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Enhanced quantum communication via optical refocusing

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We consider the problem of quantum communication mediated by a passive optical refocusing system. The model captures the basic features of all those situations in which a signal is either refocused by a repeater for long-distance communication, or it is focused on a detector prior to the information decoding process. Introducing a general method for linear passive optical systems, we determine the conditions under which optical refocusing implies information transmission gain. Although the finite aperture of the repeater may cause loss of information, we show that the presence of the refocusing system can substantially enhance the rate of reliable communication with respect to the free-space propagation. We explicitly address the transferring of classical messages over the quantum channel, but the results can be easily extended to include the case of transferring quantum messages as well.

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Although quantum information is more commonly described in terms of discrete variables (e.g., qubits), information is most naturally encoded in the electromagnetic field (EMF) via a continuous variable representation [1]. All the fundamental quantum information tools and protocols, from teleportation to quantum key distribution, have been demonstrated for such encoding [2]. Here we consider the problem of quantum communication [3–5] and compute the maximum rate at which information can be reliably transmitted through EMF signals that propagate along an optical communication line under refocusing conditions. Even though we explicitly consider classical information [6], our results are immediately extensible to the case of quantum information [7].

In the classical domain, the ultimate limits for communication via continuous variable encodings were provided by the seminal work of Shannon on Gaussian channels [8]. In the quantum domain such channels are replaced by the so-called bosonic Gaussian channels (BGCs), which describe the propagation of the EMF in linear media [9]. Their structure is notably rich [10], and a full information-theoretical characterization has been achieved only for certain special subclasses [11,12]. These results have been applied to compute the maximal rates of reliable communication via attenuating media, such as optical fibers, wave guides, and via free-space propagation [4,13,14].

Here we move further in this direction by characterizing the propagation of the EMF through a linear optical system, which induces refocusing of the transmitted signals to overcome the attenuation associated with their diffusion. For the sake of simplicity, we model this apparatus as a thin lens with a finite pupil placed between the sender of the message and the receiver under focusing conditions. Notwithstanding its relatively simple structure, this model captures the basic features of all those situations in which a signal is either refocused by a suitable passive repeater to allow long-distance communication (e.g., by means of a parabolic antenna for satellite communication [15]) or it is focused on a detector prior to the information decoding process (the latter includes, e.g., the settings in which the EMF is used for the readout process and those in which an optical system is used to interface light and matter [16]). Our analysis is complementary with those schemes that employ active strategies to improve the quality of the signaling along lossy communication lines (e.g., the active repeaters protocols of Ref. [17] and references therein, or quantum wave form conversion techniques discussed for instance in Ref. [18]).

The quantum description of the scattering of the signal by the optical system is derived by consistently applying the canonical quantization rules (see Ref. [4] and references therein). Within this framework we derive the conditions under which the refocusing properties of the optical system allow higher communication rates, hence putting on a formal and quantitative ground the benefits of optical refocusing. It turns out that, while the finiteness of the pupil limits the channel bandwidth [19], the resulting improvement with respect to the free-space scheme may be particularly advantageous for faint-pulse communication—a configuration that is close to the operative regimes of the long-distance free-space communication protocols [15,20].

The optical system. Consider a linear optical system with a set of transmitter modes, labeled by i, and receiver modes, labeled by j. In the case of rf or microwave communication, for example, the transmitter and receiver could be antennae. For optical communication, the transmitter could be a laser coupled to a telescope, and the receiver could be a telescope coupled to a charge-coupled device (CCD) array. Transmitter and receiver modes typically have both spatial characteristics determined by the optical characteristics of the transmitter and receiver and temporal characteristics determined by the frequency and bandwidth of the transmitted radiation. The transmittivity matrix T_{ji} gives the fraction of light from the ith transmitter mode that is received at the jth receiver mode. We would like to determine the maximum amount of information that can be sent from transmitter to receiver for fixed total input power.

