# Nonlinear features of quantum fluctuations in a slow-inversion laser

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We study the field correlation functions and the natural laser spectrum for a slow-inversion laser with the thresholdless intensity fluctuations. In such a laser, the spontaneous-emission-driven relaxation oscillations are responsible for some unique features: The field correlation function oscillates with the frequency of relaxation oscillations $\omega_r$ and the laser line changes its profile. We show that appreciably above threshold the laser spectrum consists of three lines. In addition to the main line at the frequency of the lasing mode $\omega_c$ appear two lines at frequencies $\omega_c + \omega_r$ and $\omega_c - \omega_r$. The spontaneous emission noise in such lasers gives rise to nonlinear stochastic effects: Quantum noise affects the oscillation frequency $\omega_r$ and the decay rates of the field correlation function.

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## I. INTRODUCTION

Quantum noise of laser radiation occurs from spontaneous emission of photons into a lasing mode. Evolution of quantum fluctuations depends on the relation between three relaxation rates in a laser: $\gamma_p$ for the atomic polarization, $\gamma_N$ for the population inversion, and $\Gamma_c$ for the field intensity in the cavity. In a class-A laser [1], when $\gamma_p,\gamma_N \gg \Gamma_c$, one can, due to fast relaxation in an active medium, adiabatically eliminate the atomic polarization and the populations of laser levels. In such lasers, the relative photon-number fluctuations undergo a sharp reduction at the laser threshold (see, for example, Refs. [2–5]). A different behavior of quantum fluctuations appears in slow-inversion lasers of small cavity size [6–10]. Such lasers belong to class B [1] ($\gamma_p \gg \Gamma_c \gg \gamma_N$). In class-B lasers, photon-number fluctuations may be very high, even far above threshold [6–10]. Such behavior of quantum fluctuations was called thresholdless [6].

In previous investigations, the intensity fluctuations and photon statistics of a slow-inversion laser were studied theoretically and experimentally [6–10]. The questions not considered before concern the optical spectrum and the field correlation function in lasers with the thresholdless intensity fluctuations. These problems were studied in previous publications using a class-A laser description (see, for example, Refs. [2–5,11–13]) and also for a bad-cavity laser (when $\Gamma_c \gg \gamma_p,\gamma_N$) [14,15]. In all these studies the relative intensity fluctuations above threshold are low and thus the effect of intensity fluctuations on the linewidth can be neglected. In this paper, we study the field correlation function and the laser spectrum for lasers with thresholdless intensity fluctuations. As is well known, class-B lasers exhibit relaxation oscillations. We show that, in lasers with the thresholdless intensity fluctuations, such oscillations excited by spontaneous-emission noise are responsible for some unique features: oscillating behavior of the field correlation function and change of the laser-line profile. In the presence of spontaneous emission noise, nonlinear stochastic effects appear in such lasers: Quantum noise affects the oscillation frequency and the decay rates of the field correlation function.

This paper is structured as follows. In Sec. II we describe the theory based on the quasilinear treatment of quantum fluctuations. In Sec. III we present the results of numerical simulations, before drawing our conclusions in Sec. IV.

## II. THEORY

For a study of quantum fluctuations we choose a solid-state laser on the Nd:$YVO_4$ (Neodymium doped Yttrium Vanadate) crystal. A laser configuration making it possible to observe the thresholdless intensity fluctuations was described in Ref. [6]. We use the same approximations as in Ref. [6]: A single-mode laser operating at maximum of a homogeneously broadened gain line in the good-cavity limit ($\gamma_p \gg \Gamma_c$), spontaneous emission is considered as the main source of the quantum noise; the decay rate of the lower laser level is assumed to be large and the lower-level population can be neglected.

