Casimir-Polder repulsion near edges: Wedge apex and a screen with an aperture

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Although repulsive effects have been predicted for quantum vacuum forces between bodies with nontrivial electromagnetic properties, such as between a perfect electric conductor and a perfect magnetic conductor, realistic repulsion seems difficult to achieve. Repulsion is possible if the medium between the bodies has a permittivity in value intermediate to those of the two bodies, but this may not be a useful configuration. Here, inspired by recent numerical work, we initiate analytic calculations of the Casimir-Polder interaction between an atom with anisotropic polarizability and a plate with an aperture. In particular, for a semi-infinite plate, and, more generally, for a wedge, the problem is exactly solvable, and for sufficiently large anisotropy, Casimir-Polder repulsion is indeed possible, in agreement with the previous numerical studies. In order to achieve repulsion, what is needed is a sufficiently sharp edge (not so very sharp, in fact) so that the directions of polarizability of the conductor and the atom are roughly normal to each other. The machinery for carrying out the calculation with a finite aperture is presented. As a motivation for the quantum calculation, we carry out the corresponding classical analysis for the force between a dipole and a metallic sheet with a circular aperture, when the dipole is on the symmetry axis and oriented in the same direction.

I. INTRODUCTION

There has been increasing interest in utilizing the quantum vacuum force or the Casimir effect in nanotechnology employing mesoscopic objects [1]. Although the original Casimir effect, between parallel conducting or dielectric plates separated by vacuum [2,3], always gives an attractive force between the plates, introducing a material (liquid) with an intermediate value of the dielectric constant can result in repulsion [4], which has now been observed [5]; for precursors, see [6–10]. [The first experimental test of the Lifshitz theory with an intermediate liquid (helium) was that of Sabisky and Anderson [11]; application of the Lifshitz theory to the melting of water ice was considered by Elbaum and Schick [12].] A recent experiment involving air bubbles in a liquid with boundary walls is described in Ref. [13]. However, this type of repulsion is unlikely to have many applications in building devices.

There are well-known repulsive quantum forces in vacuum. The first example was found by Boyer [14]. He computed the self-stress of a perfectly conducting spherical shell due to quantum electrodynamic field fluctuations and found a repulsive result, but the meaning of such a self-energy is extremely obscure. He later found [15] a more observable effect, that the force between a perfect electrically conducting plane (ε, the permittivity, goes to infinity) and a parallel perfect magnetic conducting plane (μ, the permeability, goes

to infinity) is repulsive. This, again, may be a difficult situation to approximately replicate in practice, because the unusual magnetic properties must persist over a wide frequency range.

There has been extensive interest in designing metamaterials that could give rise to Casimir repulsion by simulating a magnetic response [16–21]. Despite some early optimism, the conclusion seems to have transpired that repulsion is impossible between metamaterials made from dielectric and metallic components [22–24]. For recent attempts combining dielectric and magnetic setups, see Refs. [25–27].

Several years ago there was an interesting suggestion by Sopova and Ford [28] that the force between a small dielectric sphere and a dielectric wall was oscillatory, so there were a number of repulsive regimes. However, this effect was canceled by plasmon modes leaving the usual attractive result [29]. Earlier Ford had suggested [30] that the frequency response of materials might be manipulated in order to achieve repulsion, but this was proved to be impossible [31].

Thus it was extremely interesting when Levin *et al.* showed examples of repulsion between conducting objects, in particular between an elongated cylinder above a conducting plane with a circular aperture [32]. (An analytic counterpart is given in [33].) They first gave examples of repulsive forces between arrays of electric dipoles, and an electric dipole and a conducting plane with an aperture cut out. Then they turned to quantum vacuum forces between conducting objects, computed by quite impressive "brute force" finite-difference time-domain and boundary-element methods.

The purpose of the present paper is to try to understand these phenomena analytically. We first show, in Sec. II, that there is no repulsion possible in the weak coupling regime, where because the materials are dilute one may sum Casimir-Polder interactions between atoms [34]. However, there is repulsion in classical electrostatics between a system of three dipoles

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(Sec. III) and between a fixed dipole and a conducting plane with an aperture, which we discuss in Sec. IV, both in two and three dimensions. This is an interesting pedagogical problem, for it involves mixed coupled integral equations, like those for an electrified disk, or a plane with an aperture with different constant electric fields at large distances above and below the punctured plane [35]. These problems exhibit closed form solutions, and clearly exhibit repulsion when the dipole is directly above the aperture and is sufficiently close. In Sec. V we turn to the real problem, that of the Casimir-Polder force between an anisotropic polarizable atom and a punctured dielectric plane. Because solving the integral equations arising for the Green's dyadic for the plate with aperture is rather complicated, in Sec. VI we content ourselves with computing the Casimir-Polder interaction between a polarizable atom and a perfectly conducting wedge. When the opening angle of the wedge approaches 2π, this describes the interaction between an atom and a semi-infinite conducting plane. We exhibit situations in which repulsive forces in certain directions can arise for anisotropic atoms, in qualitative agreement with numerical work [32]. In Appendix A we give another derivation of the Casimir-Polder energy formula for the wedge, based on a closed form for the Green's dyadic, and in Appendix B we give a classical calculation of a conducting ellipsoid above a conducting plate with a circular aperture in the presence of a background field.

A word about terminology: When we say "atom" we mean any microscopic particle which may be described by a polarizability tensor. Our calculations assume that we are in the retarded regime, so that static (frequency-independent) polarizabilities may be employed. Should lower frequency transitions dominate (which could occur with some molecules), so that the separations are in the nonretarded regime, electrostatic results are valid [but for a factor of $1/2$—See Eq. (4.15) below and Ref. [36]].

In this paper we set $\hbar = c = 1$, and all results are expressed in Gaussian units except that Heaviside-Lorentz units are used for Green's dyadics.

II. WEAK COUPLING CALCULATION

A. Scalar field

We first illustrate the ideas by considering the case of a massless scalar field in two dimensions. The quantum vacuum energy between two weakly coupled potentials V_1 and V_2 is

$$U_{12} = -\frac{1}{32\pi^2} \int (d\mathbf{r})(d\mathbf{r}') \frac{V_1(\mathbf{r})V_2(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2}, \quad (2.1)$$

the scalar analog of the Casimir-Polder force between atoms. Here we consider the potentials as shown in Fig. 1, which represents a needle of length L on the symmetry axis a distance Z above a line with a gap of width a. The potentials are given by

$$V_1(x,z) = \lambda_1 \delta(x) \theta(z - Z + L/2) \theta(Z + L/2 - z), \quad (2.2a)$$

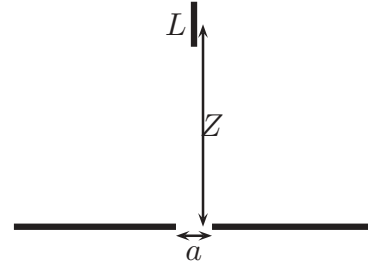$$V_2(x,z) = \lambda_2 \delta(z) [\theta(x - a/2) + \theta(-x - a/2)]. \quad (2.2b)$$

FIG. 1. Two-dimensional geometry of a needle of length L a distance Z above a line with a gap of width a.

This means that the interaction energy is

$$U_{12} = -\frac{\lambda_1\lambda_2}{32\pi^2} \int_{Z-L/2}^{Z+L/2} dz \left\{ \int_{a/2}^{\infty} + \int_{-\infty}^{-a/2} \right\} dx \frac{1}{x^2 + z^2}. \quad (2.3)$$

To get the force on the needle, we simply have to integrate on x, and differentiate with respect to the limits of the z integral:

$$F = -\frac{\partial}{\partial Z} U_{12} = \frac{\lambda_1\lambda_2}{8\pi^2 a} \left[\frac{\arctan(2Z/a + L/a)}{2Z/a + L/a} - \frac{\arctan(2Z/a - L/a)}{2Z/a - L/a} \right], \quad (2.4)$$

which, because $F < 0$, always represents an attractive force between the punctured line and the needle. Note that although the force vanishes at $Z = 0$, the energy there, which represents the work done in bringing the needle in from infinity, is not zero.

B. Electromagnetic field

Now we consider the quantum vacuum force between dilute dielectric media, which may be obtained from the Casimir-Polder potential between isotropic polarizable atoms [34],

$$U_{\text{CP}} = -\frac{23}{4\pi}\alpha_1\alpha_2 \frac{1}{r^7}, \quad (2.5)$$

where r is the distance between the atoms. We might mention that Eq. (2.5) is in general valid in the retarded limit where the atomic polarizability can be regarded as constant. (For more details, see the review [37].) The result is applicable provided that the atom-plate separation is much greater than the atomic transition wavelength (typically some hundreds of nanometers for ground-state atoms). The media have dielectric constants $\varepsilon_i = 1 + 4\pi N_i \alpha_i$, where N_i represents the density of atoms of type i. Specifically, we consider a three-dimensional configuration, in which an atom of isotropic polarizability α is placed on the symmetry axis a distance Z above a dielectric plate of thickness t with a circular hole in the middle, as shown in Fig. 2.
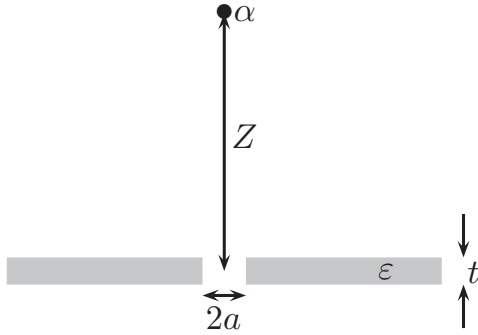
FIG. 2. Three-dimensional geometry of a polarizable atom a distance Z above a dielectric slab with a circular aperture of radius a.

The quantum interaction energy is

$$U = -\frac{23}{(4\pi)^2}\alpha(\varepsilon-1)\int_{\text{slab}}(d\mathbf{r})\frac{1}{[(z-Z)^2+r_\perp^2]^{7/2}}$$

$$= -\frac{23}{60\pi a^4}\alpha(\varepsilon-1)\left[\frac{(t+2Z)[6a^2+(t+2Z)^2]}{[4a^2+(t+2Z)^2]^{3/2}}\right.$$

$$\left.+(Z\rightarrow-Z)\right]. \qquad (2.6)$$

It is easy to see that the force $F = -\partial U/\partial Z$ is always negative (i.e., attractive).

