

## Informational power of quantum measurements

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We introduce the informational power of a quantum measurement as the maximum amount of classical information that the measurement can extract from any ensemble of quantum states. We prove the additivity by showing that the informational power corresponds to the classical capacity of a quantum-classical channel. We restate the problem of evaluating the informational power as the maximization of the accessible information of a suitable ensemble. We provide a numerical algorithm to find an optimal ensemble and quantify the informational power.

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### I. INTRODUCTION

The information stored in a quantum system is accessible only through a quantum measurement, and the postulates of quantum theory severely limit what a measurement can achieve. The problem of evaluating the informational power of a quantum measurement—that is, how much informative the measurement is—has not been addressed yet in the literature, despite the obvious practical relevance in several contexts, such as the communication of classical information over noisy quantum channels, the storage and retrieval of information from quantum memories [1], and the purification of noisy quantum measurements [2].

For which ensemble of states is a given quantum measurement more informative? To answer this question, one can consider two figures of merit: the probability of correct detection (in a discrimination scenario) and the mutual information (in a communication scenario). Up to now, the only case of optimization of an input ensemble in the literature [3] considers the former as a figure of merit, benefiting from its linearity, which simplifies calculations, and working out an explicit form for the optimal states and the corresponding detection probability. The latter case of optimization, namely, the maximization of the mutual information over input ensembles, is the aim of this work. To this purpose, we define the informational power as the maximal mutual information that a given quantum measurement is able to extract from an ensemble of quantum states. We call the optimal ensemble maximally informative.

The problem has analogies with those of quantifying classical capacity of quantum channels and of attaining accessible information [1]. In fact, as we will show, the informational power of a quantum measurement is the channel capacity of a quantum-classical (q-c) channel [4], and the evaluation of the informational power is the dual of the problem of accessible information, in a sense that we clarify later.

The paper is organized as follows. In Sec. II we introduce the informational power of quantum measurements. We show that it is the classical capacity of a q-c channel and prove additivity. We restate the problem of maximizing the

informational power of a measurement as the problem of maximizing the accessible information of a suitable ensemble and provide a bound on the minimal number of states of a maximally informative ensemble. In Sec. III, we provide a numerical algorithm to find a maximally informative ensemble for a given quantum measurement. In Sec. IV, we classify some quantum measurements according to their informational power, namely, quantum measurements with commuting elements, real-symmetric and mirror-symmetric quantum measurements, and the two-dimensional symmetric informationally complete quantum measurement (i.e., the tetrahedral measurement). We summarize our results in Sec. V.

### II. INFORMATIONAL POWER OF QUANTUM MEASUREMENTS

Let us recall some basic definitions [5] and set the notation. A *random variable*  $X = \{p_i, X_i\}$  is a set of outcomes  $\{i\}$  with values  $\{X_i\}$  and prior probabilities  $\{p_i\}$ . A joint random variable  $(X^1, \dots, X^N)$  is defined analogously.

A measure of the uncertainty associated with a random variable  $X$  is given by the *Shannon entropy*  $H(X)$

$$H(X) := - \sum_i p_i \log p_i, \quad (1)$$

where  $\log_2 x$  denotes the logarithm to base 2. A measure of the remaining uncertainty of a random variable  $Y$  given that the value of  $X$  is known is provided by the *conditional entropy*  $H(Y|X)$ :

$$H(Y|X) := H(X, Y) - H(X). \quad (2)$$

A measure of how much two random variables  $X$  and  $Y$  are correlated is given by the *mutual information*:

$$I(X : Y) := H(X) + H(Y) - H(X, Y). \quad (3)$$

The expected value of the mutual information of two random variables  $X$  and  $Y$ , given the value of a third  $Z$ , is the *conditional mutual information*:

$$I(X : Y|Z) := H(Y|Z) - H(Y|X, Z). \quad (4)$$

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Given a Markov chain  $X \rightarrow Y \rightarrow Z$ , that is, a set of three random variables  $X$ ,  $Y$ , and  $Z$ , with  $Z$  conditionally independent of  $X$ , one has the *data-processing inequality*  $I(X : Y) \geq I(X : Z)$ . In fact,

$$I(X : Z) = I(X : Y) - I(X : Y|Z), \quad (5)$$

and  $I(X : Y|Z) \geq 0$ .

An ensemble of quantum states  $R = \{p_i, \rho_i\}_{i=1}^M$  is represented by a set of  $M$  density matrices  $\rho_i$  (positive semidefinite unit-trace operators), each with a prior probability  $p_i$ . For ensembles of pure states we replace the density matrices with the normalized states, and we write  $V = \{p_i, |\psi_i\rangle\}_{i=1}^M$ . A quantum measurement is described by a *positive operator-valued measurement* (POVM)  $\Pi = \{\Pi_j\}_{j=1}^N$ , defined as a set of  $N$  positive semidefinite operators  $\Pi_j$  that sum to identity, namely,  $\sum_{j=1}^N \Pi_j = \mathbb{1}$ . If we consider an ensemble  $R = \{p_i, \rho_i\}$  and a POVM  $\Pi = \{\Pi_j\}$ , the conditional probability  $p_{j|i}$  of outcome  $j$  given state  $\rho_i$  is given by the *Born rule*, that is,  $p_{j|i} = \text{Tr}[\rho_i \Pi_j]$ . In the case of a POVM  $\Pi$  performed over an ensemble  $R$ , the mutual information is a measure of how much the outcomes of the POVM  $\Pi$  are correlated with states  $\rho_i$ ; in fact,

$$I(R, \Pi) := \sum_{i,j} p_i \text{Tr}[\rho_i \Pi_j] \log_2 \frac{\text{Tr}[\rho_i \Pi_j]}{\sum_k p_k \text{Tr}[\rho_k \Pi_j]}. \quad (6)$$

Now we can introduce the informational power of a POVM, the quantity that we analyze in the rest of this work.

*Definition 1.* The informational power  $W(\Pi)$  of a POVM  $\Pi$  is the maximum over all possible ensembles of states  $R$  of the mutual information between  $\Pi$  and  $R$ :

$$W(\Pi) = \max_R I(R, \Pi). \quad (7)$$

We call any ensemble that maximizes the mutual information a maximally informative ensemble for  $\Pi$ .

### A. Informational power as a classical capacity

Given the tensor product  $\otimes_{n=1}^N \Pi^n = \{\otimes_{n=1}^N \Pi_{j_n}^n\}$  describing the parallel use of  $N$  POVMs, by using entangled input states one may ask if the informational power is superadditive. We recall that the analogous quantity in the problem of optimization of POVMs, namely, the accessible information, is additive [6].