### A. Langevin equations

Using a class-B description one can derive the classical Langevin equations for the internal complex amplitude of the cavity mode $\alpha = \alpha_x + i\alpha_y$ and the population inversion $N$ (number of the upper-level atoms) [3,16]:

$$\begin{aligned}\dot{\alpha}_x &= \alpha_x(\beta\gamma_N N - \Gamma_c)/2 + F_x,\\ \dot{\alpha}_y &= \alpha_y(\beta\gamma_N N - \Gamma_c)/2 + F_y,\\ \dot{N} &= P - \gamma_N N - \beta\gamma_N N n,\end{aligned} \tag{1}$$

where $\beta$ is the fraction of the spontaneous emission going into the lasing mode [6], $P$ is the pump rate, and $n = \alpha_x^2 + \alpha_y^2$ is the intracavity photon number. The Langevin noise source $F = F_x + iF_y$ has the following correlation functions: $\langle F_x(t)F_x(t')\rangle = \langle F_y(t)F_y(t')\rangle = \beta\Gamma_c N\delta(t-t')/2$, $\langle F_x(t)F_y(t')\rangle = 0$.

One can derive from (1) the rate equations for $n$ and $N$, which were used in Ref. [6] under the study of quantum fluctuations:

$$\dot{n} = (\beta\gamma_N N - \Gamma_c)n + R_{\rm sp} + f_n, \tag{2a}$$

$$\dot{N} = P - \gamma_N N - \beta\gamma_N N n, \tag{2b}$$

where $R_{\rm sp} = N\beta\gamma_N$ is the spontaneous emission rate and the quantum noise source $f_n$ satisfies $\langle f_n(t)f_n(t')\rangle = 2R_{\rm sp}n\delta(t-t')$. As in Ref. [6], we neglect the inversion noise source $f_N$ in Eqs. (1) and (2b). The influence of inversion noise and justification for neglecting this noise source was discussed in Ref. [6].

 

### B. Field correlation function and laser spectrum

For a class-A laser, detailed studies of the field correlation functions were performed in Ref. [11]. Here we consider the correlation functions for a class-B laser. Writing the complex amplitude $\alpha$ in the form $\alpha = \sqrt{n}e^{i\phi}$, we have for the field correlation function $\gamma(t)$ and the amplitude correlation function $G(t)$:

$$\gamma(t) = \langle \alpha^*(0)\alpha(t)\rangle/\langle n\rangle = \langle\sqrt{n(0)n(t)}e^{i\Phi(t)}\rangle/\langle n\rangle, \quad (3)$$

$$G(t) = \langle\sqrt{n(0)n(t)}\rangle/\langle n\rangle, \quad (4)$$

where $\Phi(t) = \phi(t) - \phi(0)$.

We calculate $\gamma(t)$ and $G(t)$ using a quasilinear treatment of amplitude fluctuations. With a quasilinear treatment, the amplitude and phase fluctuations are not correlated, and one has from Eq. (3)

$$\gamma(t) = G(t)\langle e^{i\Phi(t)}\rangle. \quad (5)$$

The steady state ($\langle n\rangle = n_0, \langle N\rangle = N_0$) can be calculated from Eqs. (2). One easily finds [6]

$$n_0 = [\eta + \sqrt{\eta^2 + 4\beta(1+\eta)}]/2\beta, \quad (6a)$$

$$N_0 = \frac{\Gamma_c}{\beta\gamma_N}\frac{n_0}{n_0+1}, \quad (6b)$$

where we use the normalized pump rate $P/P_{th} = 1+\eta$, with $\eta$ the pump excess above threshold.

We write $n(t) = n_0 + \delta n(t)$. Linearizing Eqs. (2) around the steady state, one can find the following expression for the photon number fluctuations $\delta n(t)$:

$$\delta n(t) = a e^{-\gamma_r t}\cos(\omega_r t + \psi_0), \quad (7)$$

where $a$, $\psi_0$ are an amplitude and an initial phase of the relaxation oscillations, respectively, $\omega_r$ is the relaxation oscillation frequency given by

$$\omega_r = \sqrt{\beta n_0\gamma_N\Gamma_c - (\gamma_1-\gamma_2)^2}, \quad (8)$$

$\gamma_1 = \gamma_N(1+\beta n_0)$, $\gamma_2 = \Gamma_c/n_0$, and

$$\gamma_r = (\gamma_1+\gamma_2)/2 \quad (9)$$

is the damping rate of the relaxation oscillations. We do not take into account in Eq. (7) the noise source $f_n$ as the noise is delta correlated and yields no contribution in the correlation function (4) at any time $t > 0$. Such an account would be necessary only in the case of a low-band noise with a bandwidth $\Delta\omega \leqslant \gamma_r$.

