Polarization effects in the decay of orthopositronium to three photons

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We give results for the order-α radiative corrections to several polarization-related effects involving the decay of ortho-positronium (o-Ps) to three photons. Specifically, we consider the decays of spin-polarized o-Ps from states of specified spin component m into final states where the direction of the normal to the decay plane is measured, the linear polarization of one photon is observed, and the two variables giving the energies, or equivalently the relative orientations of the photons in the decay plane, are measured.

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I. INTRODUCTION

Positronium is an attractive system for the study of many aspects of fundamental physics, including the effects of polarization. Spin 1 ortho-positronium (o-Ps) is usually formed as a uniform mixture of the three magnetic substates ($m = +1$, 0, -1). Polarized samples can be obtained either by depletion of the $m = 0$ state through magnetic quenching (see below) or by preferential production in one of the $|m| = 1$ states due to the nonzero net polarization of slow positron beams produced by weakly decaying radioactive nuclei. Each of the three photons arising from the dominant decay mode of o-Ps has a polarization that is observable in principle. In practice, Compton scattering has been used to differentiate between the polarization states of the decay photons, but usually for only one of the three photons.

A number of polarization effects involving positronium have been studied. Shortly following the original discovery of positronium [1], Deutsch and Dulit [2] reported the observation of magnetic quenching where the $m = 0$ state of o-Ps mixes with spin 0 para-positronium (p-Ps) and participates in the rapid decay of p-Ps, leaving the $m = \pm 1$ states behind. The energy shift of the $m = 0$ state is related both to the magnetic field strength and to the o-Ps to p-Ps energy difference (the hyperfine splitting). Microwave excitation of the transition from an $m = \pm 1$ state to the shifted $m = 0$ state causes the $m = \pm 1$ states to quench as well [3]. Observation of this microwave-induced quenching has been the basis of many measurements of the hyperfine splitting. The angular distribution of the decay photons relative to the spin quantization axis for polarized o-Ps has been worked out to lowest order by Drisko [4] and by Bernreuther and Nachtmann [5]. Platzman and Mills [6] discussed the possibility of forming a Bose-Einstein condensate from polarized o-Ps atoms. Radiative corrections to the decay matrix for decays from polarized o-Ps were obtained by Matsukevich and Metelitsa [7], and Silenko [8] discussed a method for measuring the tensor component of the o-Ps polarization. Cassidy, Meligne, and Mills [9] have reported the production of a fully spin-polarized o-Ps ensemble.

Other effects involving polarized positronium have been vigorously pursued. Bernreuther and collaborators [10,11] suggested using angular correlations of the photons produced from polarized o-Ps as tests of *CP* and *CPT* symmetries. Several measurements using this approach have been done [12–18]. Radiative corrections to angular correlations from polarized o-Ps were computed by Adkins *et al.* [19]. Baryshevsky *et al.* [20] predicted the existence of quantum oscillations in the orientation of the decay plane of polarized o-Ps in a magnetic field, which was observed [21] and was used by Fan *et al.* [22] in a measurement of the positronium hyperfine splitting.

The linear polarization of a decay photon can be measured by subjecting the photon to a Compton scattering process. Measurements of the polarization of a single decay photon from o-Ps decay have been performed by Leipuner *et al.* [23], Ye *et al.* [24], Jia *et al.* [25], and Tang and Tang [26]. Theoretical expectations have been worked out at lowest order by Drisko [4], Bernreuther and Nachtmann [5], and Faraci and Pennisi [27].

In this work we calculate the one-loop corrections to partial decay rates from o-Ps with spin component m into a final state where the orientation of the decay plane relative to the spin quantization axis, the energies of the photons, and the linear polarization of one of the decay photons are all specified. Our results for the decay distributions have the general form

$$\rho_m \propto A_m + B_m(3\cos^2\theta - 1) + C_m \cos(2\alpha) + D_m(3\cos^2\theta - 1)\cos(2\alpha), \quad (1)$$

where θ specifies the orientation of the decay plane relative to the quantization axis and α represents the angle of the polarization of one photon relative to the decay plane normal. The coefficients depend on the value of the magnetic quantum number m of the decaying o-Ps state as well as on variables that specify the energies of the final-state photons. Our results for A_m, B_m, C_m, D_m agree with the known lowest-order expressions but go beyond them by also including the one-loop radiative corrections.