According to [4] (see also [7,8]) we provide the following definitions.

*Definition 2.* Given a channel  $\Phi$  from a Hilbert space  $\mathcal{H}$  to a Hilbert space  $\mathcal{K}$ , the *single-use channel capacity* is given by

$$C_1(\Phi) := \sup_R \sup_{\Lambda} I(\Phi(R), \Lambda), \quad (8)$$

where the suprema are taken over all ensembles  $R$  in  $\mathcal{H}$  and over all POVMs  $\Lambda$  on  $\mathcal{K}$ .

*Definition 3.* A q-c channel  $\Phi_{\Pi}$  is defined as

$$\Phi_{\Pi}(\rho) := \sum_j \text{Tr}[\rho \Pi_j] |j\rangle\langle j|, \quad (9)$$

where  $\Pi = \{\Pi_j\}$  is a POVM and  $|j\rangle$  is an orthonormal basis.

A q-c channel  $\Phi_{\Pi}$  is a decision rule that maps quantum states into classical states via a measurement  $\Pi$ .

*Proposition 1.* The informational power of a POVM  $\Pi = \{\Pi_j\}$  is equal to the single-use capacity  $C_1(\Phi_{\Pi})$  of the q-c channel  $\Phi_{\Pi}$ ; that is,

$$C_1(\Phi_{\Pi}) = W(\Pi). \quad (10)$$

*Proof.* Consider an ensemble  $R = \{p_i, \rho_i\}$  and a POVM  $\Lambda = \{\Lambda_k\}$ . Introduce the random variables  $X$ ,  $Y$ , and  $Z$ . Take  $X$  with prior probability  $p_i$ . Take  $Y$  such that the conditional probability of outcome  $j$  of  $Y$  given outcome  $i$  of  $X$  is  $p_{j|i} = \text{Tr}[\Pi_j \rho_i]$ . Take  $Z$  such that the conditional probability of outcome  $k$  of  $Z$  given outcome  $j$  of  $Y$  is  $q_{k|j} = \langle j | \Lambda_k | j \rangle$ . Clearly, the joint probability of outcome  $i$  and  $k$  of  $X$  and  $Z$ , respectively, is given by  $p_i \text{Tr}[\Lambda_k \Phi_{\Pi}(\rho_i)]$ , so  $I(X : Z) = I(\Phi_{\Pi}(R), \Lambda)$ , whereas  $I(X : Y) = I(R, \Pi)$ .

Notice that  $X \rightarrow Y \rightarrow Z$  is a Markov chain, so Eq. (5) holds. By choosing  $\Lambda_k = |k\rangle\langle k|$ , one has  $q_{k|j} = \delta_{j,k}$ , so  $H(Y|Z) = 0$ , and  $I(X : Y|Z) = H(Y|Z) - H(Y|X, Z) = 0$  for any  $\{p_i\}$ . Thus,

$$\sup_{\Lambda} I(\Phi_{\Pi}(R), \Lambda) = I(\Phi_{\Pi}(R), \{|k\rangle\langle k|\}). \quad (11)$$

Since  $p_i \langle k | \Phi_{\Pi}(\rho_i) | k \rangle = p_i \text{Tr}[\rho_i \Pi_k]$ , we have

$$C_1(\Phi_{\Pi}) = \sup_R I(\Phi_{\Pi}(R), \{|k\rangle\langle k|\}) = \sup_R I(R, \Pi) = W(\Pi). \quad (12)$$

*Proposition 2.* The informational power  $W(\Pi)$  is an additive quantity; that is,

$$W(\otimes_{n=1}^N \Pi^n) = \sum_{n=1}^N W(\Pi^n). \quad (13)$$

*Proof.* Since the tensor product of q-c channels is a q-c channel, that is,  $\otimes_{n=1}^N \Phi_{\Pi^n} = \Phi_{\otimes_{n=1}^N \Pi^n}$ , the statement follows immediately from Proposition 1 and from the additivity property of the capacity for q-c channels [4,7]. ■

### B. Duality between informational power and accessible information

According to [9], we provide the following definition.

*Definition 4.* The accessible information  $A(R)$  of an ensemble  $R = \{p_i, \rho_i\}$  is the maximum over all possible POVMs  $\Pi$  of the mutual information between  $R$  and  $\Pi$ ; namely,

$$A(R) = \max_{\Pi} I(R, \Pi). \quad (14)$$

We call any POVM that maximizes the mutual information a maximally informative POVM for  $R$ .

The accessible information of the ensemble  $R = \{p_i, \rho_i\}$  is upper bounded by the *Holevo quantity* [9],

$$A(R) \leq \chi(R) := S(\rho_R) - \sum_i p_i S(\rho_i), \quad (15)$$

where  $S(\rho) := -\text{Tr}[\rho \log_2 \rho]$  is the *von Neumann entropy* and  $\rho_R = \sum_i p_i \rho_i$ . In contrast, one has the following lower bound [10]:

$$A(R) \geq Q(\rho_R) - \sum_i p_i Q(\rho_i), \quad (16)$$

where  $Q(\rho) := -\sum_k (\prod_{l \neq k} \frac{\lambda_k}{\lambda_k - \lambda_l}) \lambda_k \log_2 \lambda_k$  is the *subentropy* of a quantum state,  $\{\lambda_k\}$  being the set of eigenvalues of  $\rho$ .

Since invertible density matrices are a dense subset, in the following we assume  $\rho$  invertible. Given the ensemble  $S = \{q_i, \sigma_i\}$ , we call  $\sigma_S = \sum_i q_i \sigma_i$ .

*Definition 5.* Given an ensemble  $S = \{q_i, \sigma_i\}$ , we define the POVM  $\Pi(S)$  as

$$\Pi(S) := \{q_i \sigma_S^{-1/2} \sigma_i \sigma_S^{-1/2}\}. \quad (17)$$

*Definition 6.* Given a POVM  $\Lambda = \{\Lambda_j\}$  and a density matrix  $\sigma$ , we define the ensemble  $R(\Lambda, \sigma)$  as

$$R(\Lambda, \sigma) := \left\{ \text{Tr}[\sigma \Lambda_j], \frac{\sigma^{1/2} \Lambda_j \sigma^{1/2}}{\text{Tr}[\sigma \Lambda_j]} \right\}. \quad (18)$$

Definition 5 corresponds to the so-called ‘‘pretty good’’ measurement [11,12]. The ensemble-measurement duality given by the definitions above was exploited in [13] to obtain measurement-dependent lower and upper bounds on  $A(R(\Lambda, \sigma))$ . The accessible information of the ensemble  $R(\Lambda, \sigma)$  has also been studied in [14], in the context of quantifying the information-disturbance trade-off of quantum measurements.