Substitution of Eq. (7) in Eq. (4) yields

$$\begin{aligned} G(t) &\cong 1 + \frac{\langle\delta n(0)\delta n(t)\rangle - \langle\delta n(0)^2\rangle}{2n_0^2} \\ &= 1 + A[e^{-\gamma_r t}\cos(\omega_r t) - 1], \end{aligned} \quad (10)$$

where

$$A = \frac{\langle a^2\cos^2\psi_0\rangle}{2n_0^2}. \quad (11)$$

A contribution in Eq. (5) from phase fluctuations $\langle e^{i\Phi(t)}\rangle$ may be calculated in a quasilinear approximation by the same way as in a class-A laser (see, for example, Ref. [3]) and is given by

$$\langle e^{i\Phi(t)}\rangle = e^{-Dt/2}, \quad (12)$$

where

$$D = \Gamma_c/2n_0 \quad (13)$$

is the phase diffusion constant.

Using Eqs. (10) and (12) we have from Eq. (5)

$$\gamma(t) = [1 + A(e^{-\gamma_r t}\cos(\omega_r t) - 1)]e^{-Dt/2}. \quad (14)$$

Equations (10) and (14) for the correlation functions may be used if $\omega_r$ is real [see Eq. (8)]:

$$\beta n_0\gamma_N\Gamma_c > (\gamma_1-\gamma_2)^2. \quad (15)$$

This is typically fulfilled in a class-B laser well above threshold. In a solid-state laser on the Nd:$YVO_4$ crystal [6] with $\beta = 1\times10^{-5}$, $\Gamma_c = 1\times10^{11}$ s$^{-1}$, $\gamma_N = 1.3\times10^4$ s$^{-1}$, one can find from (15) that $\omega_r$ is real if $\eta > 0.058$.

Near threshold (at $\eta \ll 1$), one can find an approximate expression for the field correlation function $\gamma(t)$ by the same way as in Ref. [12]. One can neglect fluctuations $\delta N$ and put $N = \langle N\rangle = N_0$ in Eqs. (1), then one finds from Eqs. (1)

$$\alpha(t) = \alpha(0)e^{-\Gamma t}, \quad (16)$$

where

$$\Gamma = (\beta\gamma_N N_0 - \Gamma_c)/2. \quad (17)$$

Again, in Eq. (16), we do not take into account at $t \neq 0$ the noise source $F = F_x + iF_y$ as it gives no contribution in $\gamma(t)$. With $\alpha(t)$ given by Eq. (16), one has from Eq. (3)

$$\gamma(t) = e^{-\Gamma t}. \quad (18)$$

Substitution of Eq. (6b) in Eq. (17) yields

$$\Gamma = D = \Gamma_c/2n_0. \quad (19)$$

The power spectrum of the laser field (the laser spectrum) is defined as the Fourier transform of the field correlation function $\gamma(t)$:

$$S(\omega) = \int dt\, e^{i\omega t}\gamma(t). \quad (20)$$

Near threshold, one can conclude from Eqs. (18) and (20) that the laser line has the Lorentz shape with a linewidth $\Delta\omega = 2D$. Appreciably above threshold, the laser-line profile changes. Using Eq. (14), one finds from Eq. (20)

$$\begin{aligned} \frac{S(\omega)}{1-A} &= \frac{D/2}{\omega^2 + D^2/4} + \frac{A}{2(1-A)}\frac{D/2+\gamma_r}{(\omega+\omega_r)^2 + (D/2+\gamma_r)^2} \\ &\quad + \frac{A}{2(1-A)}\frac{D/2+\gamma_r}{(\omega-\omega_r)^2 + (D/2+\gamma_r)^2}. \end{aligned} \quad (21)$$

It follows from Eq. (21) that the laser spectrum consists of three lines. In addition to the main line at the frequency of the lasing mode $\omega_c$, the two lines appear at frequencies $\omega_c + \omega_r$ and $\omega_c - \omega_r$.

Equations (10), (14), (18), and (21) describing the correlation functions, and the laser spectrum was derived at a quasilinear approximation. Such treatment does not take into account nonlinear stochastic effects which appear in a class-B laser in the presence of noise [17,18]. It was shown in Refs. [17] and [18] that the relaxation oscillation frequency $\omega_r$ and the decay rate of relaxation oscillations $\gamma_r$ depend on the noise intensity. In what follows we show that such effects arise in lasers with thresholdless intensity fluctuations.