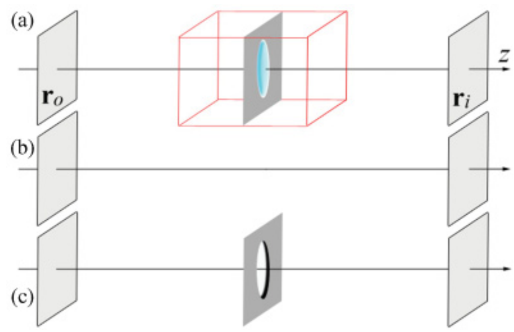

FIG. 1. (Color online) (a) Scheme for optical communication through an optical refocusing apparatus, modeled as a thin lens of radius R and focal length f. (b) Free-space propagation scenario. (c) Alternative scenario in which the lens is replaced by a hole of the same size in the absorbing screen. $\mathbf{r}_o$ and $\mathbf{r}_i$ denote Cartesian coordinates on the object and image planes.

Let us consider the purely lossy case, in which noise from the environment is negligible. This is the case, for example, for free-space optical communication in a thermal background. The addition of noise will be considered below. If the loss is η and there are ν parallel channels with total average photon number N, then the associated classical communication capacity, measured in nats, is [11]

$$C = \nu g(\eta N/\nu), \tag{1}$$

where $g(x) := (x+1)\ln(x+1) - x\ln x$, and the capacity is attained by sending coherent states down the channel.

In our case, we have a multimode lossy channel with transfer matrix T, which mixes the input modes together. Via singular value decomposition we write it as $T = \mathcal{V}\Sigma\mathcal{U}$, where $\mathcal{V}$, $\mathcal{U}$ are unitary matrices, and Σ is a nonnegative diagonal matrix whose (diagonal) entries $\sqrt{\eta_k}$ are the singular values of T. This shows that our multimode lossy channel can be decomposed into a set of parallel (noninteracting) lossy channels with input modes corresponding to the rows of $\mathcal{U}$, output modes corresponding to the columns of $\mathcal{V}$, and loss factors corresponding to the singular values η_k. The new input modes can now be quantized using annihilation and creation operators $a_i, a_i^\dagger$: $[a_i, a_{i'}^\dagger] = \delta_{i,i'}$. Similarly, the output modes can be quantized using operators $b_i, b_i^\dagger$: $[b_i, b_{i'}^\dagger] = \delta_{i,i'}$. To preserve the canonical quantization relationships, each input-output pair is also coupled to a loss mode with operators $\xi_i, \xi_i^\dagger$, $b_i = \sqrt{\eta_i} a_i + \sqrt{1-\eta_i}\xi_i$.

We consider the case of monochromatic light propagating along an optical axis. Following [4,5] the input and output signals are identified by the transverse field modes at two planes orthogonal to the optical axis [the object plane and image plane of Fig. 1(a)], and the field propagation is defined by assigning the point-spread function (PSF) $T(\mathbf{r}_i,\mathbf{r}_o)$, which connects the field amplitude at position $\mathbf{r}_o$ on the first plane with the field amplitude at position $\mathbf{r}_i$ on the second one [21]. Due to diffraction, such inputs are scattered over the whole image plane according to amplitude probability distributions defined by the PSF. This setting formally defines a Gaussian memory channel [23], in which output signals originated by distinct input fields are not mutually independent.

We model the optical refocusing system as a converging lens of focal length f, located at distance D_o from the object plane. Working in the thin-lens approximation, and neglecting aberrations, light is focused at the image plane located at distance D_i from the optical system, where $1/D_o + 1/D_i = 1/f$. Eventually the image is magnified by a factor $M = D_i/D_o$. Diffraction of light is responsible for image blurring and causes loss of information. It can be described by introducing an effective entrance pupil characterizing the optical system. Denoting $P(\mathbf{r})$ the characteristic function of the pupil that encircles the lens, the PSF for the monochromatic field at wavelength λ is obtained, in the paraxial approximation, by Fourier transforming $P(\mathbf{r})$ [21]. For a circular pupil of radius R, the PSF reads

