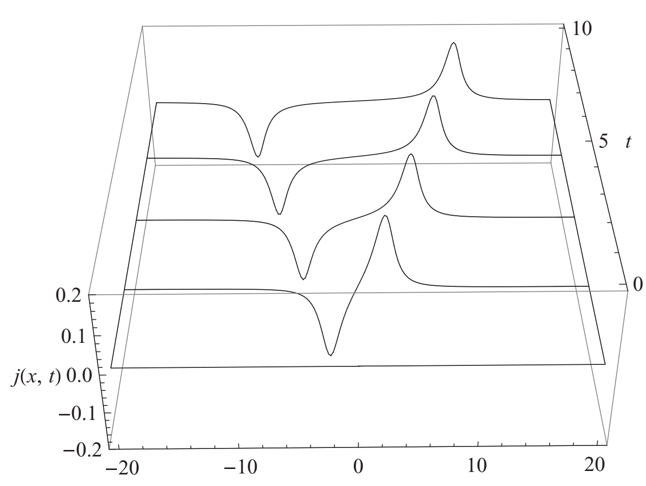

FIG. 2. The time development of the probability current (4.10) for a free massless particle moving in a line. The parameter $a=1$ m.

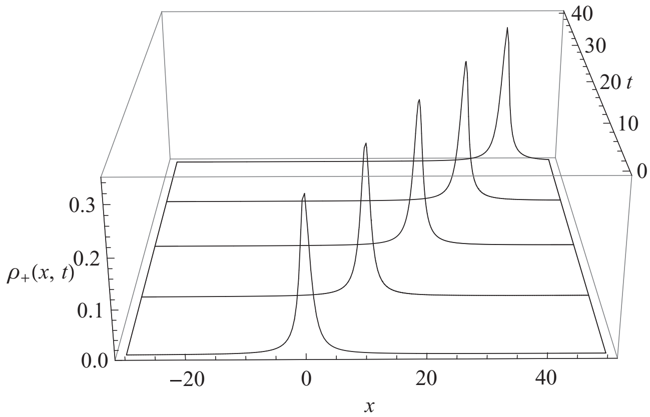

FIG. 3. The behavior of the probability density $\rho_+(x,t)$ given by (4.13), referring to the case of the free massless particle moving to the right. The stable maximum of the probability density is going with the speed of light $c = 1$.

The time development of the probability density $\rho_+(x,t)$ is shown in Fig. 3.

B. Free massive particle on a line

We now discuss a relativistic free massive particle moving in a line. The corresponding Salpeter equation is

$$\mathrm{i}\frac{\partial \phi(x,t)}{\partial t} = \sqrt{m^2 - \frac{\partial^2}{\partial x^2}}\,\phi(x,t). \tag{4.14}$$

Performing the Fourier transformation we find

$$\mathrm{i}\frac{\partial \tilde{\phi}(p,t)}{\partial t} = \sqrt{m^2 + p^2}\,\tilde{\phi}(p,t). \tag{4.15}$$

Consider now the particular solution to (4.15) of the form

$$\tilde{\phi}(p,t) = \frac{1}{\sqrt{2mK_1(2ma)}}e^{-(a+\mathrm{i}t)\sqrt{p^2+m^2}}. \tag{4.16}$$

Using the identity [37]

$$\int_0^\infty dx\ \exp(-\alpha\sqrt{x^2+\beta^2})\cos\gamma x = \frac{\alpha\beta}{\sqrt{\alpha^2+\gamma^2}}K_1(\beta\sqrt{\alpha^2+\gamma^2}),\quad \mathrm{Re}\alpha > 0,\quad \mathrm{Re}\beta > 0, \tag{4.17}$$

one can easily derive from (4.16) the following (normalized) solution to the Salpeter equation (4.14):

$$\phi(x,t) = \sqrt{\frac{m}{\pi K_1(2ma)}}\frac{a+\mathrm{i}t}{\sqrt{x^2+(a+\mathrm{i}t)^2}} \times K_1[m\sqrt{x^2+(a+\mathrm{i}t)^2}]. \tag{4.18}$$