## III. NUMERICAL SIMULATIONS

In numerical simulations, stochastic differential equations (1) were solved by the Euler method using the Box-Mueller algorithm for the generation of Gaussian noise $F_{x,y}$ [19]. We used the following values of the parameters of a solid-state laser on the Nd:$YVO_4$ crystal [6]: $\beta = 1 \times 10^{-5}$, $\Gamma_c = 1 \times 10^{11}\ \mathrm{s}^{-1}$, $\gamma_N = 1.3 \times 10^4\ \mathrm{s}^{-1}$.

At first, we describe results obtained near threshold ($\eta \ll 1$). Figure 1(a) shows the field correlation function $\gamma(t)$ at $\eta = 0.05$. The results of numerical simulation (solid line) agree with the theoretical expression (18) but, for better agreement, one should slightly modify the value of $\Gamma$ given by (19), replacing it on $\Gamma_e = 1.05\Gamma$ [see Fig. 1(a), dashed curve]:

$$\gamma(t) = e^{-\Gamma_e t}. \tag{22}$$

Figure 1(b) shows the amplitude correlation function $G(t)$ at $\eta = 0.9$. The results of numerical simulation are shown by the solid line. The dashed curve 1 is a plot of Eq. (10) with the constant $A = 0.19$ used as a fitting parameter and the values of $\omega_r$, $\gamma_r$ calculated using Eqs. (8) and (9). The results obtained in the numerical simulation for $G(t)$ may be approximated (see the dashed curve 2) by the expression

$$G(t) = 1 + A[e^{-\gamma_e t}\cos(\omega_e t) - 1]. \tag{23}$$

The oscillation frequency $\omega_e$ and the decay rate $\gamma_e$ of the amplitude correlation function $G(t)$ found in the numerical simulation (solid line) are different from $\omega_r$ and $\gamma_r$ calculated at the quasilinear approximation. In the case of $\eta = 0.9$, at the parameters used in the simulations, the shifted frequency $\omega_e$ and decay rate $\gamma_e$ are $\omega_e = 0.9\omega_r$, $\gamma_e = 3.5\gamma_r$.

Consider qualitatively the physical mechanism explaining the difference observed. The relaxation oscillations in the laser can be considered as oscillations of a nonlinear oscillator, e.g., oscillator Toda (see, for example, Refs. [10] and [20]). The oscillation frequency of a nonlinear oscillator depends on its amplitude (nonisochronism). In the oscillator Toda, the oscillation frequency decreases with increasing amplitude. The