$$T(\mathbf{r}_i,\mathbf{r}_o) = \frac{e^{j\vartheta(\mathbf{r}_i,\mathbf{r}_o)}R^2}{\lambda^2 D_o D_i}\frac{J_1(2\pi R\rho)}{R\rho}, \tag{2}$$

where J_1 is the Bessel function of first kind and order one, $\vartheta(\mathbf{r}_i,\mathbf{r}_o) = \frac{\pi}{\lambda D_o}(|\mathbf{r}_o|^2 + |\mathbf{r}_i|^2/M) + \frac{2\pi D_o}{\lambda}(1+M)$, and $\rho = |\mathbf{r}_i - M\mathbf{r}_o|/(\lambda D_i)$. The PSF in Eq. (2) accounts for two physical phenomena: (i) The focusing properties of the converging lens; (ii) The loss of the field components impinging outside the pupil of the optical system. In other words, Eq. (2) assumes the presence of an absorbing screen surrounding the lens. Eventually, one could derive a PSF describing the propagation through a lens of a radius R that is not surrounded by an absorbing screen, allowing the transfer of the field components that are not refocused by the lens. However, for the sake of conciseness, in the following we adopt the expression in Eq. (2) and show that the presence of the converging lens increases the communication capacity, with respect to the free-space propagation, in the settings in which the signal loss caused by the absorbing screen is negligible.

To characterize our optical refocusing system, we apply the singular value decomposition to the PSF, which plays the role of the transfer matrix. The system is hence characterized by a set of loss factors. In the far-field and near-field limits, they can be computed exactly in terms of the Fresnel number associated to the optical system [22]. To fix the ideas, let us assume that information is encoded in the object plane on a square of length L, creating an image on the image plane which (in the geometric optics approximation) is roughly contained in a square of size ML. The Fresnel number associated to this setup is

$$\mathcal{F} = \frac{\pi R^2 L^2}{\lambda^2 D_o^2} = \pi\left(\frac{L}{x_{\mathsf{R}}}\right)^2, \tag{3}$$

$x_{\mathsf{R}} = \lambda D_o/R$ being the Rayleigh length of the system. In the far-field limit, $\mathcal{F} \ll 1$, only one mode is transmitted with loss $\eta = \mathcal{F}^2$. In the near-field limit, $\mathcal{F} \gg 1$, a number $\nu = \mathcal{F}$ of modes are transmitted without loss.

To evaluate the effects of the refocusing system on the information transmission, we use the free-space propagation of the EMF as a term of comparison [see Fig. 1(b)]. The characterization of the free-space propagation of the field, including the quantum regime, can be found in Ref. [4] and references therein. For this scenario, the associated Fresnel number is $\mathcal{F}_{\mathrm{FS}} = (A_1A_2)/(\lambda d)^2$, where A_1 and A_2 are the areas of the surface from which the signal is emitted and on which it is detected, and d is the distance between them. For a fair comparison, we use for both the scenarios the same values for areas of the input and output surfaces and their distance, yielding the free-space Fresnel number

$$\mathcal{F}_{\rm FS} = \frac{M^2 L^4}{\lambda^2 D^2} = \frac{L^4}{\lambda^2 D_o^2}\left(\frac{M}{1+M}\right)^2, \tag{4}$$

where $D = D_o + D_i = D_o(1+M)$. Also in this case, it is possible to derive exact expressions for the effective transmissivities in the far-field and near-field limit: In the far-field limit, $\mathcal{F}_{\rm FS} \ll 1$, only one mode is transmitted, with loss $\eta_{\rm FS} = \mathcal{F}_{\rm FS}$; In the near-field limit, $\mathcal{F}_{\rm FS} \gg 1$, a number of modes $\nu_{\rm FS} = \mathcal{F}_{\rm FS}$ are transmitted without losses.

