Multisetting Bell-type inequalities for detecting genuine multipartite entanglement

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(Received 21 February 2011; published 29 June 2011)

In a recent paper, Bancal *et al.* [Phys. Rev. Lett. **106**, 250404 (2011)] put forward the concept of device-independent witnesses of genuine multipartite entanglement. These witnesses are capable of verifying genuine multipartite entanglement produced in a laboratory without resorting to any knowledge of the dimension of the state space or of the specific form of the measurement operators. As a by-product they found a multiparty three-setting Bell inequality which makes it possible to detect genuine n-partite entanglement in a noisy n-qubit Greenberger-Horne-Zeilinger (GHZ) state for visibilities as low as $2/3$ in a device-independent way. In this paper, we generalize this inequality to an arbitrary number of settings, demonstrating a threshold visibility of $2/\pi \sim 0.6366$ for number of settings going to infinity. We also present a pseudotelepathy Bell inequality achieving the same threshold value. We argue that our device-independent witnesses are optimal in the sense that for n odd the above value cannot be beaten with n-party-correlation Bell inequalities.

DOI: 10.1103/PhysRevA.83.062123 PACS number(s): 03.65.Ud, 03.67.Mn

I. INTRODUCTION

Quantum theory allows correlations between remote systems, which are fundamentally different from classical correlations [1]. Quantum entanglement is in the heart of this phenomenon [2]. Already two entangled particles give rise to correlations not reproducible within any local realistic theory [3]. However, moving to more particles a much richer structure and various types of entanglement arise [4], suggesting novel applications such as quantum computation using cluster states [5], sub-shot-noise metrology [6], or multiparty quantum networking [7]. In these tasks, genuinely entangled particles offer enhanced performance. Hence, it is a central problem to decide whether in an actual experiment genuinely multipartite entanglement has been produced, or alternatively, the entangled state prepared in the laboratory could be explained without requiring the interaction of all particles. In the latter case, we say that the state created is biseparable. Focusing on the tripartite case, a biseparable state ρ_{bs} can be written as

$$\rho_{bs} = \sum_i p_i |\phi_i\rangle\langle\phi_i|, \tag{1}$$

where the pure states ϕ_i are separable with respect to one of the three bipartitions 1|23, 12|3, 13|2, and the weights $p_i > 0$ add up to 1. For more than three parties, the generalization is straightforward.

Several experiments have been conducted so far generating multipartite entangled photonic states up to six photons (for instance, Ref. [8] generated a Dicke state of six photons). One of the traditional approaches to decide on the existence of genuine multipartite entanglement consists in performing a complete state tomography and then deducing the kind of entanglement directly from the density matrix using witness operators. Alternatively, the experimentalist may measure cleverly chosen witness operators, thereby reducing the number of correlation terms to be measured in the actual experiment [9]. However, a common drawback is that in both cases the experimentalist needs to have a precise control over the system on which the measurements are performed.

Remarkably, there is another route, avoiding the above problem, building on the seminal work of John Bell [1]: Bell expressions are linear functions of joint correlations enabling one to say important things in a black-box scenario about the dimension of the systems, the states involved, or the kind of measurements performed. In particular, it is possible to decide on the presence of genuine multipartite entanglement based on merely statistical data (that is, without relying on any knowledge of the implementation of the devices involved in the measurement process) [10,11]: If a Bell value, coming from the statistics of a Bell experiment, is bigger than a certain value achievable with measurements acting on biseparable quantum states, then we can be sure that the state in question is genuinely multipartite entangled. This approach has been formalized more recently by Bancal *et al.* [11], coining the term device-independent witnesses of genuine multipartite entanglement for such Bell expressions (for more details, we refer the reader to that paper).

As a simplest illustration of a device-independent witness of genuine tripartite entanglement, let us represent the Mermin polynomial [12] in terms of three-party correlators,

$$I_2 \equiv \langle \hat{A}_0 \otimes \hat{B}_0 \otimes \hat{C}_0\rangle - \langle \hat{A}_0 \otimes \hat{B}_1 \otimes \hat{C}_1\rangle - \langle \hat{A}_1 \otimes \hat{B}_0 \otimes \hat{C}_1\rangle - \langle \hat{A}_1 \otimes \hat{B}_1 \otimes \hat{C}_0\rangle, \tag{2}$$

where $\langle \hat{A}_\alpha \otimes \hat{B}_\beta \otimes \hat{C}_\gamma\rangle$ designate the expected value of the product of three ± 1 observables, $\hat{A}_\alpha$, $\hat{B}_\beta$, $\hat{C}_\gamma$. It has been shown in Ref. [10], that $I_2 \leqslant 2\sqrt{2}$ for biseparable quantum states ($\mathcal{B}_2 = 2\sqrt{2}$), whereas the maximum quantum value saturates the algebraic limit of 4 ($\mathcal{Q}_2 = 4$); hence, the violation of the bound $\mathcal{B}_2$ implies genuine tripartite entanglement. Note that this reasoning holds true independently on the size of the Hilbert space dimension or on the type of measurements carried out. Hence, Mermin inequality serves as a device-independent witness of genuine tripartite entanglement [11]. Let us now take the noisy three-qubit Greenberger-Horne-

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Zeilinger (GHZ) state,

$$\rho(V) = V|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-V)\frac{\mathbb{1}}{8}, \tag{3}$$

where $|\mathrm{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ is the three-qubit GHZ state [13], and V is the visibility parameter. The measurements achieving the bounds $\mathcal{Q}_2$ and $\mathcal{B}_2$ correspond to traceless observables, entailing the threshold visibility $V = \mathcal{B}_2/\mathcal{Q}_2 = 1/\sqrt{2}$. Hence, genuine tripartite entanglement in the noisy GHZ state (for $V > 1/\sqrt{2}$) can be detected in a device-independent way.