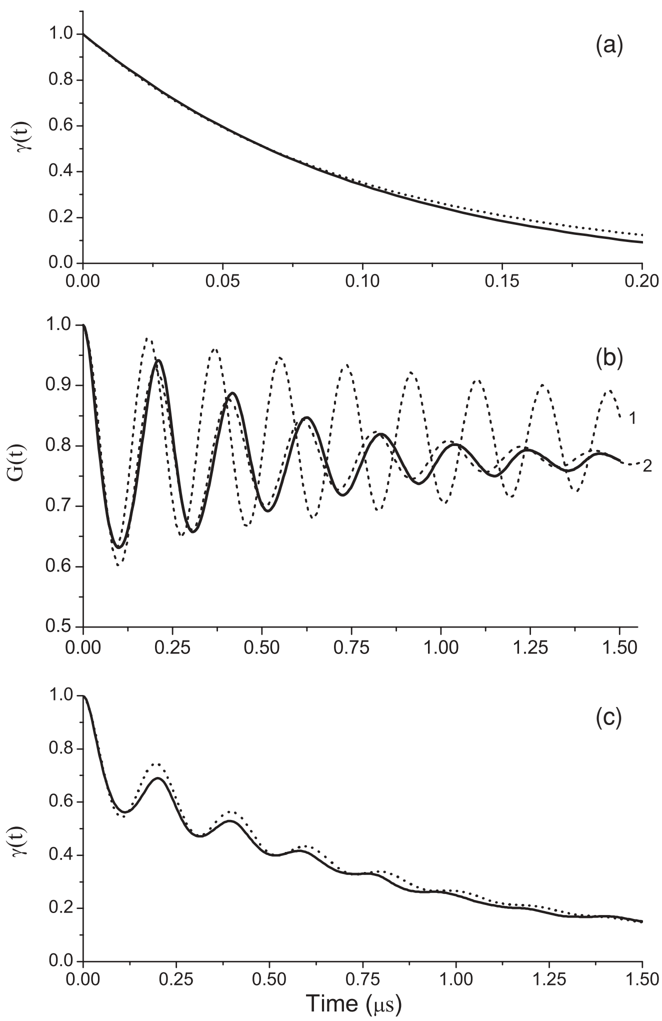


FIG. 1. The correlation functions $\gamma(t)$ and $G(t)$ at different values of the pump excess above threshold $\eta$. The solid lines show the results of numerical simulation. Function $\gamma(t)$ is shown (a) at $\eta = 0.05$, and the dashed curve is a plot of Eq. (22) with $\Gamma_e = 1.05\Gamma$. Function $G(t)$ is shown (b) at $\eta = 0.9$, and the dashed curve 1 is a plot of Eq. (10) with the constant $A = 0.19$ used as a fitting parameter; the values of $\omega_r$, $\gamma_r$ were calculated using Eqs. (8) and (9); the dashed curve 2 is a plot of Eq. (23) with $A = 0.22$, the shifted frequency $\omega_e = 0.9\omega_r$, and the decay rate $\gamma_e = 3.5\gamma_r$. Function $\gamma(t)$ is shown (c) at $\eta = 0.9$, and the dashed curve is a plot of Eq. (24) with $A = 0.22$, $\omega_e = 0.9\omega_r$, $\gamma_e = 3.5\gamma_r$, and $D_e = 4D$.

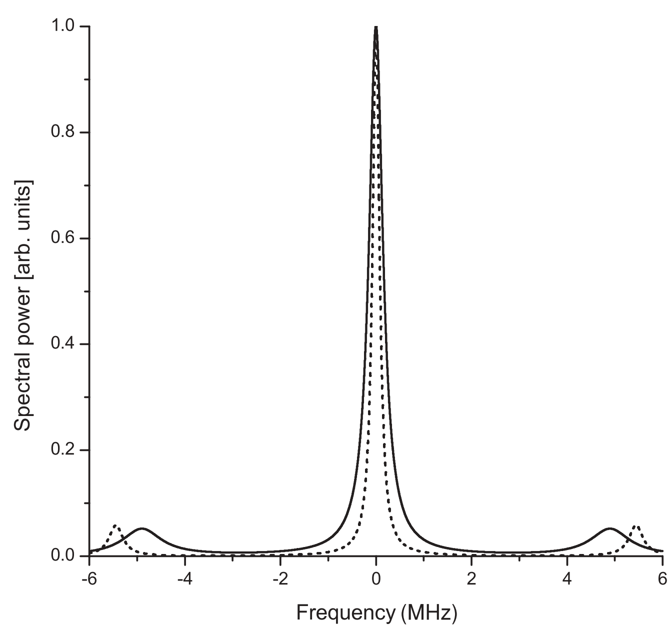


FIG. 2. Laser spectrum at the pump excess above threshold $\eta = 0.9$. The solid line is a plot of the laser spectrum $S(\omega) = \int dt\, e^{i\omega t}\gamma(t)$ with $\gamma(t)$ given by Eq. (24) with $A = 0.22$, $\omega_e = 0.9\omega_r$, $\gamma_e = 3.5\gamma_r$, and $D_e = 4D$. The dashed curve shows the analytical results (quasilinear treatment) given by Eq. (21).