II. ORTHOPOSITRONIUM DECAY

The formula for the decay rate of o-Ps with spin component m into three photons is

$$\Gamma_m = \frac{1}{3!}\frac{1}{2(2W)}\int \frac{d^3k_1}{(2\pi)^3 2\omega_1}\frac{d^3k_2}{(2\pi)^3 2\omega_2}\frac{d^3k_3}{(2\pi)^3 2\omega_3} \times (2\pi)^4\delta(P - k_1 - k_2 - k_3)\sum_{\epsilon_1,\epsilon_2,\epsilon_3}|M_m|^2, \quad (2)$$

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where W is half the o-Ps mass, $\omega_i = k_i^0 = |\vec{k}_i|$ is the ith photon energy, $P = (2W,\vec{0})$ in the o-Ps rest frame, and M_m is the decay amplitude for initial spin component m. Momentum conservation requires the three photons to emerge in a plane—the decay plane—and the normal vector $\hat{n}$ to this plane for a given decay will have some angle θ with respect to the spin quantization axis. Our approach for finding the rate for decay into states with given values for θ and α (the angle of polarization of photon 1 with respect to $\hat{n}$) will be to start with formula (2) for the rate into all final states, and just not perform the sum over polarization ϵ_1 or the integral over θ, which we identify as one of the Euler angles describing the orientation in space of the decay plane. We use the energy-momentum-conserving δ function to perform four of the nine phase space integrals. The five remaining integrals are separated into three over Euler angles and two describing the relative orientation of the three photons in the decay plane. After doing the $\vec{k}_3$ integral using the three-momentum δ function, we obtain

$$\Gamma_m = \frac{1}{3!4W}\int \frac{d^3k_1\, d^3k_2}{(2\pi)^6 8\omega_1\omega_2\omega_3}(2\pi)\delta(2W-\omega_1 - \omega_2 - \omega_3)\sum_{\epsilon_1,\epsilon_2,\epsilon_3}|M_m|^2 . \quad (3)$$

We parametrize the remaining integration variables as follows. Before applying the Euler rotations, the decay plane is taken to be the $x'y'$ plane. We orient photon 1 along the x' axis and photon 2 at angle β to that axis:

$$\hat{k}_1' = (1,0,0), \quad \hat{k}_2' = (\cos\beta, \sin\beta, 0) . \quad (4)$$

The normal to the decay plane is (0,0,1). The polarization vector for photon 1 is taken to be three-dimensional and real, and to have angle α with respect to the normal:

$$\hat{\epsilon}_1' = (0, \sin\alpha, \cos\alpha). \quad (5)$$

The xyz frame is obtained from this $x'y'z'$ frame by an Euler angle rotation $R_E = R_{z'}(\chi)R_{x'}(\theta)R_{z'}(\phi)$ where the z axis is taken to be the axis of spin quantization for describing the o-Ps state. The momentum unit vectors in the xyz frame are

$$\hat{k}_1 = R_E\hat{k}_1', \quad \hat{k}_2 = R_E\hat{k}_2', \quad \hat{k}_3 = \frac{-(\omega_1\hat{k}_1 + \omega_2\hat{k}_2)}{|\omega_1\hat{k}_1+\omega_2\hat{k}_2|}, \quad (6)$$

and the polarization vector of photon 1 is $\hat{\epsilon}_1 = R_E\hat{\epsilon}_1'$. The volume element $d^3k_1 d^3k_2$, expressed in terms of energy variables, Euler angles, and angle β, is

$$d^3k_1\, d^3k_2 = (\omega_1^2\omega_2^2 d\omega_1 d\omega_2)(\sin\beta d\beta)(d\chi\ \sin\theta d\theta\ d\phi). \quad (7)$$

