Alternative non-Markovianity measure by divisibility of dynamical maps

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(Received 1 February 2011; published 22 June 2011)

By identifying non-Markovianity with nondivisibility, we propose a measure of non-Markovianity for quantum processes. Three examples are presented, and the measure of non-Markovianity is calculated and discussed for these examples. Comparisons with other measures of non-Markovianity are made. The present non-Markovianity measure has the merit that no optimization procedure is required and it is finite for any quantum process, which greatly enhances the practical relevance of the proposed measure.

DOI: 10.1103/PhysRevA.83.062115 PACS number(s): 03.65.Yz, 03.65.Ta, 42.50.Lc

I. INTRODUCTION

A quantum process is said to be Markovian if the future states of the process depends only on the state of present time. In contrast, dependence on past states should then be a characteristic feature of non-Markovian processes. With the development of technology to manipulate quantum systems, the quantum non-Markovian process has attracted increasing attention in recent years [1–6]. On one hand the inevitable interaction of a quantum system with its environment leads to dissipation of energy and loss of quantum coherence, and on the other hand the quantum system may temporarily regain some of the previously lost energy and/or information due to non-Markovian effects in the dynamics. This motivates study of the non-Markovianity and a measure for the degree of non-Markovianity is indeed needed.

Several approaches are proposed to quantify non-Markovianity, including the measure based on the increase of trace distance [7], the measure by quantifying the increase of entanglement shared between the system and an isolated ancilla and the measure by the divisibility of the dynamical map [8], the measure based on the decay rate of the master equation itself [9], and the measure through the Fisher information flow [10]. Although several approaches to quantifying the non-Markovianity are proposed, the definitions of non-Markovianity still remain elusive and disagreeable [11].

It has been proven that all divisible dynamical maps are Markovian. This divisibility property holds for a larger class of quantum processes than those described by the Lindblad master equation, for example, the time-local master equation with positive decay rates. This indicates that the divisibility may be a good starting point to quantify non-Markovianity [8]. In this paper we propose a measure for non-Markovianity based on the divisibility of dynamical maps that was used in [8], three dynamical maps are presented, and the corresponding non-Markovian measures are calculated. These results suggest that the measure can capture the feature of non-Markovian dynamics, providing an easy way to calculate non-Markovianity.

The paper is organized as follows: In Sec. II we discuss the nondivisibility of the dynamical map and the non-Markovianity of this map. In Sec. III we introduce a measure for non-Markovianity. Three examples are presented and discussed in Sec. IV. Section V summarizes our results.

II. NON-DIVISIBILITY AND NON-MARKOVIANITY

In a quantum Markovian process, the future state of the quantum system depends only on the state of present time. However, writing this statement in a precise mathematical representation is not an easy task. Instead, we use the following description for quantum Markovian processes. A quantum evolution is Markovian if it is an element of any one-parameter continuous, completely positive semigroup [12]. The quantum evolution governed by the master equation

$$\frac{d\rho}{dt} = \mathcal{L}\rho \tag{1}$$

is an example, where $\mathcal{L}$ is a time-independent generator of the well-known Lindblad form,

$$\mathcal{L}\rho = -i[H,\rho] + \sum_\alpha \gamma_\alpha \left(V_\alpha \rho V_\alpha{}^\dagger - \frac{1}{2} V_\alpha{}^\dagger V_\alpha \rho - \frac{1}{2}\rho V_\alpha{}^\dagger V_\alpha \right) \tag{2}$$

with $\gamma_\alpha \geqslant 0$ at any time t. This generator leads to completely positive trace-preserving maps $\Lambda(t) = e^{\mathcal{L}t}$ and it satisfies the composition law

$$\Lambda(t_1 + t_2) = \Lambda(t_2)\Lambda(t_1). \tag{3}$$

If a dynamical map can be written in this decomposition with both $\Lambda(t_2)$ and $\Lambda(t_1)$ being completely positive, the dynamical map is called divisible. This composition law can be extended to a general case, where the generator in Eq. (2) is time-dependent, namely,

$$\frac{d\rho}{dt} = \mathcal{L}(t)\rho \tag{4}$$

with

$$\mathcal{L}(t)\rho = -i[H(t),\rho] + \sum_\alpha \gamma_\alpha(t)\Big[V_\alpha(t)\rho V_\alpha{}^\dagger(t) - \frac{1}{2} V_\alpha{}^\dagger(t) V_\alpha(t)\rho - \frac{1}{2}\rho V_\alpha{}^\dagger(t) V_\alpha(t)\Big] \tag{5}$$

where $\gamma_\alpha(t) \geqslant 0$. This is known as a time-dependent Markovian [12]. The solution to Eq. (4) can be written in terms of the two-parameter family of dynamical maps $\Lambda(t_2,t_1)$ $(t_2 \geqslant t_1 \geqslant 0)$. The composition law corresponding to Eq. (3) becomes