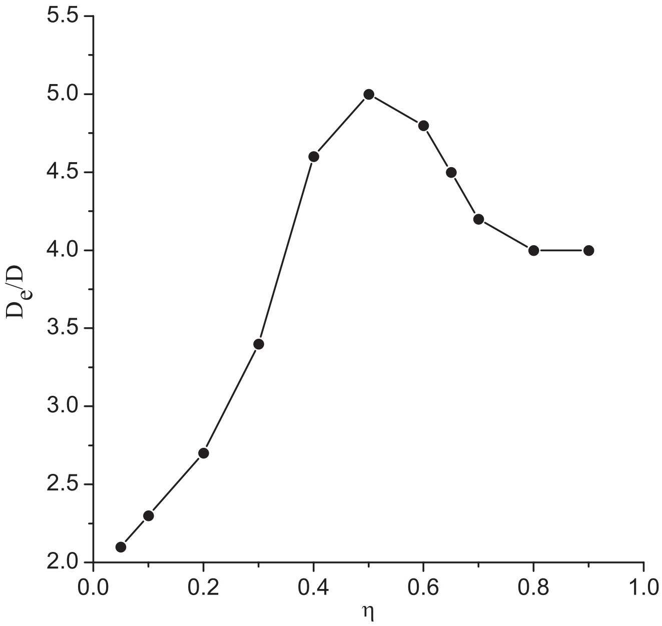


FIG. 3. The ratio $D_e/D$ representing the nonlinear stochastic broadening of the central peak as a function of the pump excess above threshold $\eta$. The points are the results of the numerical simulation.

correlation functions and the power spectra of the noise-driven nonlinear oscillators have been studied by many authors (see, for example, Ref. [21]). Usually, a decay rate of oscillations increases with increasing noise intensity.

Numerical simulations show that, in the presence of noise, the field correlation function $\gamma(t)$ may be approximated as

$$\gamma(t) = 1 + A[e^{-\gamma_e t}\cos(\omega_e t) - 1]e^{-D_e t/2}. \tag{24}$$

In the case of $\eta = 0.9$, at the parameters used in the simulations, the effective diffusion constant $D_e = 4D$, where $D$ is the phase diffusion constant found in a quasilinear approximation [see Eq. (13)].

The laser spectrum $S(\omega) = \int dt\, e^{i\omega t}\gamma(t)$ corresponding to Eq. (24) is shown in Fig. 2 by the solid line. The dashed line plots the laser spectrum at the quasilinear approximation given by Eq. (21). All peaks in the spectrum (the main peak with the width $D_e$ and two small peaks with the width $2\gamma_e + D_e$) are broader than at the quasilinear approximation. In the numerical simulations and in the quasilinear analytic treatment the laser spectrum consists of three lines. In addition to the main line at the frequency of the lasing mode $\omega_c$, the two lines appear at frequencies $\omega_c + \omega_r$ and $\omega_c - \omega_r$. It is of interest to study these results experimentally. The laser spectrum $S(\omega)$ can be measured experimentally by means of self-heterodyne detection [15].

Relative nonlinear broadening of the main peak given by the ratio $D_e/D$ is shown in Fig. 3 as a function of the pump excess above threshold $\eta$. Numerical simulations in this figure have been plotted for the values of $\eta$ up to 0.9. The reason for such a limitation is the following. As it was shown in Ref. [6], to properly describe quantum fluctuations in Nd:$YVO_4$ microchip lasers, one needs to extend the Langevin equations (1) and (2), including a dynamic equation for the lower-level population. Due to the rapid decay rate $\gamma_b$ of the lower level (compared to $\gamma_N$ of the upper level), the lower-level population is small, but it can significantly modify the photon damping term $\gamma_2 = \Gamma_c/n_0$ in (8) and (9): $\gamma_2$ is replaced by $\gamma_2 + \gamma_{NL}$, where $\gamma_{NL}$ is an additional damping term caused by the lower-level dynamics. For the considered laser on Nd:$YVO_4$, the ratio $\gamma_{NL}/\gamma_2 = \eta^2\gamma_N/\beta\gamma_b$ is of the order of 1 at $\eta = 1$. Thus, in numerical simulations at $\eta \geqslant 1$, one should take into account a dynamic equation for the lower-level population. This will be done in future studies. The thresholdless quantum fluctuations can be observed in rare-earth microchip lasers [22]. It is of interest to study the field correlation functions and the natural laser spectrum in such lasers.

## IV. CONCLUSION

In summary, we have shown that in a slow-inversion laser of small cavity size the field correlation function oscillates at the relaxation oscillation frequency, and appreciably above threshold the laser spectrum has three peaks. In such a laser, the spontaneous emission noise gives rise to nonlinear stochastic effects: a noise-induced change of the relaxation oscillation frequency and broadening of all peaks in the laser spectrum.

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