In the following we show that there exists a duality between the informational power and the accessible information that allows us to recast many results from the latter context to the former one. Notice that  $R(\Pi(S), \sigma_S) = S$  and, analogously,  $\Pi(R(\Lambda, \sigma)) = \Lambda$ . Moreover, for any ensemble  $S$  and POVM  $\Lambda$ , one has

$$I(S, \Lambda) = I(R(\Lambda, \sigma_S), \Pi(S)). \quad (19)$$

*Proposition 3.* The informational power of a POVM  $\Lambda = \{\Lambda_j\}$  is given by

$$W(\Lambda) = \max_{\sigma} A(R(\Lambda, \sigma)). \quad (20)$$

The ensemble  $S^* = \{q_i^*, \sigma_i^*\}$  is maximally informative for the POVM  $\Lambda$  if and only if  $\sigma_{S^*} = \arg \max_{\sigma} A(R(\Lambda, \sigma))$  and the POVM  $\Pi(S^*)$  is maximally informative for the ensemble  $R(\Lambda, \sigma_{S^*})$ .

*Proof.* From the definitions of informational power and accessible information, and from Eq. (19), one has

$$\begin{aligned} W(\Lambda) &= \max_{\sigma} \max_{S|\sigma_S=\sigma} I(S, \Lambda) \\ &= \max_{\sigma} \max_{\Pi(S)|\sigma_S=\sigma} I(R(\Lambda, \sigma_S), \Pi(S)) \\ &= \max_{\sigma} \max_{\Pi} I(R(\Lambda, \sigma), \Pi) \\ &= \max_{\sigma} A(R(\Lambda, \sigma)). \end{aligned} \quad (21)$$

Proposition 3 makes clear the duality between the informational power and the accessible information. A diagrammatic representation of this duality is given by

$$\begin{array}{ccc} \Lambda & \xrightarrow{\sigma_{S^*}} & R(\Lambda, \sigma_{S^*}) \\ \downarrow & & \downarrow \\ S^* & \xleftarrow{\sigma_{S^*}} & \Pi(S^*) \end{array}$$

where  $S^* = \arg \max_S I(S, \Lambda)$  and  $\Pi(S^*) = \arg \max_{\Pi} I(R(\Lambda, \sigma_{S^*}), \Pi)$ . Horizontal arrows correspond to the duality operation of Definitions 5 and 6. Moving in

the sense of the arrow corresponds to applying Eq. (18), thus requiring  $\sigma_{S^*}$ . Moving in the opposite sense corresponds to applying Eq. (17). The vertical arrow from  $\Lambda$  to  $S^*$  indicates that  $S^*$  is maximally informative for the POVM  $\Lambda$ , whereas the vertical arrow from  $R(\Lambda, \sigma_{S^*})$  to  $\Pi(S^*)$  indicates that  $\Pi(S^*)$  is maximally informative for the ensemble  $R(\Lambda, \sigma_{S^*})$ .

From Proposition 3 we can obtain a property of maximally informative ensembles using Davies’ theorem [15].

*Proposition 4.* Given a  $D$ -dimensional POVM  $\Lambda = \{\Lambda_j\}$ , there exists a maximally informative ensemble  $S^* = \{q_i^*, \sigma_i^*\}_{i=1}^M$ , with all  $\sigma_i^*$  pure and  $D \leq M \leq D^2$ .

*Proof.* By Proposition 3,  $S^*$  is maximally informative for  $\Lambda$  if and only if  $\sigma_{S^*} = \arg \max_{\sigma} A(R(\Lambda, \sigma))$  and  $\Pi(S^*)$  is maximally informative for  $R(\Lambda, \sigma_{S^*})$ . By Davies’ theorem [15], there exists a maximally informative POVM  $\Pi(S^*)$  with  $M$  rank 1 elements and  $D \leq M \leq D^2$ , so the statement follows.

For some classes of POVMs it is possible to improve the bound on the number of elements of a maximally informative ensemble as follows.

*Definition 7.* An ensemble  $S = \{q_i, \sigma_i\}$  on a Hilbert space  $\mathcal{H}$  is real if there exists a basis on  $\mathcal{H}$  relative to which all  $\sigma_i$  have real matrix elements.

*Definition 8.* A POVM  $\Lambda = \{\Lambda_j\}$  on a Hilbert space  $\mathcal{H}$  is real if there exists a basis on  $\mathcal{H}$  relative to which all  $\Lambda_j$  have real matrix elements.

*Proposition 5.* Given a  $D$ -dimensional real POVM  $\Lambda = \{\Lambda_j\}$ , there exists a maximally informative real ensemble  $S^* = \{q_i^*, \sigma_i^*\}_{i=1}^M$ , with all  $\sigma_i^*$  pure and  $D \leq M \leq D(D+1)/2$ .

*Proof.* By Proposition 3,  $S^*$  is maximally informative for  $\Lambda$  if and only if  $\sigma_{S^*} = \arg \max_{\sigma} A(R(\Lambda, \sigma))$  and  $\Pi(S^*)$  is maximally informative for  $R(\Lambda, \sigma_{S^*})$ . By Lemma 5 of [16], there exists a maximally informative POVM  $\Pi(S^*)$  with  $M$  rank 1 elements and  $D \leq M \leq D(D+1)/2$ , so the statement follows.

### III. EVALUATION OF THE INFORMATIONAL POWER

Given a POVM, it is in general a hard task to provide an explicit form for the maximally informative ensemble, due to the nonlinearity of the mutual information as a figure of merit. In the following, we prove some necessary conditions for attaining informational power, and we make use of these results to provide an iterative algorithm converging to the maximally informative ensemble. In this section it is convenient to take the states of the ensemble unnormalized, with the norm giving the prior probability of each state. Therefore we also use the notation for the ensemble  $V := \{|\psi_i\rangle\}$ , with prior probability  $p_i = \|\psi_i\|^2$ .

#### A. Necessary conditions to attain informational power

When one optimizes the informational power, considering only ensembles of pure states is not restrictive, as shown in Proposition 4. We provide here a short alternative proof of this fact, which is independent of Davies’ theorem [15].

*Proposition 6.* For any given POVM  $\Pi = \{\Pi_j\}$ , there exists a maximally informative ensemble made of pure states.

*Proof.* Consider an ensemble  $R = \{p_i, \rho_i\}$ . Each of the states can be decomposed on the basis of its orthogonal

eigenvectors as  $\rho_i = \sum_k |\psi_{ik}\rangle\langle\psi_{ik}|$ , with  $\sum_k \|\psi_{ik}\|^2 = 1, \forall i$ . Denote by  $V = \{|\psi_{ik}\rangle\}$  the ensemble of such pure states.