Optical refocusing vs free-space propagation. First of all we notice that, for given values of the physical parameters, the two scenarios, in the following denoted by (a) and (b), can independently operate in the far-field or near-field limit, or in neither of the two.

Let us consider first the case in which both the scenarios operate in the far-field regime. The ratio between the loss factors of the transmitted modes equals

$$r_1 = \frac{\eta}{\eta_{\rm FS}} = \left(\frac{\pi R^2}{\lambda D_o}\right)^2 \left(\frac{1+M}{M}\right)^2, \tag{5}$$

which is larger than one only if the parameters M, R, D_o, λ do satisfy a certain condition. We now show that such a condition is fulfilled when the loss of signal induced by absorbing screen surrounding the pupil is negligible. (Notice that for applications in quantum cryptography r_1 coincides with the gain in the secret-key rate [24].) To do so, we consider a third scenario, denoted by (c) in Fig. 1, in which the converging lens is replaced by a hole of the same size in the absorbing screen. The field propagation in this configuration can be analyzed by splitting it into two parts: the free-space propagation from the object plane to the screen ($o \to s$) and the one from the screen to the image plane ($s \to i$). These two free-space propagations are associated to the Fresnel numbers $\mathcal{F}_{\rm FS}^{o\to s} = \pi R^2 L^2/\lambda^2 D_o^2 = \mathcal{F}$, and $\mathcal{F}_{\rm FS}^{s\to i} = \pi R^2 (ML)^2/\lambda^2 D_i^2 = \mathcal{F}$. Hence the far-field condition on the scenario (a) implies that both those propagations take place in the far-field regime as well. It follows that there is, in the scenario (c), at most one mode propagating $o \to s \to i$, which is attenuated by a factor

$$\eta^{o\to s\to i} \leqslant \mathcal{F}_{\rm FS}^{o\to s}\mathcal{F}_{\rm FS}^{s\to i} = \mathcal{F}^2 = \eta. \tag{6}$$

Now, if the presence of the absorbing screen around the pupil is negligible, we must have that the losses on the $o \to s \to i$ propagation (quantified by the factor $\eta^{o\to s\to i}$) are equal to those gotten from direct free-space propagation (quantified by $\eta_{\rm FS}$). But since $\eta^{o\to s\to i}$ is not greater than η, it follows that the regime in which we can neglect the effect of the absorbing screen is the one in which $r_1 \geqslant 1$. In other words, the detrimental effects that we see for $r_1 < 1$ merely correspond to the absorptions by the screen. As a final remark we also observe that the condition $r_1 > 1$, plus the far-field condition for the scenario (a), enforces the far-field regime for the scenario (b). Finally, we compare the performances of the two scenarios in terms of capacity by computing the gain

$$G_1 = \frac{C}{C_{\rm FS}} = \frac{g(\eta N)}{g(\eta_{\rm FS} N)} = \frac{g(r_1 \eta_{\rm FS} N)}{g(\eta_{\rm FS} N)}. \tag{7}$$

We notice that in the semiclassical limit, $N \gg 1$, the gain satisfies $G_1 \simeq 1$, that is, the presence of the optical refocusing system does not affect the information transmission capacity [25]. On the other hand, the gain can be significantly greater than 1 in the quantum regime: In particular, the gain is maximum for faint signals, $N \ll 1$, in which $G_1 \simeq r_1$.

Let us now move to the case in which both scenarios operate in the near-field regime. The ratio between the numbers of modes perfectly transmitted is

$$r_2 = \frac{\nu}{\nu_{\rm FS}} = \pi\left(\frac{R}{L}\right)^2 \left(\frac{1+M}{M}\right)^2, \tag{8}$$

which can be larger or smaller than one, depending on the geometric parameters M, R, L. However, as in the previous case, we show that if the losses induced by the absorbing screen are negligible, then $r_2 \geqslant 1$. Again, to show that let us consider what happens in the scenario (c). We first notice that the near-field condition for the scenario (a) implies that both $o \to s$ and $s \to i$ propagations are in the near-field regime. The numbers of transmitted modes are $\nu_{\rm FS}^{o\to s} = \pi L^2 R^2/\lambda^2 D_o^2 = \nu$ and $\nu_{\rm FS}^{s\to i} = \pi (ML)^2 R^2/\lambda^2 D_i^2 = \nu$. Hence the number of modes that propagate from the object to the image plane in the scenario (c) satisfies the inequality

$$\nu^{o\to s\to i} \leqslant \min\left\{\nu_{\rm FS}^{o\to s}, \nu_{\rm FS}^{s\to i}\right\} = \nu. \tag{9}$$