We perform the β integral using the energy-conserving δ function. We write $x_i = \omega_i/W$, in terms of which energy conservation states

$$x_1 + x_2 + x_3 = 2 . \quad (8)$$

Moreover, momentum conservation

$$\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0 \quad (9)$$

allows us to express the angle α_{ij} between any two photons, say i and j, in the decay plane in terms of the x_i as

$$\cos(\alpha_{ij}) = \hat{k}_i \cdot \hat{k}_j = 1 - \frac{2\bar{x}_k}{x_i x_j} , \quad (10)$$

where $(ijk) = (123)$ or a permutation thereof, and $\bar{x}_k \equiv 1 - x_k$. The decay rate is

$$\Gamma_m = \frac{W}{768\pi^3}\sum_{\epsilon_1}\int_0^{\pi}\frac{\sin\theta d\theta}{2}\int d\Delta\left\{\int\frac{d\chi d\phi}{(2\pi)^2}\sum_{\epsilon_2,\epsilon_3}|M_m|^2\right\} , \quad (11)$$

where $\int d\Delta = \int_0^1 dx_1 \int_{1-x_1}^1 dx_2$ is the "phase space" integral over a triangular region in the x_1x_2 plane that specifies the energies of the photons or equivalently the relative orientations of those photons. The integrals and sums inside the curly brackets will be done initially, but we will hold off on doing the final polarization sum, the integral over θ, and the phase space integral so that the dependence of the partial rate on these variables can be studied. The final integrand depends on ϵ_1 only through the angle α between the decay plane normal and the direction of ϵ_1, so the sum over ϵ_1 can be thought of as a sum over any two values of α separated by $\pi/2$. We note that W, half of the o-Ps mass, is nearly equal to m: $W = m[1 + O(\alpha^2)]$. In our work here, which is only through $O(\alpha)$ corrections, we can always take $W \to m$.

The decay matrix element M_m can be expressed in terms of three form factors F_1, F_2, and F_3 [19,28,29]. These form factors are known to one-loop order, so all one-loop calculations involving o-Ps decay to three photons are relatively easy to set up. The matrix element is linear in each polarization vector. We choose to write the polarization vectors in a three-dimensional form with vanishing time coordinate. The matrix element then is

$$M_m = \epsilon_{1i_1}^*\epsilon_{2i_2}^*\epsilon_{3i_3}^*\epsilon_{m,a}M^{i_1i_2i_3a}(x_1,x_2,x_3) , \quad (12)$$

where $\hat{\epsilon}_j$ is the polarization vector of photon j and $\hat{\epsilon}_m$ is the o-Ps polarization vector for spin component m. Specifically, $\hat{\epsilon}_0 = (0,0,1)$ and $\hat{\epsilon}_\pm = \frac{\mp 1}{\sqrt{2}}(1, \pm i, 0)$. The Bose symmetry of M_m can be displayed explicitly through

$$M^{i_1i_2i_3a}(x_1,x_2,x_3) = \sum_{S_3}\mathcal{M}^{i_1i_2i_3a}(x_1,x_2,x_3), \quad (13)$$

where the sum is over the six photon permutations. The tensor $\mathcal{M}^{i_1i_2i_3a}$ can be written in terms of the three scalar form factors according to

$$\mathcal{M}^{i_1i_2i_3a}(x_1,x_2,x_3) = \frac{2i\pi\alpha^3}{x_1x_2x_3}\{T_1F_1 + T_2F_2 + T_3F_3\} \quad (14)$$

with

$$T_1 = \frac{x_2}{\bar{x}_2}k_{1a}k_{3i_2}\left(k_{3i_1}k_{1i_3} + 2\bar{x}_2\delta_{i_1,i_3}\right), \quad (15a)$$

$$T_2 = \left(k_{1a}k_{3i_1} + 2\bar{x}_2\delta_{a,i_1}\right)\left(k_{1i_2}k_{2i_3} + 2\bar{x}_3\delta_{i_2,i_3}\right) + 2x_2\left(\delta_{a,i_1}k_{1i_3} - \delta_{i_1,i_3}k_{1a}\right)k_{3i_2}, \quad (15b)$$

$$T_3 = \frac{\bar{x}_3}{\bar{x}_2}\left(k_{1a}k_{3i_1} + 2\bar{x}_2\delta_{a,i_1}\right)\left(k_{3i_2}k_{2i_3} + 2\bar{x}_1\delta_{i_2,i_3}\right), \quad (15c)$$

where k_{1a}, for example, is the ath component of $\vec{k}_1$. The decay matrix element is dimensionless, and in (15) the photon momentum vectors are taken to be dimensionless so that $|\vec{k}_i| = x_i$. Explicit calculation shows that the lowest-order (in