For three random variables  $X, Y$ , and  $Z$ , we have

$$I(X : Z) = H(Z) - H(Z|X) \\ \leq H(Z) - H(Z|X, Y) = I(X, Y : Z), \quad (22)$$

since conditioning reduces entropy. We take  $X$  distributed according to  $p_i$ . If we set the joint probability  $p_{i,j}$  of outcome  $i$  of  $X$  and  $j$  of  $Z$  to be  $p_{i,j} = p_i \text{Tr}[\Pi_j \rho_i]$ , we have  $I(X : Z) = I(R, \Pi)$ . If we set the joint probability  $p_{i,k,j}$  of outcomes  $i, k$ , and  $j$  of  $X, Y$  and  $Z$ , respectively, to be  $p_{i,k,j} = p_i \langle \psi_{i,k} | \Pi_j | \psi_{i,k} \rangle$ , we have  $I(X, Y : Z) = I(V, \Pi)$ , and hence  $I(V, \Pi) \geq I(R, \Pi)$ . Clearly, the maximum of  $I(R, \Pi)$  over  $R$  can be searched only among ensembles of pure states.

Now we turn to the problem of finding necessary conditions for an ensemble of pure states to be maximally informative for a given POVM  $\Pi = \{\Pi_j\}$ . For any ensemble  $V = \{|\psi_i\rangle\}$ , by defining

$$\Pi'_i := \sum_{j=1}^N \log_2 \frac{\langle \psi_i | \Pi_j | \psi_i \rangle}{\|\psi_i\|^2 \sum_{k=1}^M \langle \psi_k | \Pi_j | \psi_k \rangle} \Pi_j, \quad (23)$$

we notice that the mutual information  $I(V, \Pi)$  can be written as  $I(V, \Pi) = \sum_i \langle \psi_i | \Pi'_i | \psi_i \rangle$ .

*Proposition 7.* Given a POVM  $\Pi = \{\Pi_j\}$ , a necessary condition for the ensemble  $V = \{|\psi_i\rangle\}_{i=1}^M$  to be maximally informative is that

$$\Pi'_i |\psi_i\rangle = I(V, \Pi) |\psi_i\rangle, \quad \forall i = 1, \dots, M, \quad (24)$$

where  $\Pi'_i$  is given in Eq. (23).

*Proof.* Upon introducing a Lagrange multiplier  $\lambda$  to constrain the normalization of the input ensemble, let us consider the expression

$$C = \sum_{i=1}^M \langle \psi_i | \Pi'_i | \psi_i \rangle + \lambda \left( \sum_{i=1}^M \|\psi_i\|^2 - 1 \right). \quad (25)$$

By equating to 0 the derivative of Eq. (25) with respect to each  $\langle \psi_i |$ , we obtain  $M$  extremal equations which are necessary conditions for a maximally informative ensemble, namely,

$$\frac{\partial C}{\partial \langle \psi_i |} = [\Pi'_i + (\lambda - 1)\mathbb{1}] |\psi_i\rangle = 0, \quad \forall i = 1, \dots, M. \quad (26)$$

Upon redefining  $\mu = 1 - \lambda$ , we can rewrite the extremal equations as  $\Pi'_i |\psi_i\rangle = \mu |\psi_i\rangle$ . By multiplying both sides on the left by  $|\psi_i\rangle$  and summing over  $i$ , we notice that  $\mu = I(V, \Pi)$ .

*Corollary 1.* Given a POVM  $\Pi = \{\Pi_j\}$ , a necessary condition for  $V = \{|\psi_i\rangle\}_{i=1}^M$  to be maximally informative is that

$$I(V, \Pi) = \sqrt{\sum_{i=1}^M \langle \psi_i | \Pi_i'^2 | \psi_i \rangle}. \quad (27)$$

*Proof.* The result follows immediately by multiplying Eq. (24) on the left by its Hermitian adjoint and summing over  $i$ . ■

## B. An iterative algorithm to maximize informational power

In the following we provide a steepest-ascent iterative algorithm which is effective in finding a maximally informative

ensemble for a given POVM. A similar algorithm for the evaluation of the accessible information for a given ensemble is given in [17].

*Algorithm 1.* The following steepest-ascent algorithm converges to a maximum of the informational power. For arbitrary ensemble  $V^0 = \{|\psi_i^0\rangle\}_{i=1}^M$ , evaluate  $V^n = \{|\psi_i^n\rangle\}_{i=1}^M$  at any order  $n$  by the following steps.

(1) Given  $V^n = \{|\psi_i^n\rangle\}_{i=1}^M$ , evaluate  $\Pi^n = \{\Pi_i^n\}_{i=1}^M$  according to

$$\Pi_i^n = \sum_{j=1}^N \log_2 \frac{\langle \psi_i^n | \Pi_j | \psi_i^n \rangle}{\sum_{k=1}^M \langle \psi_k^n | \Pi_j | \psi_k^n \rangle} \Pi_j - \log_2 \|\psi_i^n\|^2 \mathbb{1}. \quad (28)$$

(2) Pick up a small enough positive  $\alpha$  and evaluate

$$|\hat{\psi}_i^{n+1}\rangle = [(1 - \alpha)\mathbb{1} + \alpha \Pi_i^n] |\psi_i^n\rangle. \quad (29)$$

(3) Obtain  $V^{n+1}$  as

$$|\psi_i^{n+1}\rangle = \frac{|\hat{\psi}_i^{n+1}\rangle}{\sqrt{\sum_{i=1}^M \|\hat{\psi}_i^{n+1}\|^2}}. \quad (30)$$

*Proof.* Consider POVM  $\Pi = \{\Pi_j\}$  and an ensemble  $V^n = \{|\psi_i^n\rangle\}_{i=1}^M$ , so Eq. (28) is just the definition given in (23).

The algorithm we are considering is a steepest-ascent algorithm. We move the ensemble in the direction of the gradient of the mutual information, namely,

$$\nabla I(V, \Pi) = \left( \frac{\partial I}{\partial \langle \psi_1 |}, \dots, \frac{\partial I}{\partial \langle \psi_M |} \right) \\ = ((\Pi'_1 - \mathbb{1})|\psi_1\rangle, \dots, (\Pi'_M - \mathbb{1})|\psi_M\rangle), \quad (31)$$

which ensures that we follow the greatest increase in the mutual information. So, if we set the iteration to be

$$(|\hat{\psi}_1^{n+1}\rangle, \dots, |\hat{\psi}_M^{n+1}\rangle) \\ = (1 - \alpha) (|\psi_1^n\rangle, \dots, |\psi_M^n\rangle) + \alpha \nabla I(\Pi, V^n), \quad (32)$$

we obtain Eq. (29).