A more favorable case for possible repulsion would be an anisotropic atom. It is easy to derive the appropriate generalization of the Casimir-Polder potential in this case, starting from the weak-coupling multiple scattering formula [38],

$$U_{12} = \frac{i}{2}\,\text{Tr}\,\mathbf{\Gamma}_0 V_1\mathbf{\Gamma}_0 V_2, \qquad (2.7)$$

where the free Green's dyadic is ($\zeta = -i\omega$),

$$\mathbf{\Gamma}_0(\mathbf{r},\mathbf{r}') = (\nabla\nabla - \mathbf{1}\zeta^2)\frac{e^{-|\zeta||\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \qquad (2.8)$$

Following the procedure given in Ref. [38], we find for an isotropic medium facing an anisotropic atom,

$$U = \frac{\varepsilon-1}{32\pi^2}\int_{\text{slab}}(d\mathbf{r})\frac{1}{|\mathbf{r}-\mathbf{R}|^7}$$

$$\times\left[13\,\text{tr}\,\boldsymbol{\alpha}+7\frac{(\mathbf{r}-\mathbf{R})\cdot\boldsymbol{\alpha}\cdot(\mathbf{r}-\mathbf{R})}{(\mathbf{r}-\mathbf{R})^2}\right], \qquad (2.9)$$

where $R = (0,0,Z)$ is the position of the atom, relative to the center of the aperture. This may be easily checked to reduce to the usual Casimir-Polder result (2.5) when $\boldsymbol{\alpha} = \alpha\mathbf{1}$.

Let us consider the extreme case when only α_{zz} is significant. Then the integrals may be easily carried out, with the result,

$$U = \frac{\alpha_{zz}(\varepsilon-1)}{60\pi a^4}\left[\frac{t+2Z}{[4+(t+2Z)^2]^{5/2}}[156a^4+70a^2(t+2Z)^2\right.$$

$$\left.+7(t+2Z)^4]+(Z\rightarrow-Z)\right]. \qquad (2.10)$$

This, again, always gives rise to an attractive force.

An interesting special case is when the aperture is small compared to the thickness of the dielectric. Then the energy is a step function,

$$U = -\frac{7}{30\pi a^4}\alpha_{zz}(\varepsilon-1)\theta(t-2|Z|), \quad a\ll t, \quad (2.11)$$

which gives rise to a δ-function force just when the atom enters and exits the aperture. If the aperture is very large compared to the thickness of the slab, $t\ll a$, the energy and force are proportional to the thickness of the slab,

$$U = -\frac{1}{80\pi a^4}\alpha_{zz}(\varepsilon-1)\frac{13a^2+18Z^2}{(a^2+Z^2)^{7/2}}a^4t. \qquad (2.12)$$

III. CLASSICAL DIPOLE INTERACTION

It is possible to achieve a repulsive force between a configuration of fixed dipoles. Consider the situation illustrated in Fig. 3. Here we have two dipoles, of strength d_2 and d_3 lying along the x axis, separated by a distance a. A third dipole of strength d_1 lies along the z axis. If the two parallel dipoles are oppositely directed and of equal strength,

$$\mathbf{d}_2 = -\mathbf{d}_3 = d_2\hat{\mathbf{x}}, \qquad (3.1)$$

and are equally distant from the z axis, and the dipole on the z axis is directed along that axis,

$$\mathbf{d}_1 = d_1\hat{\mathbf{z}}, \qquad (3.2)$$

the force on that dipole is along the z axis:

$$F_z = 3ad_1d_2\frac{a^2/4-4Z^2}{(Z^2+a^2/4)^{7/2}}, \qquad (3.3)$$

which changes sign at $Z = a/4$; that is, for distances Z larger than this, the force is attractive (in the $-z$ direction) while for shorter distances the force is repulsive (in the $+z$ direction). Evidently, by symmetry, the dipole-dipole energy vanishes at $z = 0$. Consistent with Earnshaw's theorem, the point where the force vanishes is an unstable point with respect to deviations in the x direction.

In view of this self-evident finding, it might seem surprising that the interaction between a polarizable atom and a dilute medium (made up of polarizable atoms) studied in Sec. II B failed to exhibit a repulsive regime, but this is because the medium is isotropic.
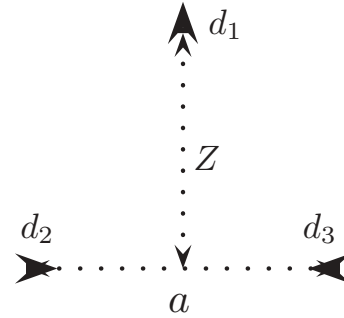

FIG. 3. Configuration of three dipoles, two of which are antiparallel, and one perpendicular to the other two.

IV. CLASSICAL INTERACTION BETWEEN A DIPOLE AND A CONDUCTING PLANE WITH AN APERTURE

In this section, we consider the interaction between a dipole and a perfectly conducting plane containing an aperture. We first consider two dimensions. (As above, we denote the Cartesian coordinates by x and z for uniformity with the three-dimensional situation.)

A. Dipole above aperture in a conducting line

If we use the Green's function which vanishes on the entire line $z = 0$,

$$G(\mathbf{r},\mathbf{r}') = -\ln[(x-x')^2 + (z-z')^2] + \ln[(x-x')^2 + (z+z')^2], \quad (4.1)$$

so

$$G(x,0;x',z') = 0, \quad (4.2)$$

we can calculate the electrostatic potential at any point above the $z = 0$ plane to be

$$\phi(\mathbf{r}) = \int_{z>0} (d\mathbf{r}') G(\mathbf{r},\mathbf{r}') \rho(\mathbf{r}') + \frac{1}{4\pi} \int_{\text{ap}} dS' \left. \frac{\partial}{\partial z'} G(\mathbf{r},\mathbf{r}') \right|_{z'=0} \phi(\mathbf{r}'), \quad (4.3)$$

where the volume integral is over the charge density of the dipole,

$$\rho(\mathbf{r}) = -\mathbf{d} \cdot \nabla \delta(\mathbf{r} - \mathbf{R}), \quad \mathbf{R} = (0,Z). \quad (4.4)$$

The surface integral extends only over the aperture because the potential vanishes on the conducting sheet. If we choose $\mathbf{d}$ to point along the z axis we easily find

$$\phi(x,z > 0) = 2d\left[\frac{z-Z}{x^2 + (z-Z)^2} + \frac{z+Z}{x^2 + (z+Z)^2}\right] + \frac{1}{\pi} \int_{-a/2}^{a/2} dx' \frac{z}{(x-x')^2 + z^2} \phi(x',0), \quad (4.5)$$

where a is the width of the aperture.

Now the free Green's function in two dimensions is

$$G_0(\mathbf{r},\mathbf{r}') = 4\pi \int \frac{(d\mathbf{k})}{(2\pi)^2} \frac{e^{ik_x(x-x')} e^{ik_z(z-z')}}{k_x^2 + k_z^2} = \int_{-\infty}^{\infty} dk_x \frac{1}{|k_x|} e^{ik_x(x-x')} e^{-|k_x||z-z'|}. \quad (4.6)$$

Then the surface integral in Eq. (4.5) is

$$\int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{ik_x x} e^{-|k_x|z} \tilde{\phi}(k_x), \quad (4.7)$$

in terms of the Fourier transform of the field,

$$\tilde{\phi}(k_x) = \int_{-\infty}^{\infty} dx' e^{-ik_x x'} \phi(x',0) = 2 \int_0^{a/2} dx' \cos k_x x' \phi(x',0), \quad (4.8)$$

since $\phi(x,0)$ must be an even function for the geometry considered. Thus we conclude

$$\phi(x,z > 0) = 2d\left[\frac{z-Z}{x^2 + (z-Z)^2} + \frac{z+Z}{x^2 + (z+Z)^2}\right] + \frac{1}{\pi} \int_0^{\infty} dk \cos kx\, e^{-kz} \tilde{\phi}(k). \quad (4.9)$$

This becomes an identity as $z \to 0$.

The electric field in the aperture is

$$E_z(x,z = 0+) = -\left. \frac{\partial}{\partial z} \phi(x,z) \right|_{z=0+} = -4d \frac{x^2 - Z^2}{(x^2 + Z^2)^2} + \frac{1}{\pi} \int_0^{\infty} dk\, k \cos kx\, \tilde{\phi}(k). \quad (4.10)$$

On the other side of the aperture, there is no charge density, so for $z < 0$ the potential is

$$\phi(x,z < 0) = \frac{1}{\pi} \int_0^{\infty} dk \cos kx\, e^{kz} \tilde{\phi}(k), \quad (4.11)$$

so the z component of the electric field in the aperture is

$$E_z(x,z = 0-) = -\left. \frac{\partial}{\partial z} \phi(x,z) \right|_{z=0-} = -\frac{1}{\pi} \int_0^{\infty} dk\, k \cos kx\, \tilde{\phi}(k). \quad (4.12)$$

Because we require that the electric field be continuous in the aperture, and the potential vanish on the conductor, we obtain the two coupled integral equations for this problem,

$$4d \frac{x^2 - Z^2}{(x^2 + Z^2)^2} = \frac{2}{\pi} \int_0^{\infty} dk\, k \cos kx\, \tilde{\phi}(k), \quad 0 < |x| < a/2, \quad (4.13a)$$

$$0 = \int_0^{\infty} dk \cos kx\, \tilde{\phi}(k), \quad |x| > a/2. \quad (4.13b)$$

In fact, these equations have a simple solution [39]:

$$\tilde{\phi}(k) = -\frac{4Zd\pi}{a} \int_0^1 dx\, x \frac{J_0(kax/2)}{(x^2 + 4Z^2/a^2)^{3/2}}. \quad (4.14)$$

From this, we can work out the energy of the system from

$$U = -\frac{1}{2} d E_z(0,Z) = \frac{1}{2} d \left. \frac{\partial \phi}{\partial z} \right|_{z=Z,x=0}, \quad (4.15)$$

where the factor of 1/2 comes from the fact that this must be the energy required to assemble the system. In computing this energy we must, of course, drop the self-energy of the dipole due to its own field. We are then left with

$$U_{\text{int}} = -\frac{d^2}{4Z^2} - \frac{d}{2\pi} \int_0^{\infty} dk\, k\, e^{-kZ} \tilde{\phi}(k) = -\frac{d^2}{4Z^2} + Z^2 d^2 \left(\frac{2}{a}\right)^4 \int_0^1 \frac{1}{2} dx^2 \frac{1}{(x^2 + 4Z^2/a^2)^3} = -\frac{4Z^2 d^2}{(a^2 + 4Z^2)^2}, \quad (4.16)$$

where to get the second line we used the derivative of Eq. (4.17). This is exactly two times larger that the result

quoted in Ref. [32].[1] Since this vanishes at $Z = 0$ and $Z = \infty$, the force must change from attractive to repulsive, which happens at $Z = a/2$.