More recently, however, Bancal *et al.* [11] managed to lower the threshold visibility in the noisy three-party GHZ state to $V = 2/3$ by considering a three-party, three-setting Bell inequality, which can be considered as a three-setting generalization of the two-setting Mermin inequality. Note that similarly to the Mermin inequality, the Bancal *et al.* inequality extends to more than three parties as well [11], detecting genuine n-partite entanglement in a noisy n-qubit GHZ state for the threshold visibility $V = 2/3$. Actually, the same inequality has already appeared in the literature [14], but it was used for a different purpose.

In the present paper, we generalize the n-party three-setting Bancal *et al.* inequality to an arbitrary number of settings m, exhibiting the threshold visibility $V = 1/[m\sin(\pi/2m)]$, which approaches $V = 2/\pi$ for large number of settings. First, we discuss the three-party case in Secs. II and III. In particular, another family of Bell inequalities is given in Sec. III, based on the extended parity game [15] yielding the same performance (for m a power of 2) as the Bell inequality of Sec. II. Then, in Sec. IV, we move on to more than three parties by generalizing the results of the preceding sections. The paper concludes in Sec. V with open questions and with a brief summary of the results.

II. MULTISETTING TRIPARTITE BELL-TYPE INEQUALITIES

Let us introduce the m-setting tripartite Bell expression, originally due to Zukowski [16],

$$I_m = \sum_{\alpha,\beta,\gamma=0}^{m-1} M_{\alpha\beta\gamma}\langle \hat{A}_\alpha \otimes \hat{B}_\beta \otimes \hat{C}_\gamma\rangle, \tag{4}$$

where the matrix of Bell coefficients is defined by

$$M_{\alpha\beta\gamma} = \cos\left[\frac{\pi}{m}(\alpha+\beta+\gamma-\Delta)\right], \tag{5}$$

where indices α, β, and γ may take the values of $0,1,\ldots,m-1$, and Δ may be any real number. By choosing $m=2$ and $\Delta = 0$, the Mermin polynomial (2) is recovered. On the other hand, for $m=3$ and $\Delta = -1/2$, we obtain the polynomial of Bancal *et al.* [11] (apart from an irrelevant multiplicative factor).

We next exhibit a lower bound on the quantum maximum, $\mathcal{Q}_m^l = m^3/2$, as a function of number of settings m. Then an upper bound is given on the biseparable quantum maximum, which is shown to be attained by von Neumann-type projective measurements, $\mathcal{B}_m = m^2/[2\sin(\pi/2m)]$. This implies the threshold visibility $V = \mathcal{B}_m/\mathcal{Q}_m \leqslant \mathcal{B}_m/\mathcal{Q}_m^l = 1/[m\sin(\pi/2m)]$ tending to $2/\pi$ in the limit of large number of measurement settings.

We wish to note that Bancal *et al.* (Appendix C in [11]) presented a biseparable model simulating all the single-party expectations $\langle\hat{A}\rangle$, $\langle\hat{B}\rangle$, $\langle\hat{C}\rangle$, two-party correlators $\langle\hat{A}\otimes\hat{B}\rangle$, $\langle\hat{A}\otimes\hat{C}\rangle$, $\langle\hat{B}\otimes\hat{C}\rangle$, and three-party correlators $\langle\hat{A}\otimes\hat{B}\otimes\hat{C}\rangle$, achievable with von Neumann measurements on the noisy three-qubit GHZ state (3) of visibility $V \leqslant 1/2$. Within this biseparable model all of the three parties may share local random variables, but at most two parties can share a quantum state at a given time. It is shown in the Appendix that if we are content with simulating only the three-party correlators, then the threshold visibility becomes a higher value, $V = 2/\pi$. This implies that it is not possible to detect genuine tripartite entanglement in the three-qubit GHZ state in the range $V \leqslant 2/\pi$ by considering Bell expressions which are sums of three-party correlators. In this sense, our family of Bell inequalities is optimal, giving $V \to 2/\pi$ when m goes to infinity.

Lower bound on the quantum maximum, $\mathcal{Q}_m^l$. If each of the participants performs a von Neumann projective measurement on one component of a shared three-qubit GHZ state, the tripartite correlation of their measurement values can be written as (see, for instance, Appendix C in Ref. [11])

$$\langle \hat{A}_\alpha \otimes \hat{B}_\beta \otimes \hat{C}_\gamma\rangle = \sin\theta_\alpha^A \sin\theta_\beta^B \sin\theta_\gamma^C \cos\left(\varphi_\alpha^A + \varphi_\beta^B + \varphi_\gamma^C\right), \tag{6}$$

where the αth, βth, and γth measurement operators $\hat{A}_\alpha$, $\hat{B}_\beta$, and $\hat{C}_\gamma$ of Alice, Bob, and Cecil, respectively, are given as

$$\begin{aligned}\hat{A}_\alpha &= \cos\varphi_\alpha^A \sin\theta_\alpha^A \hat{\sigma}_x + \sin\varphi_\alpha^A \sin\theta_\alpha^A \hat{\sigma}_y + \cos\theta_\alpha^A \hat{\sigma}_z,\\ \hat{B}_\beta &= \cos\varphi_\beta^B \sin\theta_\beta^B \hat{\sigma}_x + \sin\varphi_\beta^B \sin\theta_\beta^B \hat{\sigma}_y + \cos\theta_\beta^B \hat{\sigma}_z,\\ \hat{C}_\gamma &= \cos\varphi_\gamma^C \sin\theta_\gamma^C \hat{\sigma}_x + \sin\varphi_\gamma^C \sin\theta_\gamma^C \hat{\sigma}_y + \cos\theta_\gamma^C \hat{\sigma}_z,\end{aligned} \tag{7}$$

where $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ are the Pauli operators.