$$\Lambda(t_2,0) = \Lambda(t_2,t_1)\Lambda(t_1,0), \tag{6}$$

and the map $\Lambda(t_2,t_1)$ can be written as

$$\Lambda(t_2,t_1) = \mathcal{T} e^{\int_{t_1}^{t_2} \mathcal{L}(t')dt'} \quad (t_2 \geqslant t_1 \geqslant 0), \tag{7}$$

where $\mathcal{T}$ is the chronological operator. The composition law (divisibility of the map) implies that the dynamical map $\Lambda(t_2,t_1)$, transforming a state at t_1 into a state at t_2 [for systems governed by time-independent master equation (1), $\Lambda(t_2,t_1) = \Lambda(t_2 - t_1,0) = \Lambda(t)$], must be trace-preserving and completely positive, regardless of which dynamics it describes. Note that the starting time t_1 is not zero.

A measure for non-Markovianity should quantify the deviation of a dynamical map from Markovian evolution. Noticing that when a dynamics is non-Markovian, the dynamical map $\Lambda(t_2,t_1)$ may not be completely positive, we may use the nondivisibility to quantify the non-Markovianity. In fact, this is the underlying reason that the trace distance can increase [7], and the system gains entanglement with an isolated ancilla [8].

It is worth stressing that there is no contradiction between the requirement of incomplete positivity and that on physics. Consider a quantum evolution in a time interval $(0,t_2)$, we always have $\Lambda(t_2,0) = \Lambda(t_2,t_1)\Lambda(t_1,0)$ due to the time continuity. For $\Lambda(t_2,0)$ to be a dynamical map, it is required that $\Lambda(t_2,0)$ must be completely positive; however, $\Lambda(t_2,t_1)$ may not be completely positive. Therefore these two-parameter maps in non-Markovian dynamics do not generate a quantum dynamical semigroup. Then one may wonder: Does there exist a $\Lambda(t_2,t_1)$ that it is not completely positive but $\Lambda(t_2,0)$ does? The answer is yes. First, a wide range of non-Markovian processes can be described by time-local master equations via a time-convolutionless projection operator [13–18]. Second, it has been shown that any quantum dynamics described by a memory kernel master equation may be written in a time-local form [19]. Note that the decay rates in these time-local master equations are different from those of Eq. (5), where they can be negative. With this time-local master equation, the dynamical map with nonzero starting time in Eq. (7) may violate the complete positivity due to the negative decay rates. This implies that the incomplete positivity of the map $\Lambda(t_2,t_1)$ is an essential feature of the non-Markovian process. The time-dependent decay rate may be negatively infinite at some points of time, where the reviving of population or regaining of quantum coherence happen [7,17]. We call these points of time singular points t_s. When $t_1 = t_s$ or $t_2 = t_s$, $\Lambda(t_2,t_1)$ may not exist. However, we can use $\Lambda(t_2,t_1)$ in the limit that $t_1 \to t_s$ instead of $\Lambda(t_2,t_s)$. It is convenient to discuss $\Lambda(t_2,t_1)$ with a specific time-local master equation, but this is not necessary.

III. MEASURE FOR NON-MARKOVIANITY

To construct a measure for non-Markovianity, we resort to the Choi-Jamiołkowski isomorphism [20,21], which asserts that a linear map $\Lambda : M_d \to M_d$ is isomorphic to the Choi matrix,

$$C_\Lambda = \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes \Lambda(|i\rangle\langle j|), \tag{8}$$

where $|i\rangle$ are orthogonal bases. A familiar form of the Choi matrix is

$$\rho_\Lambda = (\Lambda \otimes \mathtt{I})|\phi\rangle\langle\phi|, \tag{9}$$

where $|\phi\rangle$ is the maximally entangled state $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle$, $\mathtt{I}$ is an identity map acting on the ancilla, and ρ_Λ is the normalized C_Λ. It turns out that Λ is completely positive if and only if ρ_Λ is positive semidefinite. In other words, the sufficient and necessary condition of incomplete positivity is the negativity of ρ_Λ. Hence, the sum of negative eigenvalues of ρ_Λ can be taken as a measure for the incomplete positivity of the dynamical map. However, the summation may sometimes be an infinite value in some models due to singular decay rates, which suggests use of a normalized quantity

$$N_{CP} = \arctan\left(-\sum \lambda_k\right) \tag{10}$$

as a measure for incomplete positivity of map Λ, where λ_k is the kth negative eigenvalue of ρ_Λ. Clearly if $\rho_\Lambda \geqslant 0$, $N_{CP} = 0$. Then we have $0 \leqslant N_{CP} \leqslant \frac{\pi}{2}$. Based on the complete positivity property of the map $\Lambda(t_1,t_2)$, a measure of non-Markovianity has been proposed [8]. This measure is different from ours in that we use an averaged negativity of the map $\Lambda(t_1,t_2)$ to quantify the non-Markovianity. Moreover, insightful examples are presented to shed light on the non-Markovianity measure.