It is clear that the presence of the pupil is negligible only if the number of modes transmitted in the propagation $o \to s \to i$ equals the number of those transmitted in the free-space propagation, that is, if $\nu^{o\to s\to i} \simeq \nu_{\rm FS}$. Equation (9) implies that this is the setting for which $r_2 \geqslant 1$. The ratio between the classical capacities of the corresponding quantum channels reads

$$G_2 = \frac{C}{C_{\rm FS}} = \frac{\nu\, g(N/\nu)}{\nu_{\rm FS}\, g(N/\nu_{\rm FS})} = r_2 \frac{g(N/\nu)}{g(r_2 N/\nu)}. \tag{10}$$

Notice that the ratio N/ν represents the number of photons per transmitted mode. In the limit $N/\nu \gg 1$ we are in the semiclassical regime, for which the gain is $G_2 \simeq r_2$. In the quantum regime $N/\nu \simeq 1$ we have $G_2 > 1$. Finally, for faint signals, $N/\nu \ll 1$, the gain tends to $G_2 \simeq 1$.

One may notice that the condition $r_2 \geqslant 1$, together with the near-field condition for the scenario (a), is not sufficient to infer the near-field condition for the scenario (b). Hence we shall compare the near-field case for the scenario (a) with the far-field case for the scenario (b), a setting which is characterized by the condition

$$\frac{L^2}{D_o}\frac{M}{M+1} \ll \lambda \ll \frac{LR}{D_o}.$$

In this case the gain becomes

$$G_3 = \frac{C}{C_{\rm FS}} = \frac{\nu g(N/\nu)}{g(\eta_{\rm FS} N)}, \tag{11}$$

which, in the semiclassical limit, $N \gg 1$, is $G_3 \simeq \nu \gg 1$, and for faint signals, $N \ll 1$, is $G_3 \simeq 1/\eta_{\rm FS} \gg 1$.

The enhancement in the transmission rate provided by the optical refocusing system persists in the presence of background thermal noise. In such a case, by encoding classical information into coherent states, Eq. (1) has to be replaced by $C = g(\eta N/\nu + N_{\rm TH}) - g(N_{\rm TH})$ [9,26], where

N_{TH} is the number of thermal photons per transmitted mode, which yields $G_1 \simeq r_1$, $G_3 \simeq 1/\eta_{\mathrm{FS}}$ for $N_{\mathrm{TH}} \gg \max\{1, \eta N/\nu\}$, and $G_2 \simeq r_2$, $G_3 \simeq \nu$ for $1 \gg N_{\mathrm{TH}} \gg \eta N/\nu$.

Conclusions. We have computed the capacity of quantum optical communication through an optical refocusing system, modeled as a thin lens with finite pupil. Despite its simplicity, the model is general enough to find application in different contexts, from refocusing antennas for long-distance communication to imaging systems on the small and medium scale, and accounts for the focusing process, light diffraction, and power loss. We have shown that, under certain conditions, the converging optical apparatus can be used to achieve, in comparison with the free-space field propagation, higher transmission rates. The tradeoff between loss and diffraction determines the conditions under which the intuitive benefits of optical refocusing can be formalized rigorously. Our results furnish the ultimate limits of quantum optical communication and may be useful for determining general bounds on the efficiency of any protocol requiring the transmission of quantum degrees of freedom of light (e.g., quantum imaging [27] and quantum discrimination [28]).

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