B. Three-dimensional aperture interacting with dipole

It is quite straightforward to repeat the above calculation in three dimensions. Again we are considering a dipole, polarized on the symmetry axis, a distance Z above a circular aperture of radius a in a conducting plate.

The free three-dimensional Green's function in cylindrical coordinates has the representation,

$$\frac{1}{\sqrt{\rho^2 + z^2}} = \int_0^\infty dk \, J_0(k\rho) \, e^{-k|z|}, \qquad (4.17)$$

and so if we follow the above procedure we find for the potential above the plate,

$$\phi(\mathbf{r}_\perp, z > 0) = d\left[\frac{z - Z}{[r_\perp^2 + (z - Z)^2]^{3/2}} + \frac{z + Z}{[r_\perp^2 + (z + Z)^2]^{3/2}}\right]$$
$$+ \int_0^\infty dk \, k \, e^{-kz} J_0(kr_\perp)\Phi(k), \qquad (4.18)$$

where the Bessel transform of the potential in the aperture is

$$\Phi(k) = \int_0^\infty d\rho \, \rho \, J_0(k\rho)\phi(\rho, 0). \qquad (4.19)$$

Thus the integral equations resulting from the continuity of the z component of the electric field in the aperture and the vanishing of the potential on the conductor are

$$d\frac{r_\perp^2 - 2Z^2}{[r_\perp^2 + Z^2]^{5/2}} = \int_0^\infty dk \, k^2 J_0(kr_\perp)\Phi(k), \quad r_\perp < a,$$
$$(4.20a)$$

$$0 = \int_0^\infty dk \, k \, J_0(kr_\perp)\Phi(k), \quad r_\perp > a. \qquad (4.20b)$$

The solution to these equations is given in Titchmarsh's book [40], and after a bit of manipulation we obtain

$$\Phi(k) = -\left(\frac{2ka}{\pi}\right)^{1/2}\frac{d}{ka}\int_0^1 dx \, x^{3/2} J_{1/2}(xka)\frac{2Z/a}{(x^2 + Z^2/a^2)^2}. \qquad (4.21)$$

Then the energy (4.15) may be easily evaluated using

$$\int_0^\infty dk \, k^{3/2} e^{-kZ} J_{1/2}(kax) = 2\sqrt{\frac{2xa}{\pi}}\frac{Z}{(x^2a^2 + Z^2)^2}. \quad (4.22)$$

The energy can again be expressed in closed form:

$$U = -\frac{d^2}{8Z^3} + \frac{d^2}{4\pi Z^3}\left[\arctan\frac{a}{Z}\right.$$
$$\left. + \frac{Z}{a}\frac{1 + 8/3(Z/a)^2 - (Z/a)^4}{(1 + Z^2/a^2)^3}\right]. \qquad (4.23)$$

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[1]This is not the factor of 1/2 in Eq. (4.15). It is not possible to trace the origin of the discrepancy, since the authors of that reference quote the result without details.

This is always negative, but vanishes at infinity and at zero:

$$Z \to 0: \quad U \to -\frac{4}{5\pi}d^2\frac{Z^2}{a^5}. \qquad (4.24)$$

This means that for some value of $Z \sim a$ the force changes from attractive to repulsive. Numerically, we find that the force changes sign at $Z = 0.742358a$.

The reason why the energy vanishes when the dipole is centered in the aperture is clear: Then the electric field lines are perpendicular to the conducting sheet on the surface, and the sheet could be removed without changing the field configuration.

Our goal is to analytically find the quantum (Casimir) analog of this classical repulsion.

V. STRONG COUPLING—FORCE BETWEEN AN ATOM AND A PUNCTURED PLANE DIELECTRIC

Now we turn to the real problem. Our starting point is the general expression for the vacuum energy [38]:

$$U = \frac{i}{2}\,\mathrm{Tr}\ln\mathbf{\Gamma}\mathbf{\Gamma}_0^{-1}, \qquad (5.1)$$

where $\mathbf{\Gamma}$ is the full Green's dyadic for the problem, and $\mathbf{\Gamma}_0^{-1}$ is the inverse of the free Green's dyadic (2.8), namely,

$$\mathbf{\Gamma}_0^{-1} = \frac{1}{\omega^2}\nabla \times \nabla \times -\mathbf{1}. \qquad (5.2)$$

In the presence of a potential $\mathbf{V}$, the full Green's dyadic has the symbolic form,

$$\mathbf{\Gamma} = (\mathbf{1} - \mathbf{\Gamma}_0\mathbf{V})^{-1}\mathbf{\Gamma}_0. \qquad (5.3)$$

Here we are thinking of the interaction between a dielectric medium, characterized by an isotropic permittivity, so $V_1 = \varepsilon - 1$, and a polarizable atom, represented by a polarizability dyadic, as shown in Fig. 2,

$$\mathbf{V}_2 = 4\pi\boldsymbol{\alpha}\delta(\mathbf{r} - \mathbf{R}), \qquad (5.4)$$

where $\mathbf{R}$ is the position of the dipole. We are only interested in a single interaction with the latter potential, so we have for the interaction energy,

$$U_{12} = \mathrm{Tr}\,\mathbf{V}_2\frac{\delta}{\delta V_1}\left[-\frac{i}{2}\ln(1 - \mathbf{\Gamma}_0 V_1)\right]$$
$$= \frac{i}{2}\,\mathrm{Tr}(\mathbf{\Gamma}_1 - \mathbf{\Gamma}_0)\mathbf{V}_2, \qquad (5.5)$$

where we have used Eq. (5.3) for the potential V_1 describing the dielectric slab plus aperture and we have subtracted the term that represents the self-energy of the atom with its own field. This subtraction happens automatically if we start from the "$TGTG$" form,

$$U_{12} = -\frac{i}{2}\,\mathrm{Tr}\ln(\mathbf{1} - \mathbf{\Gamma}_1\mathbf{V}_1\mathbf{\Gamma}_2\mathbf{V}_2)$$
$$\approx \frac{i}{2}\,\mathrm{Tr}\,\mathbf{\Gamma}_1\mathbf{V}_1\mathbf{\Gamma}_0\mathbf{V}_2 = \frac{i}{2}\,\mathrm{Tr}(\mathbf{\Gamma}_1 - \mathbf{\Gamma}_0)\mathbf{V}_2, \qquad (5.6)$$

because $\mathbf{V}_2$ is weak. This implies the Casimir-Polder expression for the interaction between the polarizable atom and the

dielectric,

$$U_{CP} = -\int_{-\infty}^{\infty} d\zeta \, \mathrm{tr} \, \boldsymbol{\alpha} \cdot (\boldsymbol{\Gamma} - \boldsymbol{\Gamma}_0)(Z,Z). \qquad (5.7)$$

We could also derive this result from the formula for the force on a dielectric body in an inhomogeneous electric field [36],

$$\mathbf{F} = -\frac{1}{8\pi} \int (d\mathbf{r}) E^2(\mathbf{r}) \nabla \varepsilon, \qquad (5.8)$$

which classically says that a dielectric body experiences a force pushing it into the region of the stronger field. This implies the interaction energy,

$$U = -\tfrac{1}{2}\alpha E^2(\mathbf{0},Z), \qquad (5.9)$$

and when we make the quantum-field-theoretic replacement,

$$\frac{1}{4\pi} \langle \mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}') \rangle \to \frac{1}{i}\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}') = \frac{1}{i}\int \frac{d\omega}{2\pi}\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}';\omega), \quad (5.10)$$

we recover the static isotropic version of Eq. (5.7) after the self-energy is subtracted.

A. No aperture

When the aperture is not present, we are considering the well-studied case of a dielectric slab, of thickness t, interacting with a polarizable atom. Because the Green's dyadic in this situation, denoted $\boldsymbol{\Gamma}^{(0)}$, then possesses translational invariance in the x-y plane, we can express it in terms of a reduced Green's dyadic,

$$\boldsymbol{\Gamma}^{(0)}(\mathbf{r},\mathbf{r}') = \int \frac{(d\mathbf{k}_\perp)}{(2\pi)^2} e^{i\mathbf{k}_\perp \cdot (\mathbf{r}-\mathbf{r}')_\perp} \mathbf{g}(z,z';k_\perp). \qquad (5.11)$$

In the case of an isotropic atom, the trace of the Green's dyadic occurs, which is for the reduced Green's dyadic,

$$\mathrm{tr}\,\mathbf{g}(Z,Z) = -\zeta^2 g^H(Z,Z) + \left(\frac{\partial}{\partial Z}\frac{\partial}{\partial Z'} + k_\perp^2\right) g^E(Z,Z')\bigg|_{Z'=Z}, \qquad (5.12)$$

in terms of the transverse electric (H) and transverse magnetic (E) Green's functions. These subtracted quantities are for z,z' above the dielectric,

$$g^{H,E}(z,z') - g_0^{H,E}(z,z') = \frac{1}{2\kappa}R^{H,E}e^{-\kappa(z+z'-t)}, \qquad (5.13)$$

in terms of the reflection coefficients,

$$R^H = \frac{\kappa - \kappa'}{\kappa + \kappa'} + 4\frac{\kappa\kappa'}{\kappa'^2 - \kappa^2}\frac{1}{D}, \qquad (5.14a)$$

$$R^E = \frac{\kappa - \bar{\kappa}'}{\kappa + \bar{\kappa}'} + 4\frac{\kappa\bar{\kappa}'}{\bar{\kappa}'^2 - \kappa^2}\frac{1}{\bar{D}}, \qquad (5.14b)$$

where

$$\kappa^2 = k_\perp^2 + \zeta^2, \quad \kappa'^2 = k_\perp^2 + \varepsilon\zeta^2, \quad \bar{\kappa}' = \kappa'/\varepsilon, \qquad (5.15)$$

and

$$D = \left(\frac{\kappa + \kappa'}{\kappa - \kappa'}\right)^2 e^{2\kappa't} - 1, \qquad (5.16)$$

with $\bar{D}$ obtained from this by replacing κ' by $\bar{\kappa}'$ except in the exponent. These results are rather trivially obtained by multiple scattering arguments.