With the choice of $\theta_\mu^A = \theta_\mu^B = \theta_\mu^C = \pi/2$ and $\varphi_\mu^A = \varphi_\mu^B = \varphi_\mu^C = \pi(\mu - \Delta/3)/m$ each tripartite correlation will take the same value as the Bell coefficient to be multiplied with, and the quantum value of the Bell expression will be easy to calculate:

$$\begin{aligned}\mathcal{Q}_m^l &= \sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{m-1}\sum_{\gamma=0}^{m-1} M_{\alpha\beta\gamma}\langle \hat{A}_\alpha \otimes \hat{B}_\beta \otimes \hat{C}_\gamma\rangle\\ &= \sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{m-1}\sum_{\gamma=0}^{m-1}\cos^2\left[\frac{\pi}{m}(\alpha+\beta+\gamma-\Delta)\right]\\ &= \frac{1}{2}\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{m-1}\sum_{\gamma=0}^{m-1}\left\{1-\cos\left[\frac{2\pi}{m}(\alpha+\beta+\gamma-\Delta)\right]\right\}\\ &= \frac{m^3}{2}.\end{aligned} \tag{8}$$

This value $\mathcal{Q}_m^l$ is a lower bound for the maximum quantum value $\mathcal{Q}_m$.

For the maximum of the biseparable value first we give an upper bound ($\mathcal{B}_m^u$), then we prove that this bound can be saturated, that is, ($\mathcal{B}_m^l = \mathcal{B}_m^u = \mathcal{B}_m$).

Upper bound on the biseparable quantum value, $\mathcal{B}_m^u$. The value to be calculated is

$$\mathcal{B}_m = \max \sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{m-1}\sum_{\gamma=0}^{m-1} M_{\alpha\beta\gamma} A_\alpha \langle \hat{B}_\beta \otimes \hat{C}_\gamma \rangle = \max \sum_{\beta=0}^{m-1}\sum_{\gamma=0}^{m-1} M_{\beta\gamma}^{\{A\}} \langle \hat{B}_\beta \otimes \hat{C}_\gamma \rangle, \quad (9)$$

where we used the fact that Bell inequality (4) is linear in the correlators and that a biseparable density matrix (1) is a convex combination of pure states; hence, it is enough to take the three-party correlators in the form $\langle \hat{A}_\alpha \rangle \langle \hat{B}_\beta \otimes \hat{C}_\gamma \rangle$. In Eq. (9) each of A_α may take the value of either $+1$ or -1, Bob and Cecil may share any quantum state and perform measurements on them, the operators of their measurement settings are $\hat{B}_\beta$ and $\hat{C}_\gamma$, respectively, and the coefficients of the two-partite Bell inequality, which depends on the actual choice of A_α, are

$$M_{\beta\gamma}^{\{A\}} = \sum_{\alpha=0}^{m-1} A_\alpha \cos\left[\frac{\pi}{m}(\alpha+\beta+\gamma-\Delta)\right]. \quad (10)$$

We note that due to the symmetry of the Bell expression under party exchange, it is enough to consider the case when it is Alice who may not share an entangled quantum object with the others.

From the work of Ref. [17] it easily follows that for bipartite correlation type Bell inequalities with an equal number of measurement settings per party, an upper bound for the maximum quantum value is the largest of the singular values of the matrix defined by the Bell coefficients multiplied by the number of measurement settings. In the present case the matrix depends on the sum of its indices. Therefore, $M_{(\beta+1)\gamma}^{\{A\}} = M_{\beta(\gamma+1)}^{\{A\}}$; that is, each row contains the elements of the preceding row, shifted to the left. From Eq. (10) it is also clear that $M_{(\beta+1)m}^{\{A\}} = -M_{\beta 1}^{\{A\}}$; that is, the last element of each row is the same as -1 times the first element of the preceding row. These properties are very similar to the properties defining circulant matrices [18], whose eigenvectors are independent of the actual values of its elements and therefore whose eigenvalues are very easy to derive. There are just two differences. In the case of the circulant matrices the elements are shifted not to the left, but to the right. Furthermore, they do it cyclically; that is, there is no change of sign when the last element takes the first place in the next row. The first difference is easily corrected if we rearrange Cecil's measurement settings into the opposite order. Fortunately, the change of sign of the matrix element poses no serious problem either, because it can be shown that the eigenvectors of these modified circulant matrices are also independent of the actual values of the elements of the matrix; they are given as

$$v_j = \left(1,\omega_j,\omega_j^2,\ldots,\omega_j^{m-1}\right)^T, \quad \omega_j = e^{\frac{2\pi i(j+1/2)}{m}}, \quad (11)$$

where $j=0,\ldots,m-1$. The difference from the circulant case [18] is the $1/2$ term in the exponent. To calculate the eigenvalues we only need the first row of the matrix. Therefore, we get the upper bound for the biseparable value as