Then Eq. (30) is just the normalization of the updated ensemble to satisfy  $\sum_{i=1}^M \|\psi_i\|^2 = 1$ . By construction, one has  $I(V^{n+1}, \Pi) \geq I(V^n, \Pi)$ .

As for all steepest-ascent algorithms, there is no protection against the possibility of convergence toward a local, rather than a global, maximum, whence one should run the algorithm for different initial ensembles to discriminate between local and global maxima.

Any ensemble can be used as a starting point, except for a subset corresponding to the minima of the mutual information (e.g., all the ensembles composed by a single quantum state). These minima are unstable fix points of the iteration, so even small perturbations let the iteration converge to some maximum. Due to Propositions 4 and 5, it is sufficient to consider ensembles with  $D^2$  states for a  $D$ -dimensional POVM and ensembles with  $D(D + 1)/2$  states for a real POVM.

The parameter  $\alpha$  controls the length of each iterative step, so for  $\alpha$  too large, an overshooting can occur. This can be kept under control by evaluating the mutual information  $I(V, \Pi)$  at the end of each step: if  $I(V, \Pi)$  decreases instead of increasing, we are warned that we have taken  $\alpha$  too large. An efficient evaluation of  $I(V, \Pi)$  can be performed through Corollary 1.

#### IV. CLASSIFICATION OF QUANTUM MEASUREMENTS

The informational power introduces a complete ordering between POVMs. In the following, we classify some POVMs according to their informational power. We consider POVMs with commuting elements (Sec. IV A), real-symmetric POVMs (Sec. IV B), mirror-symmetric POVMs (Sec. IV C), and the two-dimensional symmetric informationally complete POVM (Sec. IV D),

##### A. POVMs with commuting elements

*Proposition 8.* Given a  $D$ -dimensional POVM  $\Pi = \{\Pi_j\}_{j=1}^N$  with commuting elements, there exists a maximally informative ensemble  $V = \{p_i^*, |i\rangle\}_{i=1}^M$  of  $M \leq D$  states, where  $|i\rangle$  denotes the common orthonormal eigenvectors of  $\Pi$ , and the prior probabilities  $p_i^*$  maximize the mutual information:

$$W(\Pi) = \max_{p_i} \sum_{i,j} p_i \langle i | \Pi_j | i \rangle \log_2 \frac{\langle i | \Pi_j | i \rangle}{\sum_k p_k \langle k | \Pi_j | k \rangle}. \quad (33)$$

*Proof.* For any ensemble  $R = \{p_i, \rho_i\}$ , consider the diagonal ensemble  $S = \{p_i, \sigma_i\}$ , where  $\sigma_i = \sum_k \langle k | \rho_i | k \rangle |k\rangle \langle k|$ , with  $|k\rangle$  denoting the common eigenvectors of  $\Pi$ . Clearly,  $\text{Tr}[\Pi_j \sigma_i] = \text{Tr}[\Pi_j \rho_i]$ , whence  $I(R, \Pi) = I(S, \Pi)$ . As in Proposition 6, it is sufficient to look for the maximum over the prior probabilities  $p_i$ , with fixed states  $|i\rangle$ . Hence Eq. (33) follows.

We notice that  $M \leq D$  since some of the prior  $p_i$  obtained by optimizing Eq. (33) can be 0. Equation (33) is a concave function of the prior probabilities, and a numerical algorithm for performing the optimization is provided in [18].

As an application, we consider the POVM  $\Pi^{(\eta)} = \{\Pi_j^{(\eta)}\}_{j=1}^D$  describing the projective measurement over an orthonormal basis  $\{|j\rangle\}$  in dimension  $D$  affected by isotropic noise; that is,

$$\Pi_j^{(\eta)} = \eta |j\rangle \langle j| + (1-\eta) \frac{\mathbb{1}}{D}, \quad j = 1, \dots, D. \quad (34)$$

When  $\eta = 1$ , a maximally informative ensemble is clearly  $V = \{p_i, |i\rangle\}$ , with  $p_i = 1/D$ . For  $\eta < 1$ , by Proposition 8, ensemble  $V$  is maximally informative for  $\{p_i\}$  maximizing Eq. (33). By the Born rule, the conditional probability  $p_{j|i}$  of outcome  $j$  given state  $|i\rangle$  is  $p_{j|i} = \eta \delta_{i,j} + \frac{1-\eta}{D}$ . Consider two random variables  $X$  and  $Y$  with joint probability  $p_{i,j} = p_i p_{j|i}$  and marginal probabilities  $p_i$  and  $q_j = \sum_i p_i p_{j|i}$ , respectively. Clearly,  $I(X : Y) = I(V, \Pi^{(\eta)})$ . If  $p_i = \frac{1}{D}$ , then  $q_j = \frac{1}{D}$ , and the Shannon entropy  $H(Y)$  of  $Y$  is obviously maximized; that is,  $H(Y) = \log_2 D$ . Moreover, the conditional Shannon entropy  $H(Y|X)$  is independent of  $p_i$ , and in fact one has

$$H(Y|X) = - \left( \eta + \frac{1-\eta}{D} \right) \log_2 \left( \eta + \frac{1-\eta}{D} \right) - (D-1) \frac{1-\eta}{D} \log_2 \frac{1-\eta}{D}. \quad (35)$$

Since  $I(X : Y) = H(Y) - H(Y|X)$ , the maximum of the mutual information is attained for  $p_i = \frac{1}{D}$ , and the informational power is  $W(\Pi^{(\eta)}) = \log_2(D) - H(Y|X)$ . As expected, the informational power is an increasing function of  $\eta$  and is plotted in Fig. 1, for different values of  $D$ .

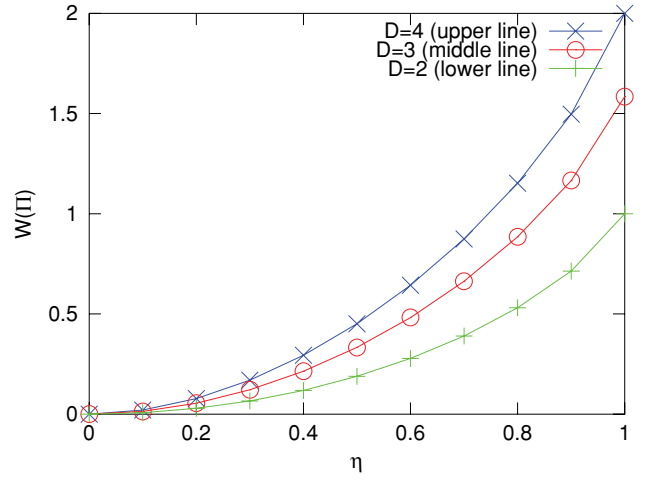


FIG. 1. (Color online) Informational power  $W(\Pi)$  of the  $D$ -dimensional POVM  $\Pi^{(\eta)}$  projecting on the orthonormal basis affected by isotropic noise parameterized by  $\eta$  [see Eq. (34)], as a function of  $\eta$ , for dimension  $D = 2, 3, 4$ .