Now the interaction energy is

$$
\begin{aligned}
U &= -\alpha \int_{-\infty}^{\infty} d\zeta \int \frac{(d\mathbf{k}_\perp)}{(2\pi)^2}[-\zeta^2 R^H + (2k^2 + \zeta^2)R^E]\frac{1}{2\kappa}e^{-\kappa(2Z-t)} \\
&= -\frac{\alpha}{4\pi}\int_0^{\infty} d\zeta \int dk_\perp^2 \frac{1}{\kappa}e^{-2\kappa(Z-t/2)}\left\{(\varepsilon-1)\zeta^4\frac{e^{2\kappa't}-1}{(\kappa+\kappa')^2 e^{2\kappa't} - (\kappa-\kappa')^2}\right. \\
&\quad \left. + \frac{\varepsilon-1}{\varepsilon}(2k^2 + \zeta^2)\left[k^2\left(1 + \frac{1}{\varepsilon}\right) + \zeta^2\right]\frac{e^{2\kappa't}-1}{(\kappa+\kappa'/\varepsilon)^2 e^{2\kappa't} - (\kappa-\kappa'/\varepsilon)^2}\right\}.
\end{aligned} \qquad (5.17)
$$

This is precisely the result found, for example, by Zhou and Spruch [41].

B. Integral equations for Green's dyadic

We now specialize to the case where the plane $z=0$ consists of a perfectly conducting screen with a circular aperture of radius a at the origin. The Green's dyadic satisfies the differential equation,

$$\left(\frac{1}{\omega^2}\nabla \times \nabla \times -\mathbf{1}\right) \cdot \boldsymbol{\Gamma}(\mathbf{r} - \mathbf{r}') = \mathbf{1}\delta(\mathbf{r} - \mathbf{r}'), \qquad (5.18)$$

subject to the boundary conditions,

$$\hat{\mathbf{z}} \times \boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')|_{|\mathbf{r}_\perp|>a,z=0} = 0, \qquad (5.19)$$

which just states that the tangential components of the electric field must vanish on the conductor. Following Levine and Schwinger [42] we introduce auxiliary electric and magnetic Green's dyadics $\boldsymbol{\Gamma}^{(1,2)}(\mathbf{r},\mathbf{r}')$ which satisfy the same differential equation (5.18) but with the boundary conditions satisfied on the entire $z=0$ plane:

$$\hat{\mathbf{z}} \times \boldsymbol{\Gamma}^{(1)}(\mathbf{r},\mathbf{r}')|_{z=0} = 0, \quad \hat{\mathbf{z}} \times [\nabla \times \boldsymbol{\Gamma}^{(2)}(\mathbf{r},\mathbf{r}')]|_{z=0} = 0. \qquad (5.20)$$

These can be constructed in terms of the free Green's dyadic $\boldsymbol{\Gamma}_0$, subject only to outgoing boundary conditions at infinity, as given in Eq. (2.8),

$$\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}') = (\mathbf{1}\omega^2 + \nabla\nabla)G(|\mathbf{r} - \mathbf{r}'|), \qquad (5.21)$$

expressed in turn in terms of the Helmholtz Green's function,

$$G(R) = \frac{e^{i|\omega|R}}{4\pi R}. \tag{5.22}$$

We can write, after the Euclidean rotation $|\omega| \to i\zeta$, the free Green's dyadic in the explicit form ($\mathbf{R} = \mathbf{r} - \mathbf{r}'$),

$$\begin{aligned} \mathbf{\Gamma}_0(\mathbf{r},\mathbf{r}') = -\frac{G(R)}{R^2}\bigg[&\mathbf{1}(1 + \zeta R + \zeta^2 R^2) \\ &- \frac{\mathbf{R}\mathbf{R}}{R^2}(3 + 3\zeta R + \zeta^2 R^2)\bigg]. \end{aligned} \tag{5.23}$$

In terms of this last dyadic, the auxilliary Green's dyadics have the form,

$$\begin{aligned} z,z' > 0: \quad &\mathbf{\Gamma}^{(1),(2)}(\mathbf{r},\mathbf{r}') \\ &= \mathbf{\Gamma}^{(0)}(\mathbf{r},\mathbf{r}') \mp \mathbf{\Gamma}^{(0)}(\mathbf{r},\mathbf{r}' - 2\hat{\mathbf{z}}z') \cdot (\mathbf{1} - 2\hat{\mathbf{z}}\hat{\mathbf{z}}). \end{aligned} \tag{5.24}$$

Now using Green's second identity, it is easy to prove

$$\nabla \times \mathbf{\Gamma}^{(2)}(\mathbf{r},\mathbf{r}') = [\nabla' \times \mathbf{\Gamma}^{(1)}]^T(\mathbf{r}',\mathbf{r}), \tag{5.25a}$$

$$\mathbf{\Gamma}^{(1),(2)}(\mathbf{r},\mathbf{r}') = [\mathbf{\Gamma}^{(1),(2)}]^T(\mathbf{r}',\mathbf{r}), \tag{5.25b}$$

where T signifies transposition. In the same way we may derive the integral equations for the Green's dyadic for the screen with the aperture,

$$\begin{aligned} z,z' > 0: \quad &\mathbf{\Gamma}(\mathbf{r},\mathbf{r}') \\ &= \mathbf{\Gamma}^{(1)}(\mathbf{r},\mathbf{r}') - \frac{1}{\zeta^2}\int_{\text{ap}} dS'' \nabla \times \mathbf{\Gamma}_+^{(2)}(\mathbf{r},\mathbf{r}'') \cdot \hat{\mathbf{z}} \times \mathbf{\Gamma}(\mathbf{r}'',\mathbf{r}'), \end{aligned} \tag{5.26a}$$

$$\begin{aligned} z < 0 < z': \quad &\mathbf{\Gamma}(\mathbf{r},\mathbf{r}') \\ &= -\frac{1}{\zeta^2}\int_{\text{ap}} dS'' \nabla \times \mathbf{\Gamma}_-^{(2)}(\mathbf{r},\mathbf{r}'') \cdot \hat{\mathbf{z}} \times \mathbf{\Gamma}(\mathbf{r}'',\mathbf{r}'), \end{aligned} \tag{5.26b}$$

where the $\pm$ subscripts on $\mathbf{\Gamma}^{(2)}$ indicate that the Green's function is defined in the domain above or below the $z = 0$ plane. The continuity of the z component of the electric field in the aperture then leads to the integral equation:

$$\begin{aligned} \hat{\mathbf{z}} \cdot \mathbf{\Gamma}^{(1)}(\mathbf{r},\mathbf{r}')|_{z\to 0+} \\ = \frac{1}{\zeta^2}\int_{\text{ap}} dS''\hat{\mathbf{z}} \cdot \nabla \times (\mathbf{\Gamma}_+^{(2)} + \mathbf{\Gamma}_-^{(2)})(\mathbf{r},\mathbf{r}'') \cdot \hat{\mathbf{z}} \times \mathbf{\Gamma}(\mathbf{r}'',\mathbf{r}'). \end{aligned} \tag{5.27}$$

The system of integral equations defining the Green's dyadic is rather more complicated than that describing the corresponding (classical) static potential problem, so we will defer the discussion of strategies for its solution to a subsequent publication. We will here turn to a situation that can be solved exactly.

VI. CASIMIR-POLDER FORCE BETWEEN ATOM AND A CONDUCTING WEDGE

The interaction between a polarizable atom and a perfectly conducting half plane is a special case of the vacuum interaction between such an atom and a conducting wedge. For the case of an isotropic atom, this was considered by Brevik *et al.* [43]. (This followed on earlier work by Brevik
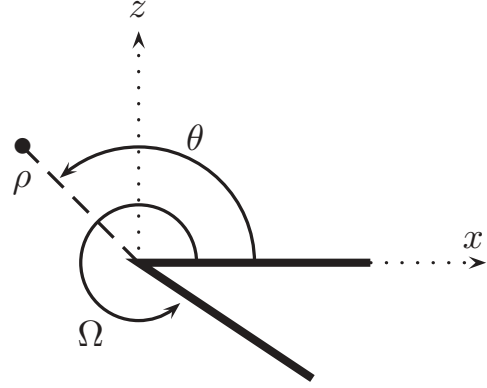

FIG. 4. Polarizable atom, located at polar coordinates ρ, θ, within a conducting wedge with dihedral angle Ω.

and Lygren [44] and DeRaad and Milton [45].) In terms of the opening dihedral angle of the wedge Ω, which we describe in terms of the variable $p = \pi/\Omega$, the electromagnetic Green's dyadic has the form (here the translational direction is denoted by y, and one plane of the wedge lies in the $z = 0$ plane, the other intersecting the xy plane on the line $\theta = \Omega$—see Fig. 4,

$$\begin{aligned} \mathbf{\Gamma}(\mathbf{r},\mathbf{r}') = 2p\sum_{m=0}^{\infty}{}' \int \frac{dk}{2\pi}\bigg[&-\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}'^*(\nabla_\perp^2 - k^2)\frac{1}{\omega^2}F_{mp}(\rho,\rho') \\ &\times \frac{\cos mp\theta \cos mp\theta'}{\pi}e^{ik(x-x')} + \boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{N}}'^*\frac{1}{\omega}G_{mp}(\rho,\rho') \\ &\times \frac{\sin mp\theta \sin mp\theta'}{\pi}e^{ik(x-x')}\bigg]. \end{aligned} \tag{6.1}$$

The first term here refers to TE (H) modes, the second to TM (E) modes. The prime on the summation sign means that the $m = 0$ term is counted with half weight. In the polar coordinates in the xz plane, ρ and θ, the H and E mode operators are

$$\boldsymbol{\mathcal{M}} = \hat{\boldsymbol{\rho}}\frac{\partial}{\rho\partial\theta} - \hat{\boldsymbol{\theta}}\frac{\partial}{\partial\rho}, \tag{6.2a}$$

$$\boldsymbol{\mathcal{N}} = ik\left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\theta}}\frac{\partial}{\rho\partial\theta}\right) - \hat{\mathbf{y}}\nabla_\perp^2, \tag{6.2b}$$

where the transverse Laplacian is

$$\nabla_\perp^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}. \tag{6.3}$$

In this situation, the boundaries are entirely in planes of constant θ, so the radial Green's functions are equal to the free Green's function,

$$\begin{aligned} \frac{1}{\omega^2}F_{mp}(\rho,\rho') &= \frac{1}{\omega}G_{mp}(\rho,\rho') \\ &= -\frac{i\pi}{2\lambda^2}J_{mp}(\lambda\rho_<)H_{mp}^{(1)}(\lambda\rho_>), \end{aligned} \tag{6.4}$$

with $\lambda^2 = \omega^2 - k^2$. We will immediately make the Euclidean rotation, $\omega \to i\zeta$, where $\lambda \to i\kappa$, $\kappa^2 = \zeta^2 + k^2$, so the free Green's functions become $-\kappa^{-2}I_{mp}(\kappa\rho_<)K_{mp}(\kappa\rho_>)$.