$$\mathcal{B}_m^u = \max m \left| \sum_{\alpha=0}^{m-1}\sum_{\gamma=0}^{m-1} A_\alpha \cos\left[\frac{\pi(\alpha-\Delta'-\gamma)}{m}\right]\omega_j^\gamma \right|, \quad (12)$$

where we introduced the notation $\Delta' \equiv \Delta - m + 1$. By substituting the ω_j from Eq. (11) and using identity

$$\cos\frac{\pi(\alpha-\Delta'-\gamma)}{m} = \cos\frac{\pi(\alpha-\Delta')}{m}\cos\frac{\pi\gamma}{m} + \sin\frac{\pi(\alpha-\Delta')}{m}\sin\frac{\pi\gamma}{m}, \quad (13)$$

we arrive at

$$\mathcal{B}_m^u = \max m \left| \sum_{\alpha=0}^{m-1} A_\alpha \Big[\cos\frac{\pi(\alpha-\Delta')}{m}\left(D_1^j + iD_2^j\right) + \sin\frac{\pi(\alpha-\Delta')}{m}\left(D_3^j + iD_4^j\right)\Big] \right|, \quad (14)$$

where

$$\begin{aligned} D_1^j &= \sum_{\gamma=0}^{m-1} \cos\frac{\pi\gamma}{m}\cos\frac{2\pi(j+1/2)\gamma}{m}, \\ D_2^j &= \sum_{\gamma=0}^{m-1} \cos\frac{\pi\gamma}{m}\sin\frac{2\pi(j+1/2)\gamma}{m}, \\ D_3^j &= \sum_{\gamma=0}^{m-1} \sin\frac{\pi\gamma}{m}\cos\frac{2\pi(j+1/2)\gamma}{m}, \\ D_4^j &= \sum_{\gamma=0}^{m-1} \sin\frac{\pi\gamma}{m}\sin\frac{2\pi(j+1/2)\gamma}{m}. \end{aligned} \quad (15)$$

However,

$$D_2^j \pm D_3^j = \sum_{\gamma=0}^{m-1} \sin\frac{2\pi(j+1/2\pm 1/2)\gamma}{m} = 0; \quad (16)$$

therefore, $D_2^j = D_3^j = 0$, and

$$D_1^j \pm D_4^j = \sum_{\gamma=0}^{m-1} \cos\frac{2\pi(j+1/2\mp 1/2)\gamma}{m}, \quad (17)$$

from which it follows that $D_1^j = D_4^j = 0$ for $1 \leqslant j \leqslant m-2$, $D_1^0 = D_4^0 = m/2$, and $D_1^{m-1} = -D_4^{m-1} = m/2$. To get the maximum we must take either $j=0$ or $j=m-1$. For $j=0$ we get

$$\mathcal{B}_m^u = \max \frac{m^2}{2} \left| \sum_{\alpha=0}^{m-1} A_\alpha e^{\frac{i\pi(\alpha-\Delta')}{m}} \right|. \quad (18)$$

If we have taken $j=m-1$ instead of $j=0$, we would have gotten the complex conjugate of the numbers whose absolute value has to be taken, which would have given the same result. In Eq. (18) we have to add m vectors on the complex plane, each pointing toward corners of a regular polygon of $2m$ sides, and then we have to take the length of this vector. Each vector lies on a different diagonal of the polygon, but may point toward either direction depending on the value of A_α. It can be shown that we get the largest value if the vectors taken in some order point toward consecutive corners. All such arrangements give obviously the same result. We get one of those arrangements if we take $A_\alpha = 1$. The result does not depend on Δ', as changing Δ' means only an overall rotation of the arrangement. Let us take $\Delta' = -1/2$. Then the set of numbers will be symmetric with respect to the imaginary axis; therefore, the real part of the sum will be zero, while the imaginary part will be positive. Then we get

$$\begin{aligned}\mathcal{B}^u_m &= \frac{m^2}{2}\sum_{\alpha=0}^{m-1}\sin\frac{\pi(\alpha+1/2)}{m} \\ &= \frac{m^2}{2\sin\frac{\pi}{2m}}\sum_{\alpha=0}^{m-1}\left(\sin\frac{\pi(\alpha+1/2)}{m}\sin\frac{\pi}{2m} \right. \\ &\quad \left. + \cos\frac{\pi(\alpha+1/2)}{m}\cos\frac{\pi}{2m}\right) \\ &= \frac{m^2}{2\sin\frac{\pi}{2m}}\sum_{\alpha=0}^{m-1}\cos\frac{\pi\alpha}{m} = \frac{m^2}{2\sin\frac{\pi}{2m}}. \end{aligned} \tag{19}$$

Here we have used that $\sum_{\alpha=0}^{m-1}\cos[\pi(\alpha+1/2)/m] = 0$ and that $\cos(\pi\alpha/m) = -\cos[\pi(m-\alpha)/m]$.

Now we show that this upper bound can be saturated.