To calculate Eq. (10) with a given time-local master equation, the exact form of $\Lambda(t_1,t_2)$ is not necessary. What we need is to extend the time-local master equation from one system to two systems, taking an isolated ancilla attached to the system. The Hilbert space is extended from H_d to $H_d \otimes H_d$ accordingly. All operators, say $\hat{O}$, are replaced by $\hat{O} \otimes \mathtt{I}$. By this extension we get a new master equation which describes the system and the ancilla. The system evolves in the same manner as before, while the ancilla is isolated from both the system and environment. The master equation can be solved starting from $|\phi\rangle$ at time t_1 and the state at time t_2 $(t_2 > t_1)$ can be obtained.

We aim at finding a measure N_M for non-Markovianity which captures the feature of incomplete positivity of all possible $\Lambda(t_2,t_1)$. Note that N_{CP} is a function of t_1 and t_2. Let S count the number of N_{CP} in all time intervals with $N_{CP} > 0$, i.e.,

$$S = \int_0^\infty dt_1 \int_{t_1}^\infty dt_2\, c(t_2,t_1), \quad c = \begin{cases} 1, & \text{if } N_{CP} > 0 \\ 0, & \text{if } N_{CP} = 0. \end{cases} \tag{11}$$

If $S = 0$, i.e., all $\Lambda(t_2,t_1)$ (for any t_1 and t_2, as long as $t_2 > t_1$) are completely positive, the non-Markovinity N_M should be defined as zero. If $S > 0$, we define

$$N_M = \lim_{T\to+\infty} \frac{\int_0^T dt_1 \int_{t_1}^T dt_2\, N_{CP}\,(t_2,t_1)}{\int_0^T dt_1 \int_{t_1}^T dt_2\, c(t_2,t_1)} \tag{12}$$

as a measure of non-Markovianity. This can be understood as an averaged incomplete positivity of all the non-completely-positive maps in interval $t = (0, +\infty)$. Therefore, from $0 \leqslant N_{CP} \leqslant \frac{\pi}{2}$, we have $0 \leqslant N_M \leqslant \frac{\pi}{2}$.

The upper limit of the integral T in the definition Eq. (12) is taken to be infinite. However, the distribution of N_{CP} in space spanned by (t_1,t_2) is often periodic or is limited in a small area, which suggests that the integration can be taken merely in one period of time or taken in a finite area. When N_{CP} is neither periodic nor limited in a small region, the upper limit T in Eq. (12) should be large enough to get a convergent N_M. The definition can be written into a simple form,

$$N_M = \mathcal{E}[N_{CP}(\Lambda_N)], \tag{13}$$

where Λ_N represents all the incompletely positive maps and $\mathcal{E}$ the expectation value. Therefore N_M can be numerically calculated by averaging a large number of incompletely positive maps with equal weight (or randomly) in the time region discussed above.

To illustrate the measure of non-Markovianity, we present three examples in the next section. We work in the interaction picture for simplicity to calculate the measure, since unitary transformation does not change the eigenvalues of ρ_Λ as well as N_{CP} of $\Lambda(t_2,t_1)$; hence the non-Markovianity measure under unitary transformation remains unchanged.

IV. EXAMPLES

A. Damping J-C model

The first example is a two-level system coupling to a reservoir at zero temperature. The reservoir consists of an infinite number of harmonic oscillators that is also referred to in the literature as the spin-boson model. This model is exactly solvable [17]. The Hamiltonian for such a system reads

$$H = H_0 + H_I \tag{14}$$

$$\text{with} \quad H_0 = \hbar\omega_0\sigma_+\sigma_- + \sum_k \hbar\omega_k b_k^\dagger b_k, \tag{15}$$

$$H_I = \sigma_+ B + \sigma_- B^\dagger, \tag{16}$$

where $B = \sum_k g_k b_k$. The Rabi frequency of the two-level system and the frequency for the kth harmonic oscillator are denoted by ω_0 and ω_k, respectively. $b_k^\dagger$ and b_k are the creation and annihilation operators of the kth oscillator, which couples to the system with coupling constant g_k.