This result can be useful to prove that the protocols proposed in [2] for the purification of noisy quantum measurements are indeed optimal. The aim of purification of noisy quantum measurements is to recast many uses of a noisy POVM to a single use of an ideal POVM. More precisely, given an ensemble  $R$  and  $N$  uses of a noisy POVM  $\Pi$ , one can ask what channel  $\Phi$  maximizes the mutual information  $I(\Phi(R), \Pi^{\otimes N})$ . For example, suppose that we have the ensemble  $V = \{1/D, |i\rangle\}_{i=1}^D$  and  $N$  uses of the  $D$ -dimensional noisy POVM  $\Pi^{(\eta)}$  as in Eq. (34). Since we have shown that the maximally informative ensemble for  $\Pi^{(\eta)}$  is  $V$ , by Proposition 2, the channel  $\Phi$  that maximizes  $I(\Phi(V), \Pi^{(\eta)\otimes N})$  is the orthogonal cloning; that is,  $\Phi(\rho) = \sum_{i=1}^D \langle i | \rho | i \rangle (|i\rangle \langle i|)^{\otimes N}$ .

##### B. Real-symmetric POVMs

In the following we parametrize any pure state as  $|\psi\rangle = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ , in the basis of eigenvectors  $|0\rangle$  and  $|1\rangle$  of the Pauli matrix  $\sigma_z$ . We denote by  $Z_N$  the group of rotations of  $\pi/N$  around the  $y$  axis, generated by  $U = \exp(-i \frac{\pi}{N} \sigma_y)$ .

*Definition 9.* A two-dimensional real ensemble  $V = \{p_i, |\psi_i\rangle\}_{i=0}^{M-1}$ , with  $|\psi_i\rangle = U^i |\psi_0\rangle$  for any fixed  $|\psi_0\rangle$ , is called real  $Z_M$  symmetric.

*Definition 10.* A two-dimensional real POVM  $\Pi = \{\Pi_j\}_{j=0}^{N-1}$ , with  $\Pi_j = \frac{2}{N} |\pi_j\rangle \langle \pi_j|$  and  $|\pi_j\rangle = U^j |\pi_0\rangle$  for any fixed  $|\pi_0\rangle$ , is called real  $Z_N$  symmetric.

Without loss of generality, we take  $|\pi_0\rangle = |0\rangle$ .

*Proposition 9.* For any real  $Z_N$ -symmetric POVM  $\Pi = \{\frac{2}{N} |\pi_j\rangle \langle \pi_j|\}_{j=0}^{N-1}$ , the ensemble  $V = \{p_i, |\psi_i\rangle\}_{i=0}^{M-1}$ , with  $|\psi_i\rangle = \begin{pmatrix} \sin \theta_i \\ \cos \theta_i \end{pmatrix}$ , is maximally informative if  $M$ ,  $\{\theta_i\}$ , and  $\{p_i\}$  are taken as either

- (1) (real  $Z_N$  symmetric)  $M = N$ ,  $\theta_i = \frac{\pi i}{N}$ , and  $p_i = \frac{1}{N}$  or
- (2) (real  $Y$  shaped)  $M = 3$ ,  $\theta_0 = 0$ ,  $\theta_1 = \frac{\pi n}{N}$ ,  $\theta_2 = -\frac{\pi n}{N}$ , and  $p_0 = 1 - 2p_1$ ,  $p_1 = p_2 = \frac{1}{4 \sin^2 \frac{\pi n}{N}}$ ,  $\forall n$  such that  $0 \leq p_0 \leq 1$ .

The informational power of  $\Pi$  is given by

$$W(\Pi) = \sum_{j=0}^{N-1} \left[ \frac{2}{N} \sin^2 \left( \frac{\pi j}{N} \right) \right] \log_2 \left[ \frac{2}{N} \sin^2 \left( \frac{\pi j}{N} \right) \right] + \log_2 N. \quad (36)$$

*Proof.* The conditional probability  $p_{j|i}$  of outcome  $j$  given state  $|\psi_i\rangle$  is  $p_{j|i} = \frac{2}{N} \sin^2(\theta_i - \frac{\pi j}{N})$ , and the probability  $q_j$  of outcome  $j$  is  $q_j = \sum_{i=0}^{M-1} p_i p_{j|i}$ .

Consider the random variables  $X$  and  $Y$ , with  $X$  distributed according to  $p_i$ , and  $Y$  such that the conditional probability of outcome  $j$  of  $Y$  given outcome  $i$  of  $X$  is  $p_{j|i}$ . Clearly,  $I(X : Y) = I(V, \Pi)$ .

By setting  $f(\theta_i) = \sum_{j=0}^{N-1} p_{j|i} \log_2 p_{j|i}$ , we have, for the joint entropy,  $H(Y|X) = -\sum_{i=0}^{M-1} p_i f(\theta_i)$ . As shown in Lemma 3 of [16],  $f(\theta)$  attains its global maximum for  $\theta = \frac{\pi k}{N}$ ,  $k \in \mathbb{N}$ . Thus by choosing  $\{\theta_i\}$  multiples of  $\frac{\pi}{N}$ ,  $H(Y|X)$  attains its minimum  $H(Y|X) = f(0)$ , independent of  $M$  and  $\{p_i\}$ .

By taking the real  $Z_N$ -symmetric or the real  $Y$ -shaped parametrizations for  $M$ ,  $\{\theta_i\}$ , and  $\{p_i\}$ , we have  $q_j = \frac{1}{N}$ , so the entropy  $H(Y)$  attains its maximum; that is,  $H(Y) = \log_2 N$ . Since  $I(X : Y) = H(Y) - H(Y|X)$ , the proposition remains proved. ■

We notice that for a real  $Z_N$ -symmetric POVM  $\Pi = \{\frac{2}{N}|\pi_j\rangle\langle\pi_j|\}$ , any maximally informative ensemble  $V = \{p_i, |\psi_i\rangle\}$  given in Proposition 9 is such that every state  $|\psi_i\rangle$  is orthogonal to one of the  $|\pi_j\rangle$ . Considering the real  $Y$ -shaped parametrization, we observe that if  $N$  is even, one can chose  $n = \frac{N}{2}$ , obtaining  $V = \{1/2, |i\rangle\}$ , with  $i = 0, 1$ . With this choice, the maximally informative real  $Y$ -shaped ensemble is minimal. For some real  $Z_N$ -symmetric POVMs, the maximally informative ensembles with a minimal number of states are represented in Fig. 2.