Lower bound on the biseparable quantum value, $\mathcal{B}^l_m$. If $\sum_{\beta=0}^{m-1}\sum_{\gamma=0}^{m-1}\bar{M}_{\beta\gamma}\vec{B}_\beta\cdot\vec{C}_\gamma$ is a certain number, where $\vec{B}_\beta$ and $\vec{C}_\gamma$ are Euclidean unit vectors, then there exist measurement operators giving the same number as the quantum value of the bipartite correlation type Bell inequality of coefficients $\bar{M}_{\beta\gamma}$, applied on the maximally entangled state [19]. In case of two-dimensional vectors pairs of real qubits are sufficient. Let $\bar{M}_{\beta\gamma} \equiv M^{\{A\}}_{\beta\gamma}$ with all $A_\alpha = +1$ [see Eq. (10)], let Cecil's vectors be

$$\vec{C}_\gamma = \begin{pmatrix}\cos\frac{\pi\gamma}{m} \\ \sin\frac{\pi\gamma}{m}\end{pmatrix}, \tag{20}$$

and let us choose $\vec{B}_\beta$ optimally, that is, $\vec{B}_\beta = \sum_{\gamma=0}^{m-1}\bar{M}_{\beta\gamma}\vec{C}_\gamma/|\sum_{\gamma=0}^{m-1}\bar{M}_{\beta\gamma}\vec{C}_\gamma|$. Then the corresponding quantum value is

$$\begin{aligned}\mathcal{B}^l_m &= \sum_{\beta=0}^{m-1}\left|\sum_{\gamma=0}^{m-1}\bar{M}_{\beta\gamma}\vec{C}_\gamma\right| \\ &= \sum_{\beta=0}^{m-1}\left|\sum_{\alpha=0}^{m-1}\sum_{\gamma=0}^{m-1}\cos\frac{\pi(\alpha+\beta+\gamma-\Delta)}{m}\begin{pmatrix}\cos\frac{\pi\gamma}{m} \\ \sin\frac{\pi\gamma}{m}\end{pmatrix}\right|.\end{aligned} \tag{21}$$

Now if we follow analogous steps to the ones we have taken calculating the value of $\mathcal{B}^u$, we will arrive at the same result. We can also easily see this if we compare Eq. (21) to Eq. (12). The maximum value of the latter expression has been attained with $A_\alpha = 1$ and $j = 0$. By substituting these values, and also ω_0 from Eq. (11), we get almost the same formula as Eq. (21), indeed. The ω_0^γ complex numbers correspond to the same vectors on the complex plane as the two-dimensional vectors appearing in Eq. (21). From the calculation of $\mathcal{B}^u$ it turns out that the value does not depend on Δ', so the absolute value in Eq. (21) does not depend on $\Delta-\beta$ either; therefore, we may replace the summation in terms of β for a multiplicative factor of m. The only remaining difference is the opposite sign of γ in the cosine, but that will not affect the result either.

As the upper and lower bound for the biseparable case are equal, the biseparable value itself is given by Eq. (19). For the quantum value we have only proven a lower bound [see Eq. (8)]. Therefore, for this family of Bell inequalities the ratio of the quantum and the biseparable values (which equals the visibility threshold) satisfies

$$V = \frac{\mathcal{B}_m}{\mathcal{Q}_m} \leqslant \frac{1}{m\sin\frac{\pi}{2m}}. \tag{22}$$

We believe that the lower bound $\mathcal{Q}^l_m$ we have given in (8) is actually the quantum maximum itself, and the above expression is valid as an equality.

III. BELL-TYPE INEQUALITIES BASED ON THE EXTENDED PARITY GAME

Now we define another family of multisetting tripartite inequalities giving the same ratio of $\mathcal{B}_m/\mathcal{Q}_m$ as the right-hand side of Eq. (22), at least when m is a power of 2.

The Bell coefficients may only take the values of 0, 1, and -1, namely,

$$M_{\alpha\beta\gamma} = \begin{cases} 0 & \text{if } (\alpha+\beta+\gamma) \bmod m \neq 0, \\ 1 & \text{if } (\alpha+\beta+\gamma)/m \text{ is even}, \\ -1 & \text{if } (\alpha+\beta+\gamma)/m \text{ is odd}, \end{cases} \tag{23}$$

and $\alpha,\beta,\gamma = 1,\ldots,m-1$. These Bell coefficients correspond to the so-called extended parity game considered in Ref. [15]. An equivalent definition, more similar to the definition of the Bell inequality (5) treated in Sec. II, is that $M_{\alpha\beta\gamma} = \cos[\pi(\alpha+\beta+\gamma)/m]$, whenever the absolute value of this expression is 1, and $M_{\alpha\beta\gamma} = 0$ otherwise.

Maximum quantum value, $\check{\mathcal{Q}}_m$. We get a lower bound for the quantum value with measurement operators given in Eq. (7) applied to components of a three-qubit GHZ state, with $\theta^A_\mu = \theta^B_\mu = \theta^C_\mu = \pi/2$ and $\varphi^A_\mu = \varphi^B_\mu = \varphi^C_\mu = \pi\mu/m$. Using Eq. (6) it is clear that each nonzero Bell coefficient will be multiplied by the same value as itself; therefore, the quantum value will be equal to the sum of the absolute values of the Bell coefficients, that is with the no-signaling limit, which is an upper bound for the quantum value. Hence, it has the property of pseudotelepathy [20]. From this it follows, that the quantum value will be nothing else than the number of nonzero Bell coefficients, which is actually m^2. To see this, it is enough to note that however we slice up the $m\times m\times m$ arrangement, each resulting $m\times m$ matrix will have exactly one nonzero number ($+1$ or -1) in each of its rows and columns. We can get such a row or column by fixing two of the indices of $M_{\alpha\beta\gamma}$. The sum of the indices we get this way are m consecutive numbers; exactly one of them will be divisible by m. Such a matrix will have m nonzero elements; the m slices together will contain m^2 such elements. Therefore, the quantum value and the no-signaling limit will be $\check{\mathcal{Q}}_m = m^2$.