We start by considering the most favorable case for CP repulsion, where the atom is only polarizable in the z direction, that is, only $\alpha_{zz} \neq 0$. In the static limit, then the only

component of the Green's dyadic that contributes is

$$\int \frac{d\zeta}{2\pi} \Gamma_{zz} = \frac{2p}{4\pi^3} \int dk\, d\zeta \left\{ [\zeta^2 \sin^2\theta \sin^2 mp\theta - k^2 \cos^2\theta \cos^2 mp\theta] \frac{m^2 p^2}{\kappa^2 \rho_< \rho_>} I_{mp}(\kappa\rho_<) K_{mp}(\kappa\rho_>) \right.$$
$$\left. - [k^2 \sin^2\theta \sin^2 mp\theta - \zeta^2 \cos^2\theta \cos^2 mp\theta] I'_{mp}(\kappa\rho_<) K'_{mp}(\kappa\rho_>) \right\}. \tag{6.5}$$

Here we note that the off diagonal ρ-θ terms in $\mathbf{\Gamma}$ cancel. We have regulated the result by point-splitting in the radial coordinate. At the end of the calculation, the limit $\rho_< \to \rho_> = \rho$ is to be taken.

Now the integral over the Bessel functions is given by

$$\int_0^\infty dk\, \kappa\, I_\nu(\kappa\rho_<) K_\nu(\kappa\rho_>) = \frac{z^\nu}{\rho_>^2 (1 - \xi^2)}, \tag{6.6}$$

where $\xi = \rho_< / \rho_>$. After that the m sum is easily carried out by summing a geometrical series. Care must also be taken with the $m = 0$ term in the cosine series. The result of a straightforward calculation leads to

$$\int \frac{d\zeta}{2\pi} \Gamma_{zz} = -\frac{\cos 2\theta}{\pi^2 \rho^4} \frac{1}{(\xi - 1)^4} + \text{finite}, \tag{6.7}$$

where the divergent term, as $\xi \to 1$, may, through a similar calculation, be shown to be that corresponding to the vacuum in the absence of the wedge, that is, that obtained from the free Green's dyadic. Therefore, we must subtract this term off, leaving for the static Casimir energy (5.7):

$$U_{\text{CP}}^{zz} = -\frac{\alpha_{zz}(0)}{8\pi} \frac{1}{\rho^4 \sin^4 p\theta} \left[p^4 - \frac{2}{3} p^2 (p^2 - 1) \sin^2 p\theta \right.$$
$$\left. + \frac{(p^2 - 1)(p^2 + 11)}{45} \sin^4 p\theta \cos 2\theta \right]. \tag{6.8}$$

This result is derived by another method in Appendix A.

A small check of this result is that as $\theta \to 0$ (or $\theta \to \Omega$) we recover the expected Casimir-Polder result for an atom above an infinite plane:

$$U_{\text{CP}}^{zz} \to -\frac{\alpha_{zz}(0)}{8\pi Z^4}, \tag{6.9}$$

in terms of the distance of the atom above the plane, $Z = \rho\theta$. This limit is also obtained when $p \to 1$, for when $\Omega = \pi$ we are describing a perfectly conducting infinite plane.

A very similar calculation gives the result for an isotropic atom, $\boldsymbol{\alpha} = \alpha\mathbf{1}$, which was first given in Ref. [43]:

$$U_{\text{CP}} = -\frac{3\alpha(0)}{8\pi \rho^4 \sin^4 p\theta} \left[p^4 - \frac{2}{3} p^2 (p^2 - 1) \sin^2 p\theta \right.$$
$$\left. - \frac{1}{3} \frac{1}{45} (p^2 - 1)(p^2 + 11) \sin^4 p\theta \right]. \tag{6.10}$$

Note that this is not three times U_{CP}^{zz} in Eq. (6.8) because the $\cos 2\theta$ factor in the last term in the latter is replaced by $-1/3$ here. This case was reconsidered recently, for example, in Ref. [46].

A. Repulsion by a conducting half plane

Let us consider the special case $p = 1/2$, that is $\Omega = 2\pi$, the case of a semi-infinite conducting plane. This was the situation considered, for anisotropic atoms, in recent papers by Eberlein and Zietal [47–49]. Note that in such a case, for the completely anisotropic atom, $U_{\text{CP}}^{zz} = 0$ at $\theta = \pi/2$, that is, there is no force on the dipole when it is polarized perpendicular to the half sheet and directly above the edge, as observed in Refs. [48,49].

Consider a particle free to move along a line parallel to the z axis, a distance X to the left of the semi-infinite plane; see Fig. 5. The half plane $x < 0$, $z = 0$ constitutes an aperture of infinite width. With X fixed, we can describe the trajectory by $u = X/\rho = -\cos\theta$, in which the variable ranges from 0 to 1. The polar angle is given by

$$\sin^2 \frac{\theta}{2} = \frac{1 + u}{2}. \tag{6.11}$$

The energy for an isotropic atom is given by

$$U_{\text{CP}} = -\frac{\alpha(0)}{32\pi} \frac{1}{X^4} V(u), \tag{6.12}$$

where

$$V(u) = 3u^4 \left[\frac{1}{(1 + u)^2} + \frac{1}{u + 1} + \frac{1}{4} \right]. \tag{6.13}$$

The energy for the completely anisotropic atom is

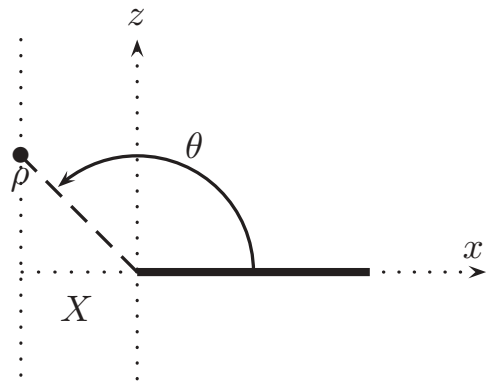$$V_{zz} = \frac{1}{3} V(u) + \frac{u^4}{2} (1 - 3u^2). \tag{6.14}$$

FIG. 5. Polarizable atom, above a half conducting plane, free to move on a line perpendicular to the plane but a distance X to the left of the plane.
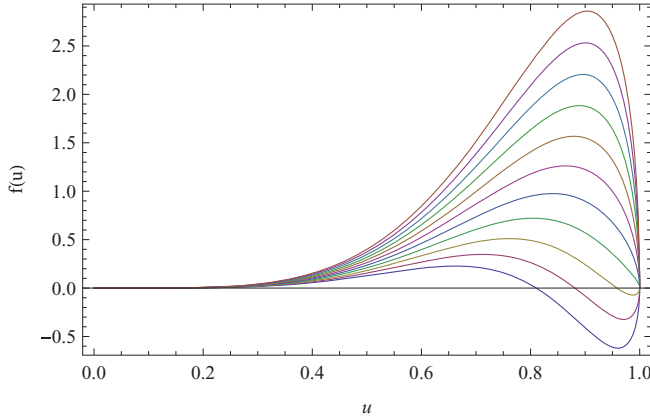
FIG. 6. (Color online) The z component of the force between an anisotropic atom (with ratio of transverse to longitudinal polarizabilities γ) and a semi-infinite perfectly conducting plane, $z = 0$, $x > 0$. $F_z = -\alpha_{zz}/(32\pi X^5) f(u)$ in terms of the variable $u = X/\rho = -\cos\theta$. Here the atom lies on the line $y = 0$, $x = -X$, and ρ is the distance from the edge of the plane and the atom. Here, $f > 0$ corresponds to an attractive force on the z direction, and $f < 0$ corresponds to a repulsive force. The different curves correspond to different values of γ, $\gamma = 0$ to 1 by steps of 0.1, from bottom to top. For $\gamma < 1/4$ a repulsive regime always occurs when the atom is sufficiently close to the plane of the conductor.

If we consider instead a cylindrically symmetric polarizable atom in which

$$\boldsymbol{\alpha} = \alpha_{zz}\hat{\mathbf{z}}\hat{\mathbf{z}} + \gamma\alpha_{zz}(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}}) = \alpha_{zz}(1-\gamma)\hat{\mathbf{z}}\hat{\mathbf{z}} + \gamma\alpha_{zz}\mathbf{1},$$
(6.15)

where γ is the ratio of the transverse polarizability to the longitudinal polarizability of the atom, then the effective potential is

$$(1-\gamma)V_{zz} + \gamma V,$$
(6.16)

and the z component of the force on the atom is

$$F_z^\gamma = -\frac{\alpha_{zz}(0)}{32\pi}\frac{1}{X^5}u^2\sqrt{1-u^2}\frac{d}{du}$$
$$\times\left[\frac{1}{2}u^4(1-\gamma)(1-3u^2) + \frac{1}{3}(1+2\gamma)V(u)\right],$$
(6.17)

where V is given by Eq. (6.13). Note that the energy (6.16), or the quantity in square brackets in Eq. (6.17), only vanishes at $u = 1$ (the plane of the conductor) when $\gamma = 0$. Thus, the argument given in Ref. [32] applies only for the completely anisotropic case.