α) contributions to F_1, F_2, and F_3 are 0, 1, and 0, respectively [29]. The expansions in α of the form factors F_i are

$$F_1 = 0 + \frac{\alpha}{\pi}F_1^{(1)} + O(\alpha^2), \tag{16a}$$

$$F_2 = 1 + \frac{\alpha}{\pi}F_2^{(1)} + O(\alpha^2), \tag{16b}$$

$$F_3 = 0 + \frac{\alpha}{\pi}F_3^{(1)} + O(\alpha^2). \tag{16c}$$

The one-loop form factors are combinations of rational functions of the x_i times one of the functions

$$h_1(x_1) = \log(2x_1), \tag{17a}$$

$$h_2(x_1) = \sqrt{\frac{x_1}{\bar{x}_1}}\theta_1, \tag{17b}$$

$$h_3(x_1) = \frac{1}{2x_1}\{\zeta(2) - \mathrm{Li}_2(1-2x_1)\}, \tag{17c}$$

$$h_4(x_1) = \frac{1}{2x_1}\left\{\left(\frac{\pi}{2}\right)^2 - \theta_1^2\right\}, \tag{17d}$$

$$h_5(x_1) = \frac{1}{2\bar{x}_1}\theta_1^2, \tag{17e}$$

$$h_6(x_1,x_3) = \frac{1}{\sqrt{x_1\bar{x}_1x_3\bar{x}_3}}\{\mathrm{Li}_2(r^+,\bar{\theta}_1) - \mathrm{Li}_2(r^-,\bar{\theta}_1)\}, \tag{17f}$$

where

$$\theta_1 = \arctan(\sqrt{\bar{x}_1/x_1}), \tag{18a}$$

$$\bar{\theta}_1 = \arctan(\sqrt{x_1/\bar{x}_1}), \tag{18b}$$

$$r^{\pm} = \sqrt{\bar{x}_1}\left(1 \pm \sqrt{\frac{x_1\bar{x}_3}{\bar{x}_1x_3}}\right). \tag{18c}$$

The dilogarithm functions are defined by Lewin [30]. Explicit expressions for the one-loop form factors $F_i^{(1)}$ in terms of the x_i and h_i can be found in the supplemental material depository [31].

III. RESULTS

In this section we report our results and give some numerical examples. We performed the algebra and integrals using routines written with MATHEMATICA [32]. Our general result for the decay rate of o-Ps with spin component m is

$$\Gamma_m = \frac{2m\alpha^6}{9\pi}\frac{1}{2}\sum_{\epsilon_1}\int_0^{\pi}\frac{\sin\theta d\theta}{2}\int d\Delta\frac{1}{(x_1x_2x_3)^2} \times\{A_m + B_m(3\cos^2\theta - 1) + C_m\cos(2\alpha) + D_m(3\cos^2\theta - 1)\cos(2\alpha)\}. \tag{19}$$

This formula displays explicitly the dependence of Γ_m on the angle θ between the normal to the decay plane and the quantization axis and the angle α between the polarization direction of photon 1 and the normal to the decay plane. The polarization sum over ϵ_1 is really just a sum over any two values of α that differ by $\pi/2$. The coefficients A_m, B_m, C_m, and D_m have contributions at all orders in α. Including contributions through order α, the coefficients have the form

$$A_m = A_m^{(0)} + \frac{\alpha}{\pi}A_m^{(1)} = A_m^{(0)} + \frac{\alpha}{\pi}(\bar{x}_1\bar{x}_2\bar{x}_3)\sum_{n=1}^{3}\sum_{S_3}\mathcal{A}_{m,n}^{ijk}F_n^{(1)}(x_i,x_j,x_k), \tag{20}$$