We now place an upper bound on the maximum of the biseparable value ($\breve{\mathcal{B}}^l_m$), and then we prove that this bound can be saturated, that is, ($\breve{\mathcal{B}}^l_m = \breve{\mathcal{B}}_m$).

Upper bound on the biseparable quantum value, $\breve{\mathcal{B}}^u_m$. A further property of the slices of the present $m \times m \times m$ arrangement is that they are modified circulant matrices like in the case of the previous family, which can be shown exactly the same way as we have shown there. To get the matrices relevant to the biseparable value, we have to add up the slices with different signs. If it is Alice who is not allowed to share entangled state with the others, this sum is $M^{\{A\}}_{\beta\gamma} = \sum_{\alpha=0}^{m-1} A_\alpha M_{\alpha\beta\gamma}$. Due to the property of the arrangement, for each matrix element, all terms of the sum but one will be zero. Therefore, each entry of $M^{\{A\}}_{\beta\gamma}$ will either be 1 or -1. Moreover, this matrix will also be a modified circulant one, and its first line, which determines all the others, may contain any combination of $+1$ and -1 values, depending on A_α. Let $a_{m-1-\gamma} \equiv \sum_{\alpha=0}^{m-1} A_\alpha M_{\alpha 0\gamma}$, that is, the first line of $M^{\{A\}}_{0\gamma}$ written in opposite order. Then an upper bound for the biseparable value may be written as

$$\breve{\mathcal{B}}^u_m = \max m \left| \sum_{\gamma=0}^{m-1} a_\gamma e^{\frac{i\pi(2j+1)\gamma}{m}} \right|. \tag{24}$$

Let us consider the case of $j = 0$. Then what we get is the same as Eq. (18) but with $\Delta' = 0$ (which is irrelevant), and a prefactor of m instead of $m^2/2$. Then, according to Eq. (19), the result is $m \sin(\pi/2m)$. We show that this actually is the upper bound, whenever m is a power of two. As we have discussed earlier, $\exp(i\pi\gamma/m)$, which corresponds to $j = 0$, will point toward consecutive corners of a regular polygon of $2m$ edges on the complex plane while γ takes all values between zero and $m-1$. If m is a power of two, then for any j, $\exp[i\pi(2j+1)\gamma/m]$ will point toward different corners for the different γ values; moreover, if one of them will point toward one corner, there will be none pointing toward the opposite corner. The reason is that $(2j+1)\gamma$ is never divisible with m in this case. Choosing a_γ appropriately one can achieve that the terms to be added point toward consecutive corners, if taken in some order, which maximizes the absolute value of the sum. This is not true if m is divisible with an odd number. When $2j+1$ is equal to this number, for $\gamma = m/(2j+1)$ the value of $\exp[\pi i(2j+1)\gamma/m] = -1$, which lies opposite to $+1$, the value for $\gamma = 0$. In this case not all corners can be reached with appropriate choices of a_γ, and the other corners can be reached more than once, and $\breve{\mathcal{B}}^u_m$ may be larger than what we have calculated. If m is odd, for $2j+1 = m$ with $a_\gamma = -1^\gamma$ we even reach the no-signaling limit.

Now we show that we can actually reach the value of $\breve{\mathcal{B}}^u_m = m/\sin(\pi/2m)$.

Lower bound on the biseparable quantum value, $\breve{\mathcal{B}}^l_m$. The coefficients of the reduced Bell inequality are the elements of the modified circulant matrix $\tilde{M}_{\beta\gamma}$ whose entries in the first line are all $+1$. The appropriate Euclidean vectors $\vec{C}_\gamma$ are the same as the ones already defined in Eq. (20), and analogously to Eq. (21) we may write

$$\breve{\mathcal{B}}^l_m = \sum_{\beta=0}^{m-1} \left| \sum_{\gamma=0}^{m-1} \tilde{M}_{\beta\gamma} \begin{pmatrix} \cos\frac{\pi\gamma}{m} \\ \sin\frac{\pi\gamma}{m} \end{pmatrix} \right|. \tag{25}$$

For $\beta = 0$, $\tilde{M}_{0\gamma} = 1$, and we have to sum m unit vectors pointing toward consecutive corners of a polygon of $2m$ sides, the usual formation, the length of the resulting vector is $1/\sin(\pi/2m)$. For $\beta = 1$ only the last element of the row will be -1. However, if we change the sign of just the last vector of the formation, we get the same formation rotated by an angle of $\pi/2m$. This formation will give the same result. The next line will give a formation rotated further by $\pi/2m$, and so on; therefore, the result is $m/\sin(\pi/2m)$. This is a lower bound for the biseparable value, which is equal to the upper bound if m is a power of 2. In this case the ratio of the quantum and the biseparable limits is $\breve{\mathcal{Q}}_m/\breve{\mathcal{B}}_m = m\sin(\pi/2m)$, resulting in the threshold visibility $V = \breve{\mathcal{B}}_m\breve{\mathcal{Q}}_m = 1/[m\sin(\pi/2m)]$. This is the same visibility obtained under Eq. (22) by means of the Bell inequality (5) of Sec. II.

However, we would like to mention that this family of inequalities is more economical than our previous one. Namely, the number of joint measurements involved in Bell inequality (5) scales as m^3, whereas the present Bell inequality defined by (23) consists of only m^2 joint measurements. Even for smaller number of measurements, the case which is more relevant to experiments, the difference is not negligible: Inequality (5) (or equivalently the Bancal *et al.* inequality [11]) gives the threshold visibility $V = 0.666$, requiring 18 joint correlation terms. On the other hand, the inequality defined by (23) yields for three settings the lower threshold $V = 0.653$, using only 16 joint terms.