The force is plotted in Figs. 6 and 7. It will be seen that if γ is sufficiently small, when the atom is sufficiently close to the plane of the plate the z component of the force is repulsive rather than attractive. The critical value of γ is $\gamma_c = 1/4$. This is a completely analytic exact analog of the numerical calculations shown in Ref. [32], where the interaction was considered between a conducting plane with an aperture (circular hole or slit), and a conducting cylindrical or ellipsoidal object. Our calculation demonstrates that three-body effects are not required to exhibit Casimir-Polder repulsion.

It is interesting to observe that the same critical value of γ occurs for the nonretarded regime of a circular aperture,
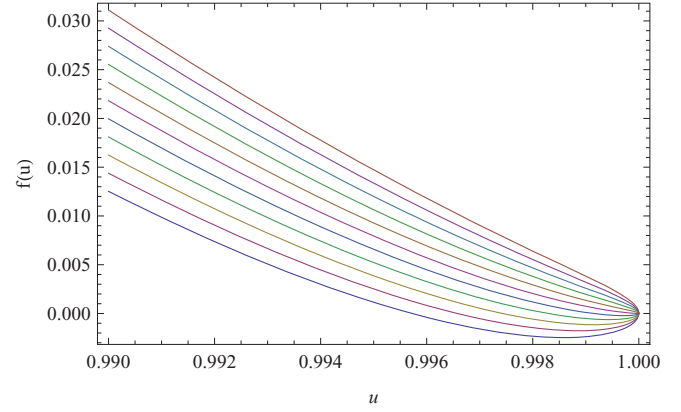

FIG. 7. (Color online) Same as Fig. 6. The region close to the plane, $1 \geqslant u \geqslant 0.99$, with γ near the critical value of 1/4. Here from bottom to top are shown the results for values of γ from 0.245 to 0.255 by steps of 0.001.

as follows from a simple computation based on the result of Ref. [49]. For example, applying the result there for an atom with polarizability given by Eq. (6.15) placed a distance Z along the symmetry axis of an circular aperture of radius a in a conducting plane gives an energy,

$$U = -\frac{1}{16\pi^2}\int_{-\infty}^{\infty}d\zeta\,\alpha_{zz}(\zeta)$$
$$\times\frac{1}{Z^3}\left\{(1+\gamma)\left(\frac{\pi}{2} + \arctan\frac{Z^2-a^2}{2aZ}\right) + \frac{2aZ}{(Z^2+a^2)^3}\right.$$
$$\times\left.\left[(1+\gamma)(Z^4-a^4) - \frac{8}{3}(1-\gamma)a^2Z^2\right]\right\}.$$
(6.18)

It is easy to see that this has a minimum for $z > 0$, and hence there is a repulsive force close to the aperture, provided $\gamma < \gamma_c = 1/4$.

B. Repulsion by a wedge

It is very easy to generalize the above result for a wedge, $p > 1/2$; that is, we want to consider a strongly anisotropic atom, with only α_{zz} significant, to the left of a wedge of the opening angle,

$$\beta = 2\pi - \Omega,$$
(6.19)

as shown in Fig. 8. We want the z axis to be perpendicular to the symmetry axis of the wedge so the relation between the polar angle of the atom and the angle to the symmetry line is

$$\phi = \theta + \beta/2,$$
(6.20)

where, as before, θ is the angle relative to the top surface of the wedge. Then, it is obvious that the formula for the Casimir-Polder energy (6.8) is changed only by the replacement of $\cos 2\theta$ by $\cos 2\phi$, with no change in $\sin p\theta$. Now we can ask how the region of repulsion depends on the wedge angle β.

Write for an atom on the line $x = -X$,

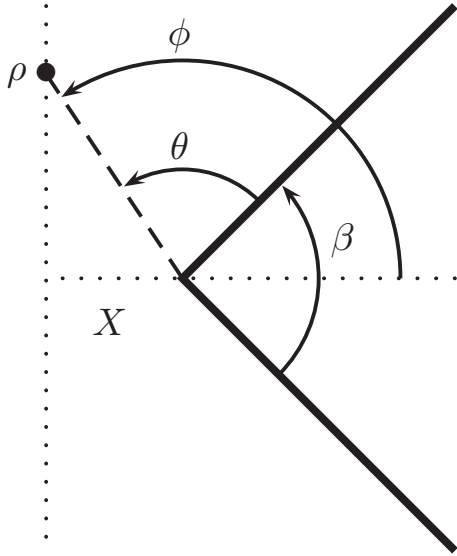$$U_{\text{CP}}^{zz} = -\frac{\alpha_{zz}(0)}{8\pi X^4}V(\phi),$$
(6.21)

FIG. 8. A polarizable atom outside a perfectly conducting wedge of interior angle β. The atom is located at polar angles ρ, ϕ relative to the symmetry plane of the wedge.

where

$$V(\phi) = \cos^4 \phi \left[\frac{p^4}{\sin^4 \left(\frac{\pi}{2} \frac{\phi - \beta/2}{\pi - \beta/2} \right)} - \frac{2}{3} \frac{p^2(p^2 - 1)}{\sin^2 \left(\frac{\pi}{2} \frac{\phi - \beta/2}{\pi - \beta/2} \right)} \right.$$
$$\left. + \frac{1}{45}(p^2 - 1)(p^2 + 11) \cos 2\phi \right]. \quad (6.22)$$

At the point of closest approach,

$$V(\pi) = \tfrac{1}{45}(4p^2 - 1)(4p^2 + 11), \quad (6.23)$$

so the potential vanishes at that point only for the half-plane case, $p = 1/2$. The force in the z direction is

$$F_z = -\frac{\alpha_{zz}}{8\pi} \frac{1}{X^5} f(\phi), \quad (6.24a)$$

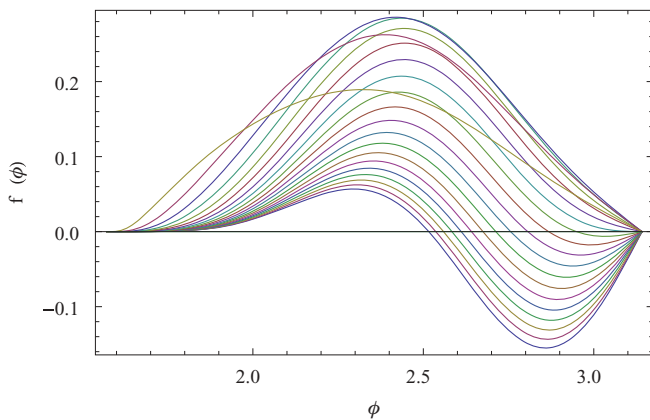$$f(\phi) = \cos^2 \phi \frac{\partial V(\phi)}{\partial \phi}. \quad (6.24b)$$

FIG. 9. (Color online) The z component of the force on a completely anisotropic atom moving on a line perpendicular to a wedge. The different curves are for various values of β from 0 to π by steps of $\pi/20$, from bottom up. The last few values of β have a markedly different character from the others.

Figure 9 shows the force as a function of ϕ for fixed X. It will be seen that the force has a repulsive region for angles close enough to the apex of the wedge, provided that the wedge angle is not too large. The critical wedge angle is actually rather large, $\beta_c = 1.87795$, or about $108°$. For larger angles, the z component of the force exhibits only attraction. Of course, the force is zero for $\beta = \pi$ because then the geometry is translationally invariant in the z direction.

VII. CONCLUSIONS

This paper may be thought of as a counterpart to Ref. [32]. While that reference proceeded on the basis of numerical calculations, we have used analytic approaches. After some examples indicating that Casimir-Polder attraction is typical, and always seems to occur in weak coupling, we demonstrate that the quantum-vacuum Casimir-Polder interaction for a sufficiently anisotropic atom above a conducting half plane can exhibit regimes of repulsive forces for motion confined to certain specified directions. This directly translates into repulsion between such an atom and a plane with an aperture for motion along a line perpendicular to the plane. More complete analysis of that case will be presented elsewhere.

Recently, Ref. [49] appeared, which demonstrates in the *nonretarded* (van der Waals) regime, repulsion could occur between an anisotropically polarizable atom and a conducting plate with an aperture. The critical value of the anisotropy is the same as found here.

Perhaps most remarkable here is that not only can we achieve repulsion with a half plane, but also with a wedge geometry, even when the interior angle of the wedge is greater than $90°$. This indicates that while anisotropy in both the atom and the conductor must be present for repulsion, the anisotropy in the latter need not be too extreme, and that repulsion in other geometries may be readily achievable. Three-body forces are not required, nor is a high degree of symmetry, as was present in Refs. [32,49].

APPENDIX A: DERIVATION OF ANISOTROPIC WEDGE CP FORCE FROM CLOSED-FORM GREEN'S FUNCTION

Many years ago Lukosz gave a closed form for the Green's functions for a perfectly conducting wedge [50]. The four-dimensional Euclidean Green's dyadic has the closed form,

$$\mathbf{\Gamma}(\tau - \tau', y - y', \rho, \rho', \phi, \phi') = -\mathbf{M}\mathbf{M}' G^{\mathrm{H}} + \mathbf{N}\mathbf{N}' G^{\mathrm{E}}, \quad (A1)$$

where the transverse differential operators are [cf. Eq. (6.2)]

$$\mathbf{M} = \hat{\boldsymbol{\rho}} \frac{1}{\rho} \frac{\partial}{\partial \phi} - \hat{\boldsymbol{\phi}} \frac{\partial}{\partial \rho} \equiv \mathcal{M}, \quad \mathbf{N} = \hat{\boldsymbol{\rho}} \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{1}{\rho} \frac{\partial}{\partial \phi}, \quad (A2)$$

where there is an additional contribution to **N** in the y direction. This Green's dyadic is the frequency Fourier transform of that discussed in Sec. VI. Here the E (TM) and H (TE) Green's functions have the form,

$$
\begin{aligned}
G^{\mathrm{H,E}} &= \chi(y,\rho,\tau;y',\rho',\tau';\phi-\phi') \\
&\quad \pm \chi(y,\rho,\tau;y'\rho',\tau';\phi+\phi'-\Omega),
\end{aligned}
\tag{A3}
$$

for a wedge of dihedral angle Ω, with $\phi \in [-\Omega/2, \Omega/2]$. Here

$$
\begin{aligned}
&\chi(y,\rho,\tau;y',\rho',\tau';\psi) \\
&= \frac{1}{8\pi\Omega\rho\rho'\sinh\upsilon}\frac{\sinh(\pi\upsilon/\Omega)}{\cosh(\pi\upsilon/\Omega)-\cos(\pi\psi/\Omega)},
\end{aligned}
\tag{A4}
$$

where

$$
\sinh\frac{\upsilon}{2} = \frac{1}{2}\left[\frac{(\tau-\tau')^2+(y-y')^2+(\rho-\rho')^2}{\rho\rho'}\right]^{1/2}.
\tag{A5}
$$