Taking into account the asymptotic formula (2.25) we find that the solution (4.18) is a generalization of the solution (4.8) obtained for the massless particle to the case $m > 0$. More precisely, we have

$$\lim_{m\to 0}\phi(x,t) = \sqrt{\frac{2a}{\pi}}\frac{a+\mathrm{i}t}{x^2+(a+\mathrm{i}t)^2}. \tag{4.19}$$

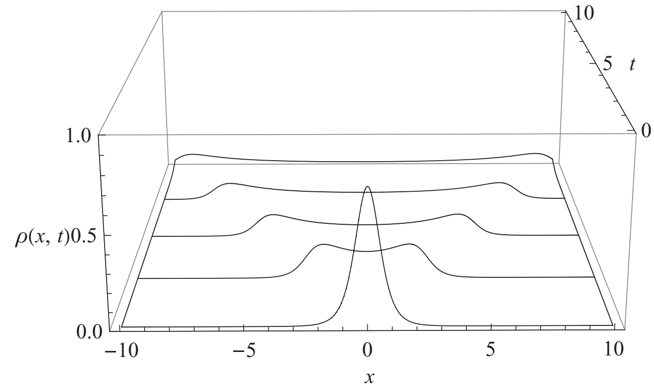

FIG. 4. The time development of the probability density $\rho(x,t) = |\phi(x,t)|^2$ corresponding to the case of the free massive particle, where $\phi(x,t)$ is given by (4.18). The mass $m = 0.5$ and $a = 1$ m.

The solution (4.18) was independently derived in Ref. [12]. The probability density $\rho(x,t) = |\phi(x,t)|^2$ corresponding to the the wave function (4.18) is presented in Fig. 4. It turns out that, opposite to the massless case, the wave function $\phi(x,t)$ spreads out as time passes. Finally, taking into account (3.34), (2.10), the identity

$$\frac{1}{z}K_1(z) = -\frac{1}{2}[K_0(z) - K_2(z)], \tag{4.20}$$

and the fact that $\phi(x,t)$ is an even function of x [see the discussion below Eq. (3.22)] we find the following formula for the probability current:

$$j(x,t) = \frac{m^2}{\pi K_1(2ma)}\int_0^x dx\,\mathrm{Im}\left(\left\{\frac{x^2-(a+\mathrm{i}t)^2}{x^2+(a+\mathrm{i}t)^2} \times K_2[m\sqrt{x^2+(a+\mathrm{i}t)^2}] - K_0[m\sqrt{x^2+(a+\mathrm{i}t)^2}]\right\} \times \frac{a-\mathrm{i}t}{\sqrt{x^2+(a-\mathrm{i}t)^2}}K_1[m\sqrt{x^2+(a-\mathrm{i}t)^2}]\right). \tag{4.21}$$

The time development of the probability current (4.21) is shown in Fig. 5. As expected in view of the behavior of wave functions, the probability current spreads out.

C. Massless particle in a linear potential

Our purpose now is to analyze the case of a relativistic massless particle moving in a line in a linear potential. Classically, this system is defined by the Hamiltonian

$$H = c|p| + \mu x, \tag{4.22}$$

where $\mu > 0$ is a parameter. The corresponding Hamilton equations lead to a solution of the form

$$x(t) = -\frac{c}{\mu}[|-\mu t + p(0)| - p(0)] + x(0). \tag{4.23}$$

In the following we put $c = 1$. Under the simplest initial conditions $x(0) = 0$, $p(0) = 0$ we have $x(t) = -|t|$