IV. MORE THAN THREE PARTIES

Both families of tripartite inequalities considered in Secs. II and III are easy to generalize to more than three parties. The m-setting n-partite Bell expression can be written as [16,21,22]

$$I^n_m = \sum_{\alpha_1,\ldots,\alpha_n=0}^{m-1} M_{\alpha_1,\ldots,\alpha_n} \langle \hat{A}^1_{\alpha_1} \otimes \ldots \otimes \hat{A}^n_{\alpha_n} \rangle, \tag{26}$$

which corresponds to Eq. (4). The A^i_α denotes the αth measurement operator of the ith participant, and each index α_i ($1 \leqslant i \leqslant n$) may take the values of $0,1,\ldots,m-1$. For both families the Bell coefficients given by Eqs. (5) and (23) depend only on the sum of their indices $\alpha + \beta + \gamma$. For more parties we simply keep the same functional dependences on the sum of all indices $\sum_{i=1}^n \alpha_i$. Now we show that Eq. (22) for the ratio of the maximum quantum and biseparable values will hold in the case of more than three parties as well.

First we give lower bounds for the maximum quantum values. Like in the tripartite case, let each participant perform a von Neumann projective measurement on one component of a shared n-qubit GHZ-like state $(|00\cdots 0\rangle + |11\cdots 1\rangle)/\sqrt{2}$. Then the n-partite correlation of their measurement values may be written as

$$\langle \hat{A}^1_{\alpha_1} \otimes \ldots \otimes \hat{A}^n_{\alpha_n} \rangle = \prod_{i=1}^{n} \sin\theta^{A^i}_{\alpha_i} \cos\left(\sum_{i=1}^{n} \varphi^{A^i}_{\alpha_i}\right) + (n+1 \bmod 2) \prod_{i=1}^{n} \cos\theta^{A^i}_{\alpha_i}, \quad (27)$$

where the operators are characterized by the angles θ and ϕ the same way as in Eq. (7). Let us choose the same set of measurement operators for each party. Let all θ angles be $\pi/2$, then the product of the sines will be 1, while the product of the cosines appearing for even numbers of parties will become 0. If we choose the angle φ_μ characterizing the μth measurement operator of each party to be $\varphi_\mu = \pi(\mu - \Delta/n)/m$, the correlator will become

$$\langle \hat{A}^1_{\alpha_1} \otimes \ldots \otimes \hat{A}^n_{\alpha_n} \rangle = \cos\left[\frac{\pi}{m}\left(\sum_{i=1}^{n} \alpha_i - \Delta\right)\right]. \quad (28)$$

For the case of the first family this expression is the same as the Bell coefficient to be multiplied with. For the second family let us choose $\Delta = 0$; then the correlator will again be equal to the corresponding Bell coefficient whenever the latter has a nonzero (that is ± 1) value. Therefore, for both families the lower quantum bound will be the sum of the squares of all Bell coefficients.

For the first family the bound may be calculated the same way as in Eq. (8). The only difference is that instead of three summations for three indices there will be n ones for n indices, with the result of $m^n/2$. In the case of the second family the absolute value of each Bell coefficient is either one or zero; therefore, the sum of their squares is equal to the sum of their absolute value, that is, the no-signaling limit, which is an upper bound. Therefore, here the lower bound is actually the exact quantum maximum, and these multipartite Bell inequalities also have the property of pseudotelepathy. It is easy to see that this quantum maximum will be nothing else than m^{n-1}. For the same reason as in the tripartite case, if we fix $n-1$ indices, the resulting column, that is, the m numbers we get while varying the remaining index, will contain exactly one nonzero number. We may fix the $n-1$ indices in m^{n-1} ways; therefore, there are m^{n-1} such columns, each contributing one to the sum. We note that for both families the lower bound for the quantum maximum is multiplied by m when the number of participants is increased by one.

To give upper bounds for the maxima of the biseparable values we may use the argument given by Bancal *et al.* in the Appendix B in Ref. [11]. The argument does not use the actual form of their inequality, only that the elements of the Bell matrix depend only on the sum of their indices, and this dependence is given by a function whose absolute value is periodic in the number of measurement settings per party, which is three in their case. Their argument may be repeated for larger numbers of measurement settings by replacing three for m appropriately. If we do that, we get that the maximum of the biseparable value for $n+1$ participants will be at most m times the maximum for n participants. Our families of inequalities satisfy the conditions of the proof; therefore, the result is readily applicable. As the lower bound for the quantum maximum and the upper bound for the biseparable maximum are multiplied by the same factor when the number of participants n is increased, their ratio will not depend on n.

Taking together the above results on upperbounding the biseparable value and lowerbounding the true quantum maximum on n-qubit GHZ states, we have that their ratio (and hence the visibility) can achieve the value of $1/[m\sin(\pi/2m)]$ for any number of parties $n > 2$. In particular, when m goes to infinity the ratio becomes $2/\pi$. We argue that for odd number of parties n this value is the smallest possible one if the witness is based on a Bell inequality which consists of only n-party correlators. Indeed, it is straightforward to extend the biseparable model of Appendix A to more than three parties. This results in a biseparable model for the particular bipartition of 1 and $n-1$ parties, which simulates equatorial von Neumann measurements on the noisy n-qubit GHZ state for visibility $2/\pi$. However, looking at the form of Eq. (27), von Neumann measurements which are not on the equator cannot increase the quantum Bell value for any odd n (similarly to the case of three parties). This proves the optimality of our threshold visibility $2/\pi$ for number of settings going to infinity and for odd number of parties.