with analogous expressions for B_m, C_m, and D_m. The index n labels the three form factors, and the permutation sum is over the six elements of the permutation group S_3, e.g., $ijk \to 123$, etc. The lowest-order coefficients are

$$A_0^{(0)} = (x_1\bar{x}_1)^2 + (x_2\bar{x}_2)^2 + (x_3\bar{x}_3)^2, \tag{21a}$$

$$B_0^{(0)} = \frac{1}{4}A_0^{(0)}, \tag{21b}$$

$$C_0^{(0)} = 2\bar{x}_1\bar{x}_2\bar{x}_3, \tag{21c}$$

$$D_0^{(0)} = \tfrac{1}{2}\bar{x}_2\bar{x}_3(x_2x_3 + 2\bar{x}_2\bar{x}_3), \tag{21d}$$

for $m = 0$. The $|m| = 1$ lowest-order coefficients are closely related to the $m = 0$ coefficients:

$$A_1^{(0)} = A_0^{(0)}, \tag{22a}$$

$$B_1^{(0)} = -\frac{1}{2}B_0^{(0)}, \tag{22b}$$

$$C_1^{(0)} = C_0^{(0)}, \tag{22c}$$

$$D_1^{(0)} = -\tfrac{1}{2}D_0^{(0)}. \tag{22d}$$

The order-α factors $\mathcal{A}_{0,n}^{ijk}$, etc., are given in Table I for the $m = 0$ state. The factors for $|m| = 1$ are given in terms of the corresponding $m = 0$ factors by equations analogous to (22).

There are a number of checks that we can make of our results. If we go ahead and perform the sum over ϵ_1 using

$$\frac{1}{2}\sum_{\epsilon_1}\{G + H\cos(2\alpha)\} = G \tag{23}$$

(for any G and H independent of ϵ_1) we find

$$\Gamma_m = \frac{2m\alpha^6}{9\pi}\int_0^{\pi}\frac{\sin\theta d\theta}{2}\int d\Delta\frac{1}{(x_1x_2x_3)^2} \times\{A_m + B_m(3\cos^2\theta - 1)\}. \tag{24}$$

At lowest order this is

$$\Gamma_m^{(0)} = \frac{2m\alpha^6}{9\pi}\int_0^{\pi}\frac{\sin\theta d\theta}{2}f_m(\theta)\int d\Delta\frac{1}{(x_1x_2x_3)^2}A_0^{(0)}, \tag{25}$$

where the angular probability distributions are $f_0(\theta) = \frac{3}{4}(1 + \cos^2\theta)$ and $f_{\pm1}(\theta) = \frac{3}{8}(3 - \cos^2\theta)$. This agrees with the result given in Appendix D of Bernreuther and Nachtmann [5]. If instead we complete the integral over the Euler angle θ but leave the polarization sum unevaluated, we find

$$\Gamma_m = \frac{2m\alpha^6}{9\pi}\frac{1}{2}\sum_{\epsilon_1}\int d\Delta\frac{1}{(x_1x_2x_3)^2}\{A_m + C_m\cos(2\alpha)\}. \tag{26}$$

TABLE I. Coefficients of the three form factors at order α for decays from the $m = 0$ state. The $|m| = 1$ coefficients are obtained from these by equations analogous to (22).