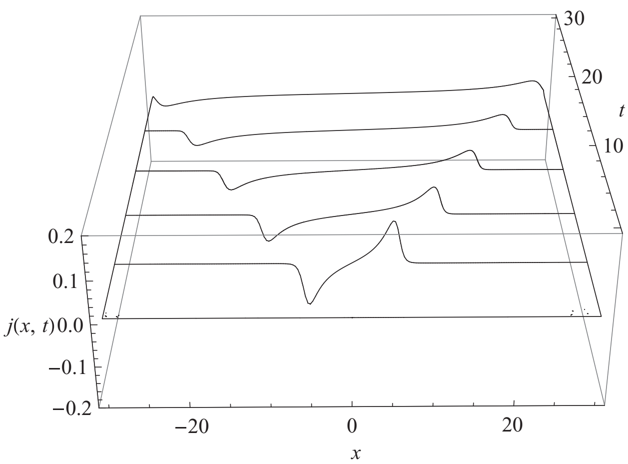

FIG. 5. The plot of the probability current (4.21) versus time, where $m = 0.5$ and $a = 1$ m.

(a Λ-shaped trajectory). On the quantum level this system is described by the Salpeter equation of the form

$$\mathrm{i}\frac{\partial\phi(x,t)}{\partial t} = \sqrt{-\frac{\partial^2}{\partial x^2}}\,\phi(x,t) + \mu x\phi(x,t), \tag{4.24}$$

where we set $c = 1$ and $\hbar = 1$. The Fourier transform of $\phi(x,t)$ satisfies

$$\mathrm{i}\frac{\partial\tilde{\phi}(p,t)}{\partial t} = |p|\tilde{\phi}(p,t) + \mathrm{i}\mu\frac{\partial\tilde{\phi}(p,t)}{\partial p}. \tag{4.25}$$

The following solution of Eq. (4.25) can be derived easily:

$$\tilde{\phi}(p,t) = e^{\mathrm{i}\frac{\varepsilon(p)p^2}{2\mu}}\chi(p+\mu t), \tag{4.26}$$

where $\chi(p)$ is an arbitrary function. Now, we choose the initial wave packet so that

$$\chi(p) = Ce^{-\frac{\lambda p^2}{2\mu}}, \tag{4.27}$$

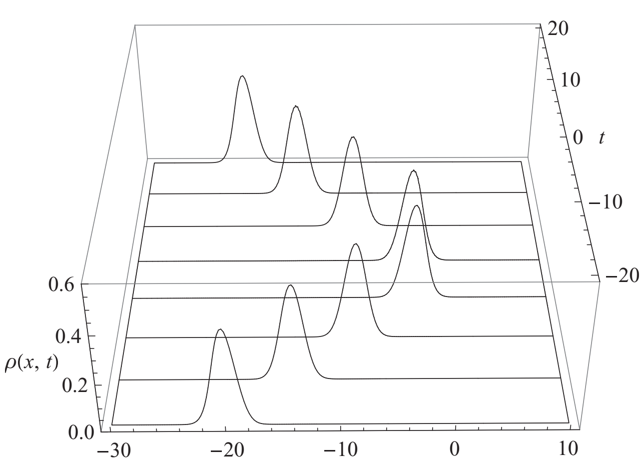

FIG. 6. The plot of the probability density $\rho(x,t) = |\phi(x,t)|^2$, referring to the wave function (4.29) of the massless particle in a linear potential. The parameter $\mu = 1$ m^{-2} and $\lambda = 1$. The classical Λ-shaped dynamics of the maxima of the probability density is easily observed.

where $\lambda > 0$ is a parameter and C is a normalization constant. Hence, the normalized wave function in the momentum representation is

$$\tilde{\phi}(p,t) = \left(\frac{\lambda}{\mu\pi}\right)^{\frac{1}{4}} e^{\mathrm{i}\frac{\varepsilon(p)p^2}{2\mu}}\,e^{-\frac{\lambda(p+\mu t)^2}{2\mu}}. \tag{4.28}$$