V. CONCLUSION

In this paper, we extend the n-party three-setting inequality of Bancal *et al.* [11], which serves as a device-independent genuine multipartite entanglement witness, to an arbitrary number of settings. In particular, in the case of three parties our Bell inequalities are defined by Eqs. (5) and (23), which can detect genuine tripartite entanglement in the noisy three-qubit GHZ state with a visibility threshold of $V = 1/[m\sin(\pi/2m)]$, where m denotes the number of settings per party. This is generalized in Sec. IV to any number of parties ($n > 3$) and can detect genuine multipartite entanglement in the noisy n-qubit GHZ state with the same visibility threshold of $V = 1/[m\sin(\pi/2m)]$. For $m = 2,3$ our result recovers the threshold values corresponding to the Mermin inequality [12] and the Bancal *et al.* inequality [11], respectively. Numerical optimization suggests that in case of three parties these threshold values are optimal for $m = 2$ and $m = 3$. However, it is still an open question whether by considering inequalities as a sum of full-n-party correlators, our family of inequalities [defined by Eq. (5) and its straightforward generalization to more than three parties] is optimal for any values of m and n. The optimality of these inequalities for any $n,m > 2$ is supported by the fact that for m going to infinity the visibility V of the noisy n-qubit GHZ state approaches the value of $2/\pi$, achievable by a biseparable model simulating n-party correlators arising from equatorial von Neumann measurements. It would also be desirable to generalize our families to more than two outcomes in case of n-qudit GHZ states. One may also wonder whether the inequalities presented in this work are optimal for important states different from the n-qubit GHZ state. Furthermore, it would be of interest to find a Bell inequality, which is not the sum of n-party correlators, giving a threshold visibility lower than $2/\pi$ for the noisy n-qubit GHZ state.

Note added. Recently, very similar inequalities have been found as well by Bancal and his collaborators, giving the same threshold visibilities in number of settings and parties as our inequalities [23].

ACKNOWLEDGMENTS

The authors are indebted to an anonymous referee for initiating the generalization of our witnesses to more than three parties. T.V. has been supported by a János Bolyai Programme of the Hungarian Academy of Sciences.

APPENDIX: BISEPERABLE MODEL FOR THREE-PARTY CORRELATORS ON THE NOISY THREE-QUBIT GHZ STATE

The biseparable model based on the model of Appendix C of Bancal *et al.* [11] looks as follows. There are three parties, Alice, Bob, and Cecil; each of them receives the respective inputs $\vec{x} \in S^1$, $\vec{y} \in S^1$, and $\vec{z} \in S^1$, which can be considered as the measurement directions of a planar von Neumann measurement.

On the other hand, the source provides Alice and Bob with a quantum state,

$$|\psi\rangle = (|00\rangle + e^{-i\beta}|11\rangle)/\sqrt{2}, \tag{A1}$$

in which case the bipartite correlation is

$$\langle\psi|\hat{A}(\vec{x}) \otimes \hat{B}(\vec{y})|\psi\rangle = \cos(\phi_x + \phi_y + \beta), \tag{A2}$$

where $\hat{A}(\vec{x})$ and $\hat{B}(\vec{y})$ are ± 1 observables and ϕ_x, ϕ_y, ϕ_z are the angles corresponding to the two-dimensional Euclidean unit vectors $\vec{x}, \vec{y}, \vec{z}$.

At the same time, the source also provides Cecil with a vector $\vec{\lambda} = (\cos\beta, \sin\beta)$ picked uniformly from S^1, and as a result Cecil outputs the binary value $C = \mathrm{sgn}(\vec{\lambda} \cdot \vec{z})$. By averaging over λ we obtain the following expression for the three-partite correlator,

$$\langle ABC\rangle = \frac{1}{2\pi}\int_{\beta=0}^{2\pi} \mathrm{sgn}[\vec{\lambda}(\beta) \cdot \vec{z}] \cos(\phi_x + \phi_y + \beta) d\beta$$
$$= \frac{2}{\pi}\cos(\phi_x + \phi_y + \phi_z). \tag{A3}$$

Here we used the fact that $\vec{\lambda}(\beta) \cdot \vec{z} = \cos(\beta - \phi_z)$, and therefore the integration over β can be performed in two parts depending on the sign of $\cos(\beta - \phi_z)$. Note that apart from a factor of $2/\pi$ this correlation reproduces Eq. (6) with $\theta_A = \theta_B = \theta_C = \pi/2$. This corresponds to correlations where the parties carry out equatorial von Neumann measurements on a noisy three-qubit GHZ state with visibility $V = 2/\pi$.

Now we argue that for any Bell inequality which is the sum of three-party correlators, the maximum quantum violation can be attained by equatorial von Neumann measurements. This is due to the fact that in Eq. (6) the non-$\pi/2$ angles $\theta_A, \theta_B, \theta_C$ arising in a nonequatorial von Neumann measurement have only a shrinking effect on the correlators, which can only decrease the quantum Bell value. Therefore, the biseparable model presented above guarantees that a threshold visibility lower than $2/\pi$ for noisy three-qubit GHZ states is not possible for any genuine tripartite entanglement witness based on three-party-correlator Bell inequalities.

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