For the interaction with an atom possessing only an α_{zz} polarizability, we need

$$
\begin{aligned}
\Gamma_{zz} &= \cos(\phi+\phi')\left(\frac{1}{\rho\rho'}\frac{\partial}{\partial\phi}\frac{\partial}{\partial\phi'}-\frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho'}\right)\chi(\phi-\phi') \\
&\quad + 2\left(\sin\phi\cos\phi'\frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{\partial}{\partial\rho'}+\sin\phi'\cos\phi\frac{1}{\rho'}\frac{\partial}{\partial\phi'}\frac{\partial}{\partial\rho}\right) \\
&\quad \times \chi(\phi-\phi')-\cos(\phi-\phi') \\
&\quad \times \left(\frac{1}{\rho\rho'}\frac{\partial}{\partial\phi}\frac{\partial}{\partial\phi'}+\frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho'}\right)\chi(\phi+\phi'-\Omega).
\end{aligned}
\tag{A6}
$$

Here, we have suppressed all the arguments in χ except for the angular ones. For our application here, we are interested in the coincidence limit, so from the outset we can set $\tau=\tau'$ and $x=x'$. Then,

$$
\sinh\frac{\upsilon}{2}=\frac{1}{2}\frac{1-\xi}{\sqrt{\xi}}, \quad \xi=\frac{\rho_<}{\rho_>},
\tag{A7}
$$

which implies

$$
\upsilon = -\ln\xi.
\tag{A8}
$$

Now we expand first in $\phi-\phi'$, then after the differentiations set $\phi=\phi'$, and then expand in υ, that is, in $1-\xi$. We immediately note that the mixed derivative term in Eq. (A6) does not contribute, because there is no linear term in $\phi-\phi'$. The result of a straightforward calculation is

$$
\begin{aligned}
\Gamma_{zz} &= -\frac{\cos 2\theta}{16\pi^2\rho^4}\left\{\frac{16}{(1-\xi)^4}-\frac{1}{45}(p^2-1)(p^2+11)\right\} \\
&\quad + \frac{1}{16\pi^2\rho^4}\left\{\frac{p^4}{\sin^4 p\theta}-\frac{2}{3}\frac{p^2(p^2-1)}{\sin^2 p\theta}\right\},
\end{aligned}
\tag{A9}
$$

where $p=\pi/\Omega$, and we have switched to the angle from the "upper" plate, $\theta=\phi+\Omega/2$, which is chosen to run from 0 to Ω. The first term in Eq. (A9) corresponds to the $\chi(\phi-\phi')$ contribution, and the second to the $\cos(\phi+\phi'-\Omega)$ contribution. Note, the divergent term (as $\xi\to 1$) is precisely

the vacuum term given in Eq. (6.7), and should be subtracted off, and the rest, when multiplied by $-2\pi\alpha_{zz}$, coincides with Eq. (6.8).

APPENDIX B: ELECTROSTATIC ASPECTS: CONDUCTING ELLIPSOID OUTSIDE A CONDUCTING PLATE WITH A CIRCULAR HOLE

Consider a conducting uncharged solid ellipsoid with semi-axes $c>a>b$, centered at $X=Y=Z=0$. The ellipsoid is orientated such that the major semiaxis c lies along the Z axis. To describe the electrostatic potential ϕ in the external region, one can make use of ellipsoidal coordinates ξ,η,ζ, corresponding to solutions for u of the cubic equation,

$$
\frac{Z^2}{c^2+u}+\frac{X^2}{a^2+u}+\frac{Y^2}{b^2+u}=1.
\tag{B1}
$$

The coordinate intervals are

$$
\infty > \xi \geqslant -b^2, \quad -b^2 \geqslant \eta \geqslant -a^2, \quad -a^2 \geqslant \zeta \geqslant -c^2.
\tag{B2}
$$

The relationships between the ellipsoidal and the Cartesian coordinates are given in Ref. [51] and will not be reproduced here. We shall, however, need the line element,

$$
dl^2 = h_1^2\,d\xi^2 + h_2^2\,d\eta^2 + h_3^2\,d\zeta^2,
\tag{B3}
$$

where

$$
h_1=\frac{1}{2R_\xi}\sqrt{(\xi-\eta)(\xi-\zeta)}, \quad h_2=\frac{1}{2R_\eta}\sqrt{(\eta-\zeta)(\eta-\xi)},
\tag{B4}
$$

$$
h_3=\frac{1}{2R_\zeta}\sqrt{(\zeta-\xi)(\zeta-\eta)}, \quad R_u^2=(u+c^2)(u+a^2)(u+b^2),
\tag{B5}
$$

with $u=\xi,\eta,\zeta$.

In the following we assume axial symmetry around the Z axis. Then $a\to b$, $\eta\to -b^2$, and the equation for the surface of the ellipsoid becomes

$$
\frac{Z^2}{c^2}+\frac{R^2}{b^2}=1,
\tag{B6}
$$

with $R^2=X^2+Y^2$. We now have

$$
Z=\pm\left[\frac{(\xi+c^2)(\zeta+c^2)}{c^2-b^2}\right]^{1/2}, \quad R=\left[\frac{(\xi+b^2)(\zeta+b^2)}{b^2-c^2}\right]^{1/2}.
\tag{B7}
$$

The ellipsoidal coordinates ξ,η,ζ reduce in the case of axisymmetry to so-called prolate spheroidal coordinates ξ and ζ, lying in the intervals,

$$
\infty > \xi \geqslant -b^2, \quad -b^2 \geqslant \zeta \geqslant -c^2.
\tag{B8}
$$

Surfaces of constant ξ and ζ are prolate spheroids and hyperboloids of revolution, the surfaces intersecting orthogonally. On the Z axis ($R=0$) one has $\zeta=-b^2, Z=\pm\sqrt{\xi+c^2}$,

whereas in the XY plane ($Z = 0$) one has $\zeta = -c^2, R = \sqrt{\xi + b^2}$. On the surface of the ellipsoid, $\xi = 0$.

In free space outside the ellipsoid the Laplace equation reads

$$\nabla^2 \phi \equiv \frac{4}{\zeta - \xi}\left[\frac{R_\xi}{\xi + b^2}\frac{\partial}{\partial \xi}\left(R_\xi\frac{\partial \phi}{\partial \xi}\right)\right.$$
$$\left. - \frac{R_\zeta}{\zeta + b^2}\frac{\partial}{\partial \zeta}\left(R_\zeta\frac{\partial \phi}{\partial \zeta}\right)\right] = 0. \quad (B9)$$

Assume now that the ellipsoid is placed in an external potential ϕ_0, axisymmetric with respect to the Z axis so that $\phi_0 = \phi_0(\xi, \zeta)$. We write the resulting potential ϕ in the form,

$$\phi(\xi, \zeta) = \phi_0(\xi, \zeta)[1 + F(\xi)], \quad (B10)$$

so that $\phi_0 F$ is the perturbation of the external field. As the boundary condition $\xi = 0$ on the surface has to hold for all values of ζ, it is natural to make the ansatz that F depends on ξ only.

Inserting Eq. (B10) into Eq. (B9) we find that the terms containing F as a factor sum up to zero, the reason being the validity of Eq. (B9) also when ϕ is replaced by ϕ_0. The remaining terms containing $F'(\xi)$ and $F''(\xi)$ yield the equation,

$$\frac{d^2 F}{d\xi^2} + \frac{dF}{d\xi}\frac{d}{d\xi}\ln\left(R_\xi \phi_0^2\right) = 0. \quad (B11)$$

When integrating this equation, in order to preserve the validity of the ansatz $F = F(\xi)$, the coordinate ζ in ϕ_0 has to be regarded as a parameter. The integration thus has to extend from $\xi = 0$ (the surface) in the outward direction, along a line on the hyperboloid $\zeta = $ constant.

The solution of Eq. (B11) can be written as

$$F = A\int_\xi^\infty \frac{d\xi}{R_\xi \phi_0^2}, \quad (B12)$$

where the constant A is determined from the condition $F(0) = -1$ on the ellipsoid surface. That means

$$\phi = \phi_0\left[1 - \frac{\int_\xi^\infty \frac{d\xi}{R_\xi \phi_0^2}}{\int_0^\infty \frac{d\xi}{R_\xi \phi_0^2}}\right]. \quad (B13)$$

We now specify the form of ϕ_0, as the potential from a grounded conducting plate lying in the xy plane, when far from the plate there are constant electric fields, directed normal to the plate, having different values on either side. In the plate there is a circular opening with radius a (this radius is not to be confused with the semiaxis a mentioned above). The center of the opening is at position $x = y = z = 0$. It is known (Ref. [35], Sec. 3.13) that on the z axis,

$$\phi_0(z) = \Phi_{00}\left[1 - \frac{|z|}{a}\arctan\frac{a}{|z|}\right], \quad (B14)$$

where Φ_{00} is a constant. At the origin, $\phi_0 = \Phi_{00}$. At infinity, $|z| \to \infty$, $\phi_0 \to 0$.

The center of the vertically oriented ellipsoid is at position $z = z_0$. Thus $z = z_0 + Z$. We will assume that the ellipsoid

is so slender that the variation of ϕ_0 in the transverse x and y directions can be neglected. Thus we adopt the expression (B14) in the external field region of interest, $\phi_0 = \phi_0(\xi, \zeta)$, ξ and ζ being restricted to the same intervals (B8) as before.

We consider now the upper half of the ellipsoid, $z \geqslant z_0$ or $Z \geqslant 0$. The nonperturbed potential, called ϕ_{0+}, is then

$$\phi_{0+} = \Phi_{00}\left[1 - \frac{z_0 + \sqrt{\xi + c^2}}{a}\arctan\frac{a}{z_0 + \sqrt{\xi + c^2}}\right]. \quad (B15)$$

Thus the potential ϕ_+ in Eq. (B13) can be found numerically, inserting ϕ_{0+} together with $R_\xi = (\xi + b^2)\sqrt{\xi + c^2}$. [In practice the following expansion can here be useful [52]:

$$\frac{1}{x}\arctan x = 1 + \sum_{k=1}^8 a_{2k} x^{2k} + O(10^{-8}), \quad 0 \leqslant x \leqslant 1, \quad (B16)$$

with coefficients a_{2k} of order unity or less.]