Equations (4.28) and (4.3) taken together yield the normalized wave function such that

$$\begin{aligned}\phi(x,t) = {} & \frac{1}{2}\left(\frac{\lambda\mu}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\lambda\mu}{2}t^2}\Bigg\{\frac{1}{\sqrt{\lambda+\mathrm{i}}}e^{\frac{\mu(-\lambda t+\mathrm{i}x)^2}{2(1+\mathrm{i})}} \\ & \times\mathrm{erfc}\left[\sqrt{\frac{\mu}{2(1+\mathrm{i})}}(-\lambda t+\mathrm{i}x)\right] \\ & + \frac{1}{\sqrt{\lambda-\mathrm{i}}}e^{\frac{\mu(\lambda t-\mathrm{i}x)^2}{2(1-\mathrm{i})}}\,\mathrm{erfc}\left[\sqrt{\frac{\mu}{2(1-\mathrm{i})}}(\lambda t-\mathrm{i}x)\right]\Bigg\},\end{aligned} \tag{4.29}$$

where $\mathrm{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$ is the complementary error function. A remarkable property of the corresponding probability density $\rho(x,t) = |\phi(x,t)|^2$ presented in Fig. 6 is the behavior of its maxima following the classical Λ-shaped trajectory. Moreover, the expectation value of the position operator, calculated easily in the momentum representation, is of the form

$$\langle\hat{x}(t)\rangle = -\frac{e^{-\lambda\mu t^2}}{\sqrt{\lambda\mu\pi}} - t\,\mathrm{erf}(\sqrt{\lambda\mu}t), \tag{4.30}$$

where $\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-\tau^2}d\tau$ is the error function. Hence, we find the asymptotic formula

$$\langle\hat{x}(t)\rangle = -|t|, \quad |t| \gg 1; \tag{4.31}$$

i.e., for $|t| \gg 1$ the average value of the position operator behaves classically (see Fig. 7). We also point out that the expectation value of the particle velocity $\langle v(t)\rangle = -\mathrm{erf}(\sqrt{\lambda\mu}t)$ lies between -1 and 1, that is, it does not exceed the light speed; for $|t| \gg 1$, $|\langle v(t)\rangle| \to 1$, i.e., it reaches asymptotically the speed of light. This is yet more evidence of the correctness of our approach based on the Salpeter equation. The probability

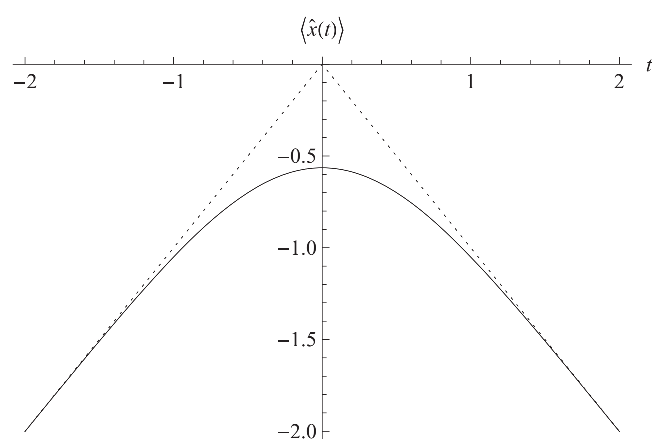

FIG. 7. The plot of the expectation value of the position operator in the state (4.29), given by Eq. (4.30) with $\mu = 1$ m^{-2} and $\lambda = 1$ versus time (solid line). The dotted line refers to the classical trajectory $x(t) = -|t|$.

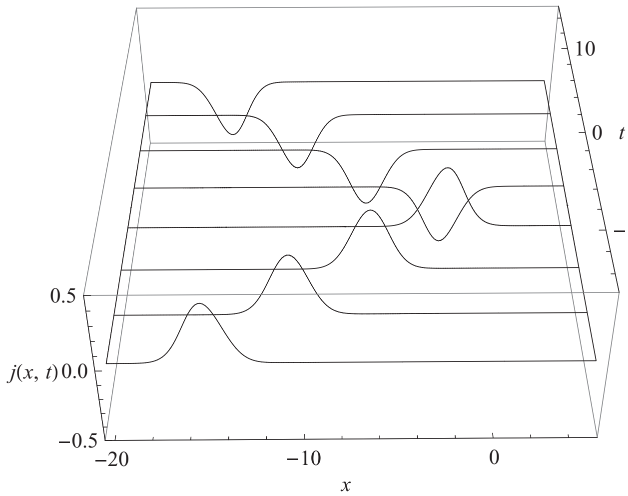

FIG. 8. The plot of the probability current referring to the wave function (4.29) with $\lambda = 1$ versus time.

current that can be derived with the help of (3.36) is too complex to be reproduced herein. The time development of the probability current obtained from computer simulations is shown in Fig. 8.