The real  $Z_3$ -symmetric POVM  $\Pi$  is usually called the *trine* measurement. The informational power of  $\Pi$  is  $W(\Pi) = \log_2 3/2$  by Proposition 9. The maximally informative ensemble for  $\Pi$  parameterized as in Proposition 9 is usually called *antitrine*. The analogous problem of maximization of the accessible information for real-symmetric ensembles has been addressed by Holevo [9] and by Sasaki *et al.* [16].

### C. Mirror-symmetric POVMs

In this subsection we apply the duality shown in Proposition 3 between the informational power and the accessible information to mirror-symmetric POVMs.

*Definition 11.* We call a mirror-symmetric ensemble any two-dimensional real ensemble  $S = \{p_i, |\psi_i\rangle\}$  such that for any  $|\psi_i\rangle$ , there exists a  $|\psi_k\rangle = \sigma_z |\psi_i\rangle$  and  $p_i = p_k$ .

*Definition 12.* We call a mirror-symmetric POVM any two-dimensional real POVM  $\Lambda = \{\Lambda_j\}$  with  $\Lambda_j = n_j |\lambda_j\rangle\langle\lambda_j|$  such that for any  $|\lambda_j\rangle$ , there exists a  $|\lambda_i\rangle = \sigma_z |\lambda_j\rangle$  and  $n_j = n_i$ .

The problem of accessible information for mirror-symmetric POVMs has been addressed in [19]. From Definitions 5 and 6, it immediately follows that if the ensemble  $S$  is mirror symmetric, the POVM  $\Pi(S)$  is mirror symmetric, and that if the POVM  $\Lambda$  is mirror symmetric, the ensemble  $R(\Lambda, \sigma)$  is mirror symmetric, for any density matrix  $\sigma$ .

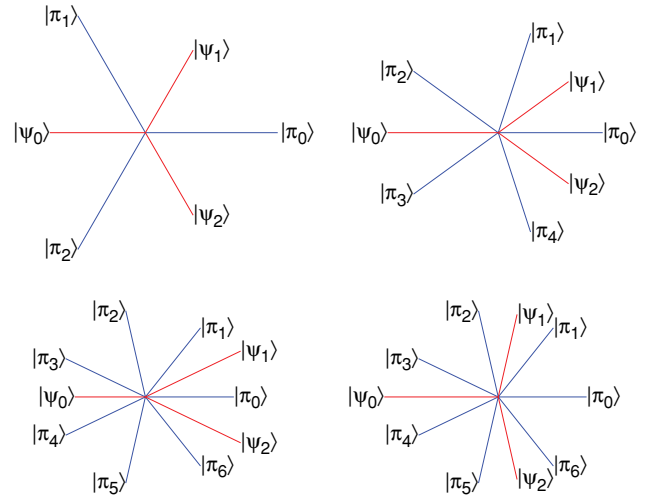


FIG. 2. (Color online) Real  $Z_N$ -symmetric POVMs  $\Pi = \{\frac{2}{N}|\pi_j\rangle\langle\pi_j|\}_{j=0}^{N-1}$  (blue vectors labeled  $|\pi_j\rangle$ ) for  $N = 3$  (upper left),  $N = 5$  (upper right), and  $N = 7$  (lower left and lower right). A maximally informative ensemble (red vectors labeled  $|\psi_i\rangle$ )  $V = p_i, |\psi_i\rangle_{i=0}^{M-1}$  with a minimal number of states is represented for each POVM. For  $N = 7$ , there are two inequivalent maximally informative ensembles (lower left and lower right). The angle between state  $|\psi\rangle = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$  and state  $|\psi'\rangle = \begin{pmatrix} \cos \theta' \\ \sin \theta' \end{pmatrix}$  is  $2(\theta_1 - \theta_0)$ , as in the Bloch sphere representation. The length of the vector corresponding to state  $|\psi_i\rangle$  is proportional to  $\sqrt{p_i}$ .

*Proposition 10.* Given a mirror-symmetric POVM  $\Lambda = \{\Lambda_j\}$ , there exists a maximally informative ensemble  $S = \{p_i, |\psi_i\rangle\}_{i=0}^{M-1}$  such that  $S$  is mirror symmetric and  $M \leq 4$ .

*Proof.* By Proposition 3,  $S^*$  is maximally informative for  $\Lambda$  if and only if  $\sigma_{S^*} = \arg \max_{\sigma} A(R(\Lambda, \sigma))$  and  $\Pi(S^*)$  is maximally informative for  $R(\Lambda, \sigma_{S^*})$ . By Proposition 2 in [19], there exists a maximally informative mirror-symmetric four-element POVM  $\Pi(S^*)$ , so the statement follows.

As an application we consider the mirror-symmetric POVM  $\Pi = \{n_j |\pi_j\rangle\langle\pi_j|\}_{j=0}^2$ , with

$$|\pi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\pi_1\rangle = \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}, \quad |\pi_2\rangle = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}, \quad (37)$$

and  $n_0 = \frac{\cos 2\theta}{\cos^2 \theta}$  and  $n_1 = n_2 = \frac{1}{2 \cos^2 \theta}$ . Figure 3 shows the informational power  $W(\Pi)$  as a function of  $\theta$ , as obtained by Algorithm 1. ■

### D. Two-dimensional SIC POVM

According to [20] and [21], we provide the following definition.

*Definition 13.* A  $D$ -dimensional POVM  $\Pi = \{\Pi_j\}_{j=0}^{N-1}$  with  $N = D^2$  elements  $\Pi_j = \frac{1}{D} |\pi_j\rangle\langle\pi_j|$  with invariant inner product  $\text{Tr}[\Pi_j \Pi_l] = [D^2(D+1)]^{-1}$ , for any  $i \neq j$ , is called a symmetric informationally complete (SIC) POVM.