Coefficient of $F_1^{(1)}(x_i,x_j,x_k)$	Coefficient of $F_2^{(1)}(x_i,x_j,x_k)$	Coefficient of $F_3^{(1)}(x_i,x_j,x_k)$
$\mathcal{A}_{(0)1}^{ijk} = -\bar{x}_j$	$\mathcal{A}_{(0)2}^{ijk} = \frac{x_j\bar{x}_j+x_k\bar{x}_k}{\bar{x}_i}$	$\mathcal{A}_{(0)3}^{ijk} = \frac{x_i\bar{x}_i}{\bar{x}_j}$
$\mathcal{B}_{(0)1}^{ijk} = -\frac{1}{2}\mathcal{A}_{(0)1}^{ijk}$	$\mathcal{B}_{(0)2}^{ijk} = \frac{1}{4}\mathcal{A}_{(0)2}^{ijk}$	$\mathcal{B}_{(0)3}^{ijk} = \frac{1}{4}\mathcal{A}_{(0)3}^{ijk}$
$\mathcal{C}_{(0)1}^{1jk} = -\bar{x}_k$	$\mathcal{C}_{(0)2}^{1jk} = x_1$	$\mathcal{C}_{(0)3}^{1jk} = 0$
$\mathcal{C}_{(0)1}^{i1k} = \bar{x}_1$	$\mathcal{C}_{(0)2}^{i1k} = \bar{x}_k$	$\mathcal{C}_{(0)3}^{i1k} = \frac{\bar{x}_k(\bar{x}_1+x_k)}{\bar{x}_1}$
$\mathcal{C}_{(0)1}^{ij1} = -\bar{x}_1$	$\mathcal{C}_{(0)2}^{ij1} = \bar{x}_1 + x_i$	$\mathcal{C}_{(0)3}^{ij1} = \frac{\bar{x}_1(x_1+\bar{x}_j)}{\bar{x}_j}$
	$\mathcal{D}_{(0)2}^{1jk} = \frac{x_1\bar{x}_1+6\bar{x}_2\bar{x}_3}{4\bar{x}_1}$	$\mathcal{D}_{(0)3}^{1jk} = \frac{3x_1\bar{x}_1}{4\bar{x}_j}$
$\mathcal{D}_{(0)1}^{ijk} = -\frac{1}{2}\mathcal{C}_{(0)1}^{ijk}$	$\mathcal{D}_{(0)2}^{i1k} = \frac{2\bar{x}_1+x_i}{4}$	$\mathcal{D}_{(0)3}^{i1k} = \frac{2x_1\bar{x}_1+x_i\bar{x}_i}{4\bar{x}_1}$
	$\mathcal{D}_{(0)2}^{ij1} = \frac{x_1-x_j}{4}$	$\mathcal{D}_{(0)3}^{ij1} = \frac{x_i\bar{x}_i+2x_j\bar{x}_j}{4\bar{x}_j}$

This expression is the same for any value of m. At lowest order this becomes

$$\Gamma_m^{(0)} = \frac{2m\alpha^6}{9\pi}\frac{1}{2}\sum_{\epsilon_1}\int d\Delta\left\{\left(\frac{\bar{x}_1}{x_2x_3}\right)^2 + \left(\frac{\bar{x}_2}{x_3x_1}\right)^2 + \left(\frac{\bar{x}_3}{x_1x_2}\right)^2 + \frac{2\bar{x}_1\bar{x}_2\bar{x}_3}{(x_1x_2x_3)^2}\cos(2\alpha)\right\}, \tag{27}$$

which agrees with the result of Eq. (5) of Drisko [4]. Finally, if we do both the final polarization sum and the integral over θ, we find the decay rate

$$\Gamma_m = \frac{2m\alpha^6}{9\pi}\int d\Delta\frac{1}{(x_1x_2x_3)^2} \times\left\{A_m^{(0)} + \frac{\alpha}{\pi}(\bar{x}_1\bar{x}_2\bar{x}_3)\sum_{n=1}^{3}\sum_{S_3}\mathcal{A}_{m,n}^{ijk}F_n^{(1)}(x_i,x_j,x_k)\right\}. \tag{28}$$

Again, this is independent of initial state m. The lowest-order rate is

$$\Gamma^{(0)} = \frac{2m\alpha^6}{9\pi}\int d\Delta\left\{\left(\frac{\bar{x}_1}{x_2x_3}\right)^2 + \left(\frac{\bar{x}_2}{x_3x_1}\right)^2 + \left(\frac{\bar{x}_3}{x_1x_2}\right)^2\right\} = \frac{2}{9\pi}(\pi^2 - 9)m\alpha^6, \tag{29}$$

in accord with the result of Ore and Powell [33]. The first-order correction to the rate is

$$\Gamma^{(1)} = \frac{2m\alpha^6}{9\pi}\int d\Delta\frac{\bar{x}_1\bar{x}_2\bar{x}_3}{(x_1x_2x_3)^2}\sum_{n=1}^{3}\sum_{S_3}\mathcal{A}_{m,n}^{ijk}F_n^{(1)}(x_i,x_j,x_k) = \Gamma^{(0)}I_1, \tag{30}$$

where the numerical value is $I_1 = -10.286\,6148$, in agreement with the known result [28,34–36].