D. Plane wave solutions

An easy inspection (compare Ref. [22]) shows that the Salpeter equation (2.2) with $V = 0$, i.e., in the case of a free particle, possesses plane-wave solutions such that

$$\phi(\boldsymbol{x},t) = Ce^{-\frac{i}{\hbar}(\mathcal{E}t-\boldsymbol{k}\cdot\boldsymbol{x})}, \tag{4.32}$$

where $\mathcal{E} = \sqrt{m^2c^4 + \boldsymbol{k}^2c^2}$ and C is a normalization constant. Taking into account (2.8) and (3.8) or inserting (4.32) into (3.23) we obtain the following formula for the corresponding probability current:

$$\boldsymbol{j} = \rho\boldsymbol{v}, \tag{4.33}$$

where $\rho = |\phi|^2 = |C|^2$, and $\boldsymbol{v}$ is the relativistic three-velocity given by

$$\boldsymbol{v} = \frac{c^2\boldsymbol{k}}{\mathcal{E}}. \tag{4.34}$$

The relation (4.33) is a natural generalization of the analogous formula on the probability current, which is well known in the nonrelativistic quantum mechanics. We also recall that the nonrelativistic counterpart of (4.33) is a point of departure for the hydrodynamical formulation of quantum mechanics.

We point out that in the case of the Klein-Gordon equation the probability current for plane waves has the same form as (4.33), that is,

$$\boldsymbol{j}_{\mathrm{KG}} = \rho_{\mathrm{KG}}\boldsymbol{v}. \tag{4.35}$$

However, the probability density ρ_{KG} such that

$$\rho_{\mathrm{KG}} = \frac{|\phi|^2\mathcal{E}}{mc^2} \tag{4.36}$$

can be either positive or negative depending on the sign of the energy.

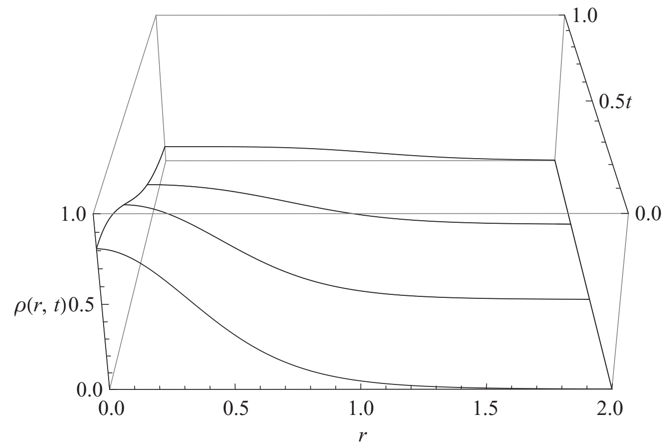

FIG. 9. The time evolution of the probability density (4.40), where $[\rho] = \mathrm{m}^{-3}$, $[r] = \mathrm{m}$, and $a = 1$ m, showing the spreading of the wave function (4.39).