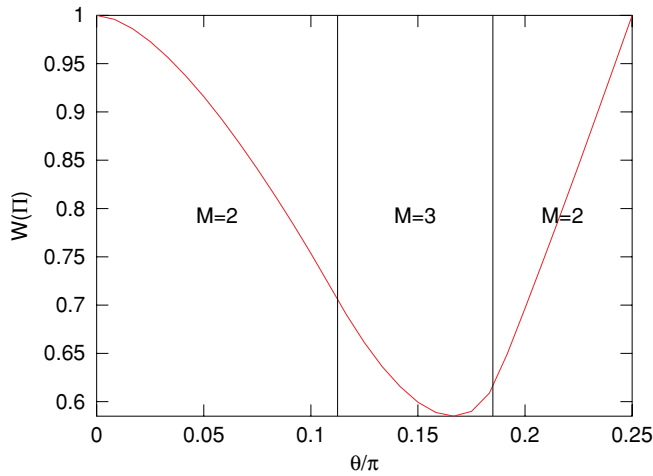


FIG. 3. (Color online) Informational power of a mirror-symmetric POVM  $\Pi = \{n_j |\pi_j\rangle\langle\pi_j|\}_{j=0}^2$ , with  $|\pi_j\rangle$  as in Eq. (37), as a function of  $\theta$ . The minimum is attained for  $\theta = \pi/6$ , where  $\Pi$  corresponds to the trine POVM and a maximally informative ensemble is the antitrine. The maxima at  $\theta = 0$  and  $\theta = \pi/4$  correspond to the degenerate case of the POVM projecting on an orthonormal basis. The label  $M = 2$  or  $M = 3$  denotes the minimum number  $M$  of states of a maximally informative ensemble in each of the three regions.

For  $D = 2$  there exists only one SIC POVM  $\Pi = \{\frac{1}{2}|\pi_j\rangle\langle\pi_j|\}_{j=0}^3$ , with

$$\begin{aligned} |\pi_0\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & |\pi_1\rangle &= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}, \\ |\pi_2\rangle &= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ e^{i\frac{2}{3}\pi} \sqrt{\frac{2}{3}} \end{pmatrix}, & |\pi_3\rangle &= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ e^{i\frac{4}{3}\pi} \sqrt{\frac{2}{3}} \end{pmatrix}. \end{aligned} \quad (38)$$

Since these states lie on the four vertex of a tetrahedron, this POVM is usually called the *tetrahedron*.

*Proposition 11.* Given the two-dimensional SIC POVM  $\Pi = \{\frac{1}{2}|\pi_j\rangle\langle\pi_j|\}_{j=0}^3$ , the ensemble  $V = \{\frac{1}{4}|\psi_i\rangle\}_{i=0}^3$  with

$$\begin{aligned} |\psi_0\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & |\psi_1\rangle &= \begin{pmatrix} \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \\ |\psi_2\rangle &= \begin{pmatrix} \sqrt{\frac{2}{3}} \\ e^{i\frac{1}{3}\pi} \frac{1}{\sqrt{3}} \end{pmatrix}, & |\psi_3\rangle &= \begin{pmatrix} \sqrt{\frac{2}{3}} \\ e^{i\frac{2}{3}\pi} \frac{1}{\sqrt{3}} \end{pmatrix} \end{aligned} \quad (39)$$

is maximally informative, and the informational power is  $W(\Pi) = \log_2 \frac{4}{3}$ .

*Proof.* Consider an ensemble  $V = \{p_i, |\psi_i\rangle\}$  parameterized as  $|\psi_i\rangle = \begin{pmatrix} \sin \theta_i \\ e^{i\theta_i} \cos \theta_i \end{pmatrix}$ . Call  $p_{j|i} = |\langle\psi_i|\pi_j\rangle|^2$  the conditional probability of outcome  $j$  given state  $|\psi_i\rangle$ , and  $q_j = \sum_{i=0}^3 p_i p_{j|i}$  the probability of outcome  $j$ .

Consider the random variables  $X$  and  $Y$ , with  $X$  distributed according to  $p_i$ , and  $Y$  such that the conditional probability of outcome  $j$  of  $Y$  given outcome  $i$  of  $X$  is  $p_{j|i}$ . Clearly,  $I(X : Y) = I(V, \Pi)$ .

By setting  $f(\theta_i, \phi_i) = \sum_{j=0}^{N-1} p_{j|i} \log_2 p_{j|i}$ , we have, for the joint entropy,  $H(Y|X) = -\sum_{i=0}^{M-1} p_i f(\theta_i, \phi_i)$ . As it is easy to show,  $f(\theta, \phi)$  attains its global maximum  $\log_2 3$  at  $\theta = 0$  for any  $\phi$ , and at  $\theta = \arccos(\frac{1}{\sqrt{3}})$  for  $\phi = \frac{\pi}{3}$ ,  $\phi = \pi$ , and  $\phi = \frac{5\pi}{3}$ . Thus, making one of these choices for  $\{\theta_i, \phi_i\}$ ,  $H(Y|X)$  attains its minimum  $H(Y|X) = \log_2 3$ .

Moreover, by setting  $M = 4$  and  $p_i = 1/4$ , we have  $q_j = \frac{1}{4}$ , so the entropy  $H(Y)$  attains its maximum; that is,  $H(Y) = \log_2 4$ . Since  $I(X : Y) = H(Y) - H(Y|X)$ , the proposition remains proved.

We notice that for the two-dimensional SIC POVM  $\Pi = \{\frac{1}{2}|\pi_j\rangle\langle\pi_j|\}_{j=0}^3$ , the maximally informative ensemble  $V = \{\frac{1}{4}|\psi_i\rangle\}_{i=0}^3$  in Proposition 11 is such that any state  $|\psi_i\rangle$  is orthogonal to one state  $|\pi_j\rangle$ . Since the states of  $V$  lie on the vertexes of a tetrahedron, this ensemble is usually called *antitetrahedron*. The accessible information of the ensemble which enjoys the same symmetry as  $\Pi$  has been proven in [15] to be  $\log_2 4/3$ . We want to comment that generally SIC POVMs have low informational power, as happens for overcomplete measurements: for informational completeness one must pay the price of low informational power. ■

## V. CONCLUSIONS

In this work we have introduced the informational power of a quantum measurement as the maximum amount of classical information that the POVM can extract from any ensemble of states. We have shown that it is the classical capacity of a q-c channel and proved additivity. We have restated the problem of maximizing the informational power of a POVM as the problem of maximizing the accessible information of a suitable ensemble and provided a bound on the minimal number of states of a maximally informative ensemble. Then we have provided a numerical algorithm to find a maximally informative ensemble for a given POVM. Finally, we have classified some POVMs according to their informational power, namely, POVMs with commuting elements, real-symmetric POVMs, and mirror-symmetric POVMs.

The results presented have obvious practical relevance in several contexts, such as the communication of classical information over quantum channels and the storage and retrieval of information from quantum memories.

*Note added in proof.* Recently, two related papers have appeared on arXiv [22,23]. In particular, Holevo [23] studied the informational power in the relevant infinite-dimensional case.

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