Specific results of physical interest can be computed readily from formulas (19) and (20) using the coefficients listed in Table I. For example, at the symmetry point $x_1 = x_2 = x_3 = 2/3$, the $m = 0$ order-α coefficients are [37]

$$A_0^{(1)} \to -1.4060, \tag{31a}$$

$$B_0^{(1)} \to -0.3696, \tag{31b}$$

$$C_0^{(1)} \to -0.8107, \tag{31c}$$

$$D_0^{(1)} \to -0.4652. \tag{31d}$$

These are roughly 10 times the size of the corresponding lowest-order coefficients. Since they must be multiplied by α/π, the order-α corrections contribute about 2% to the total result.

The polarization of one photon (photon 1) relative to the decay plane normal has been measured in the configuration where the angles between the measured photon and the other two are equal ($\alpha_{12} = \alpha_{13}$, or equivalently, $x_2 = x_3$) [23–26]. The theoretical distribution is obtained from (19) by performing the θ average:

$$\rho_m \propto A_m + C_m\cos(2\alpha). \tag{32}$$

We note that this distribution is independent of m because the A_m and C_m coefficients are. The degree of linear polarization is defined as

$$P = \frac{N_\perp - N_\parallel}{N_\perp + N_\parallel}, \tag{33}$$

TABLE II. Linear polarization for a single photon (photon 1) as a function of the angle α_{23} between photons 2 and 3 (measured in degrees). The lowest-order prediction is given in the second column. The third column contains the order-α correction. The final column is the full prediction through order-α: $P(\alpha_{23}) = P^{(0)}(\alpha_{23}) + (\alpha/\pi)P^{(1)}(\alpha_{23})$.

α_{23}	$P^{(0)}(\alpha_{23})$	$P^{(1)}(\alpha_{23})$	$P(\alpha_{23})$
0	0	0	0
15	0.01703	−0.0261	0.01697
30	0.06683	−0.1031	0.06659
45	0.14477	−0.2249	0.14425
60	0.24134	−0.3740	0.24047
75	0.34143	−0.5199	0.34022
90	0.42678	−0.6301	0.42531
105	0.48189	−0.6923	0.48028
120	0.50000	−0.7269	0.49831
135	0.48438	−0.7785	0.48257
150	0.44457	−0.9090	0.44246
165	0.39131	−1.2607	0.38838

where $N_\perp$, $N_\parallel$ are the numbers of photons with polarization perpendicular and parallel to the decay plane, and are proportional to $\rho(\alpha = 0)$ and $\rho(\alpha = \pi/2)$, respectively. So one has $P = C_m/A_m$. The lowest-order approximation is $P^{(0)} = C_m^{(0)}/A_m^{(0)}$, and at order-$\alpha$:

$$P^{(1)} = \frac{C_m^{(0)}}{A_m^{(0)}}\left(\frac{C_m^{(1)}}{C_m^{(0)}} - \frac{A_m^{(1)}}{A_m^{(0)}}\right). \tag{34}$$

Numerical values for the lowest order, order-α correction, and total polarization are shown in Table II for a number of angles, including those for which measurements have been made.

IV. DISCUSSION

We have obtained a convenient form for the distribution for the decay of o-Ps with spin component m into three photons as a function of the angle (relative to the spin quantization axis) of the decay plane normal, the angle (relative to the decay plane normal) of the linear polarization of one photon, and the two parameters giving the energies or relative orientation of the three photons in the decay plane. Our result generalizes the known lowest-order expression to include one-loop radiative corrections. These corrections are typically smaller by a factor of $\approx 10\alpha/\pi$ than the lowest-order results, so they are not crucial for the interpretation of experiments to date. There has been significant ongoing interest in polarized positronium and in the polarization properties of the decay photons from positronium, so the results given in this paper should be useful as experimental uncertainties continue to decrease.

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