The induced surface charge density σ_+ on the ellipsoid is

$$\sigma_+ = -\left[\frac{\epsilon_0}{h_1}\frac{\partial \phi_+}{\partial \xi}\right]_{\xi=0} = -\left[\frac{2\epsilon_0 bc}{\sqrt{-\zeta}}\frac{\partial \phi_+}{\partial \xi}\right]_{\xi=0}, \quad (B17)$$

since on the surface $h_1 = (b/2R_\xi)\sqrt{-\zeta} = (1/2bc)\sqrt{-\zeta}$. In view of the relationships between the ellipsoidal and Cartesian coordinates this can be reexpressed as

$$\sigma_+ = -2\epsilon_0\left[\frac{Z^2}{c^4} + \frac{R^2}{b^4}\right]^{-1/2}\left[\frac{\partial \phi_+}{\partial \xi}\right]_{\xi=0}. \quad (B18)$$

From Eq. (B13) it follows that the derivative $[\partial \phi_{0+}/\partial \xi]_{\xi=0}$ does not contribute to σ_+ [recall that $F(0) = -1$]. The remaining term is

$$\left[\frac{\partial \phi_+}{\partial \xi}\right]_{\xi=0} = \frac{1}{b^2 c}\frac{1}{[\phi_{0+}]_{\xi=0}}\left[\int_0^\infty \frac{d\xi}{R_\xi \phi_{0+}^2}\right]^{-1}. \quad (B19)$$

Thus for $z \geqslant z_0$ we get as solution,

$$\sigma_+ = \frac{\sigma_{0+}}{c}\left[\frac{Z^2}{c^4} + \frac{R^2}{b^4}\right]^{-1/2}, \quad (B20)$$

where σ_{0+} is the constant,

$$\sigma_{0+} = -\frac{2\epsilon_0}{b^2}\frac{1}{\Phi_{00}}\frac{\left[\int_0^\infty \frac{d\xi}{R_\xi \phi_{0+}^2}\right]^{-1}}{\left[1 - \frac{z_0+c}{a}\arctan\frac{a}{z_0+c}\right]} \quad (B21)$$

(recall again that a is the radius of the hole). The dependence of σ_+ on the coordinates Z and R in Eq. (B20) is actually the same as for a charged ellipsoid in free space [51]. The surface force density on the ellipsoid is $(\sigma^2/2\epsilon_0)\mathbf{n}$, $\mathbf{n}$ being the outward normal. The slope of the tangent to the surface is $dZ/dR = -(c^2/b^2)R/Z$; the slope of $\mathbf{n}$ is accordingly $(b^2/c^2)Z/R$. Denoting this as $\tan\theta$, we get, when going over to ellipsoidal coordinates,

$$\tan\theta = \frac{b}{c}\left[\frac{\zeta + c^2}{-\zeta - b^2}\right]^{1/2}. \quad (B22)$$

The component of **n** along the Z axis is, then,

$$n_Z = \sin\theta = \frac{b}{\sqrt{c^2 - b^2}}\left[\frac{\zeta + c^2}{-\zeta}\right]^{1/2}, \qquad \text{(B23)}$$

and we can now find the total vertical force F_{Z+} on the upper half of the ellipsoid by integrating over the actual surface. The line element along the meridian is

$$h_3 d\zeta = \frac{1}{2}\left[\frac{\zeta}{(\zeta + b^2)(\zeta + c^2)}\right]^{1/2} d\zeta, \qquad \text{(B24)}$$

and the surface element dA becomes

$$dA = 2\pi R\, h_3\, d\zeta = \frac{\pi b}{\sqrt{c^2 - b^2}}\left[\frac{-\zeta}{\zeta + c^2}\right]^{1/2} d\zeta. \qquad \text{(B25)}$$

As σ_+ in Eq. (B20) can be reexpressed as

$$\sigma_+ = \sigma_{0+}\frac{b}{\sqrt{-\zeta}}, \qquad \text{(B26)}$$

we can calculate F_{Z+} as

$$F_{Z+} = \int_{Z\geqslant 0}\frac{\sigma_+^2}{2\epsilon_0}n_Z\, dA = \frac{\sigma_{0+}^2}{2\epsilon_0}\frac{\pi b^4}{c^2 - b^2}\int_{b^2}^{c^2}\frac{d(-\zeta)}{(-\zeta)}$$
$$= \frac{\sigma_{0+}^2}{\epsilon_0}\frac{\pi b^4}{c^2 - b^2}\ln\frac{c}{b}. \qquad \text{(B27)}$$

The expression is positive as expected; the force is acting upward. The only dependence on the position z_0 lies in σ_{0+}, as $\sigma_{0+} = \sigma_{0+}(z_0)$ according to Eq. (B21).

The lower half of the ellipsoid, $Z < 0$, can be treated in an analogous way. A complicating element is here the presence of the conducting plate in the xy plane, for radii $\rho \geqslant a$. It means that we can no longer extend the integration over ξ in the solution (B12) to infinity in a straightforward way. We observe that the undisturbed potential in the xy plane can be written as

$$\phi_0(\rho, 0) = \begin{cases} \Phi_{00}\sqrt{1 - \rho^2/a^2}, & \rho \leqslant a \\ 0, & \rho > a, \end{cases} \qquad \text{(B28)}$$

where $\rho^2 = x^2 + y^2$, Φ_{00} being the potential at the center.

Our approach will be based on the following two assumptions:

(1) The ξ integration will be terminated on the xy plane, this implying that the effect of the perturbation is assumed to be small at that level. This approximation is expected to be good except when the distance between the lower end of the ellipsoid and the plane is small.

(2) Secondly, the integration over ξ will be assumed to run over trajectories lying close to the z axis, corresponding to $\zeta = -b^2$. This assumption simplifies the mathematical analysis. It is supported by physical considerations also, since when the ellipsoid is slender the hyperboloids $\zeta = \text{constant}$ emerging from the surface of the ellipsoid near its lower end become concentrated in the vicinity of the z axis.

As according to Eq. (B7) the plane position $z = 0$ in general corresponds to

$$z_0 = \left[\frac{(\xi + c^2)(\zeta + c^2)}{c^2 - b^2}\right]^{1/2}, \qquad \text{(B29)}$$

our approximations imply that the ξ integration is terminated at

$$\xi_{\text{plane}} = z_0^2 - c^2, \qquad \text{(B30)}$$

that is, the same constant for the whole lower half of the ellipsoid.

As solution for the perturbed potential we thus get

$$\phi_- = \phi_{0-}\left[1 - \frac{\int_\xi^{\xi_{\text{plane}}}\frac{d\xi}{R_\xi\, \phi_{0-}^2}}{\int_0^{\xi_{\text{plane}}}\frac{d\xi}{R_\xi\, \phi_{0-}^2}}\right], \qquad \text{(B31)}$$

where

$$\phi_{0-} = \Phi_{00}\left[1 - \frac{z_0 - \sqrt{\xi + c^2}}{a}\arctan\frac{a}{z_0 - \sqrt{\xi + c^2}}\right]. \qquad \text{(B32)}$$

The force F_{Z-} on the lower half can now be calculated. As before, $R_\xi = (\xi + b^2)\sqrt{\xi + c^2}$. Equation (B26) becomes replaced by

$$\sigma_- = \sigma_{0-}\frac{b}{\sqrt{-\zeta}}, \qquad \text{(B33)}$$

where now

$$\sigma_{0-} = -\frac{2\epsilon_0}{b^2}\frac{1}{\Phi_{00}}\frac{\left[\int_0^{\xi_{\text{plane}}}\frac{d\xi}{R_\xi\, \phi_{0-}^2}\right]^{-1}}{\left[1 - \frac{z_0 - c}{a}\arctan\frac{a}{z_0 - c}\right]}. \qquad \text{(B34)}$$

The total force on the ellipsoid becomes

$$F_Z = F_{Z+} + F_{Z-} = \frac{\sigma_{0+}^2 - \sigma_{0-}^2}{\epsilon_0}\frac{\pi b^4}{c^2 - b^2}\ln\frac{c}{b}, \qquad \text{(B35)}$$

which can be rewritten as

$$F_Z = \frac{4\pi\epsilon_0}{\Phi_{00}^2}\frac{1}{c^2 - b^2}\left\{\frac{\left[\int_0^\infty\frac{d\xi}{R_\xi\, \phi_{0+}^2}\right]^{-2}}{\left[1 - \frac{z_0 + c}{a}\arctan\frac{a}{z_0 + c}\right]^2} - \frac{\left[\int_0^{\xi_{\text{plane}}}\frac{d\xi}{R_\xi\, \phi_{0-}^2}\right]^{-2}}{\left[1 - \frac{z_0 - c}{a}\arctan\frac{a}{z_0 - c}\right]^2}\right\}\ln\frac{c}{b}. \qquad \text{(B36)}$$

In the limiting case of a sphere, $b \to c$, the expression becomes somewhat simpler,

$$F_Z = \frac{2\pi\epsilon_0}{\Phi_{00}^2}\frac{1}{c^2}\left\{\frac{\left[\int_0^\infty\frac{d\xi}{(\xi + c^2)^{3/2}\phi_{0+}^2}\right]^{-2}}{\left[1 - \frac{z_0 + c}{a}\arctan\frac{a}{z_0 + c}\right]^2} - \frac{\left[\int_0^{\xi_{\text{plane}}}\frac{d\xi}{(\xi + c^2)^{3/2}\phi_{0-}^2}\right]^{-2}}{\left[1 - \frac{z_0 - c}{a}\arctan\frac{a}{z_0 - c}\right]^2}\right\}. \qquad \text{(B37)}$$

We have made some numerical checks of these expressions (using MAPLE). They indicate that there is no change in the sign of the force for various input parameters for the geometry. The force is attractive, as expected. It turns out that the dependence on the upper integration limit $\xi_{\text{plane}} = z_0^2 - c^2$ is weak, as anticipated above.

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