E. Massless particle in three dimensions

In this section we investigate a free massless quantum particle in three dimensions described by the Salpeter equation

$$i\frac{\partial\phi(\boldsymbol{x},t)}{\partial t} = \sqrt{-\Delta}\,\phi(\boldsymbol{x},t), \tag{4.37}$$

where we set $c = 1$. Taking into account the form of the Fourier transformation (4.7) of the solution to (4.1) corresponding to the case of a free massless particle in one dimension, one can easily guess the following solution to (4.37):

$$\phi(\boldsymbol{x},t) = \frac{C}{(2\pi)^{\frac{3}{2}}}\int d^3\boldsymbol{p}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}}e^{-it|\boldsymbol{p}|}e^{-a|\boldsymbol{p}|}, \tag{4.38}$$

where C is a normalization constant and $a > 0$ is a parameter. Hence, we get the normalized solution to (4.37), which is a plausible three-dimensional generalization of the solution (4.8), such that

$$\phi(\boldsymbol{x},t) = \frac{(2a)^{\frac{3}{2}}}{\pi}\frac{a+it}{[r^2+(a+it)^2]^2}, \tag{4.39}$$

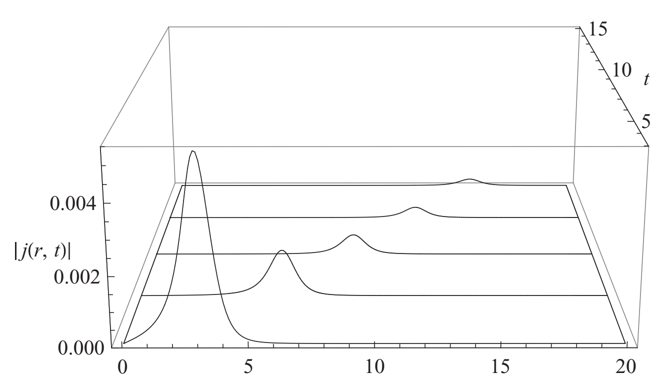

FIG. 10. The time development of the norm of the probability current (4.41), where $[j] = \mathrm{m}^{-3}$ and $a = 1$ m.

where $r = |\boldsymbol{x}|$. From (4.39) it follows immediately that the probability density is

$$\rho(\boldsymbol{x},t) = |\phi(\boldsymbol{x},t)|^2 = \frac{(2a)^3}{\pi^2}\frac{a^2+t^2}{[(r^2-t^2+a^2)^2+4a^2t^2]^2}. \tag{4.40}$$

The time evolution of the probability density (4.40) is presented in Fig. 9. It appears that, unlike the case of a free particle in one dimension (see Fig. 1), the wave function (4.39) spreads out.

Furthermore, using the definition (3.8) one can derive the following formula for the probability current:

$$\boldsymbol{j}(\boldsymbol{x},t) = \left\{-\frac{a^3}{2\pi^2r^2t^3}\frac{32r^2t^4+(r^2+t^2+a^2)[3(r^2-t^2+a^2)^2+12a^2t^2-8r^2t^2]}{[(r^2-t^2+a^2)^2+4a^2t^2]^2}+\frac{3a^3}{8\pi^2r^3t^4}\ln\frac{(r+t)^2+a^2}{(r-t)^2+a^2}\right\}\boldsymbol{x}. \tag{4.41}$$

The plot of the norm of the probability current (4.41) is shown in Fig. 10. As expected the norm approaches zero as time passes.

V. CONCLUSION

In this work we study the probability current for a quantum spinless relativistic particle based on the Salpeter equation as the fundamental equation of the relativistic quantum mechanics. The introduced probability current and related probability density show a very good behavior free of pathologies occurring in the case of the Klein-Gordon equation such as the negative probability density or the lack of the massless limit of the probability current. Referring to the nonlocality, which is often indicated as a grave flaw of the Salpeter equation, we have not observed any ill behavior of the discussed general as well as particular exact solutions. Quite the opposite, the introduced probability current satisfying Eq. (3.10) excludes in view of (3.12) the particle velocities greater than the speed of light. We realize that the point remains concerning the lack of the manifest covariance of the theory, which is usually pointed out as the second main disadvantage of the Salpeter equation. However, it is our belief that this flaw can be circumvented by using the preferred frame approach, which has been already successfully applied for the Lorentz covariant localization in quantum mechanics [38], relativistic EPR correlations [39], and Lorentz covariant formulation of classical and quantum statistical mechanics [40,41].

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