Assuming the system and the reservoir are initially uncorrelated, we can obtain a time-dependent master equation in the interaction picture,

$$\begin{aligned}\dot\rho = &-i\frac{s(t)}{2}[\sigma_+\sigma_-,\rho]\\ &+\gamma(t)\left(\sigma^-\rho\sigma^+ - \frac{1}{2}\sigma^+\sigma^-\rho - \frac{1}{2}\rho\sigma^+\sigma^-\right),\end{aligned} \tag{17}$$

where $s(t) = -2\mathrm{Im}[\frac{\dot c(t)}{c_0(t)}]$ and $\gamma(t) = -2\mathrm{Re}[\frac{\dot c(t)}{c_0(t)}]$. $\Omega(t)$ plays the role of Lamb shift and $\gamma(t)$ is the decay rate. Both $\Omega(t)$ and $\gamma(t)$ are time dependent, and $c(t)$ is determined by $\dot c(t) = -\int_0^t f(t-\tau)c(\tau)d(\tau)$, where $f(t-\tau) = \int d\omega J(\omega)\exp[i(\omega_0-\omega)(t-\tau)]$ is the environmental correlation function. In the derivation of the master equation, the reservoir is assumed in its vacuum at $t = 0$.

Consider the following spectral density

$$J(\omega) = \frac{1}{\pi}\frac{\gamma_0\lambda^2}{(\omega_0-\omega)^2+\lambda^2}, \tag{18}$$

where γ_0 represents the coupling constant between the system and reservoir, and λ defines the spectral width of the coupling at the resonance point ω_0. For the spectral density (18), we have $s(t) = 0$, $c(t) = c_0 e^{-\lambda t/2}[\cosh(\frac{dt}{2}) + \frac{\lambda}{d}\sinh(\frac{dt}{2})]$, and $\gamma(t) = \frac{2\gamma_0\lambda\sinh(dt/2)}{d\cosh(dt/2)+\lambda\sinh(dt/2)}$ with $d = \sqrt{\lambda^2 - 2\gamma_0\lambda}$. Note that Eq. (17) is derived without any approximations; hence it is non-Markovian and it exactly describes the dynamics of the open system.

It is well known that λ characterizes the correlation time τ_R of the reservoir through $\tau_R = \lambda^{-1}$. The time scale τ_S on which the state of the system changes is given by $\tau_S = \gamma_0^{-1}$, so the degree of non-Markovianity should be relevant to the rate $R = \tau_R/\tau_S$. Namely, when R is very small, the evolution is Markovian, and when τ_R is comparable with τ_S, the memory effect of the reservoir should be taken into account and the dynamics of the open system is then non-Markovian.

Let us first analyze the non-Markovianity of the dynamics by examining $\gamma(t)$. For $R < \frac{1}{2}$, $\gamma(t)$ is always positive and it is a monotonically increasing function of time; all $\Lambda(t_2,t_1)$ are completely positive, and hence the dynamics is Markovian. When $R > \frac{1}{2}$, $\gamma(t)$ is a periodic function of time, it takes negative values sometimes. In particular, $\gamma(t)$ has discrete singular points where the upper level gains population, which is a typical feature of non-Markovianity.

Now we see if our measure can capture all these features of non-Markovianity. In order to apply our measure, we have to extend the time-local master equation to the compound system, i.e., the operators $\sigma_\pm$ in Eq. (17) are replaced by $\sigma_\pm \otimes \mathtt{I}$, with $\mathtt{I}$ being the ancilla's 2×2 identity operator. For a given time interval (t_1,t_2), a straightforward calculation yields

$$N_{CP}(t_1,t_2) = \begin{cases}\arctan\left[\frac{1}{2}\left(\frac{c(t_2)^2}{c(t_1)^2}-1\right)\right], & \frac{c(t_2)^2}{c(t_1)^2} > 1\\ 0, & \frac{c(t_2)^2}{c(t_1)^2} \leqslant 1,\end{cases} \tag{19}$$

where $c(t)$ was defined below Eq. (17) and $t_2 \geqslant t_1 \geqslant 0$.

For a typical non-Markovian case ($R = 5$), we plot N_{CP} as a function of t_1 and Δt in Fig. 1(a) ($\Delta t = t_2 - t_1$, since $t_2 \geqslant t_1$, we use Δt instead of t_2 for convenience). This plot shows the nonzero area and its value of N_{CP} versus t_1 and $t_1 - t_2$. As N_{CP} is a periodic function of t_1 and the area where $N_{CP} > 0$ decays very fast with Δt, N_M can be given by averaging all N_{CP} in one period, yielding $N_M = 0.835$.

As expected, when $R < 0.5$, $N_M = 0$, and when $R > 0.5$ the non-Markovianity is finite. We plot the measure of non-Markovianity with different R in Fig. 1(b). Here R is chosen from 0.5 to 10. Note that when $R = 0.5$, $\gamma(t)$ does not exist due to the zero denominator. Our result for $R = 0.5$ is obtained at $R = 0.5 + \varepsilon$, with ε an infinitesimal positive number. Intuitively, the larger R is, the stronger the non-Markovianity. The results in Fig. 1(b) demonstrate that this is indeed the case.

B. J-C model with detuning

The second example is similar to the first one, but here we consider the system in a cavity whose center frequency is

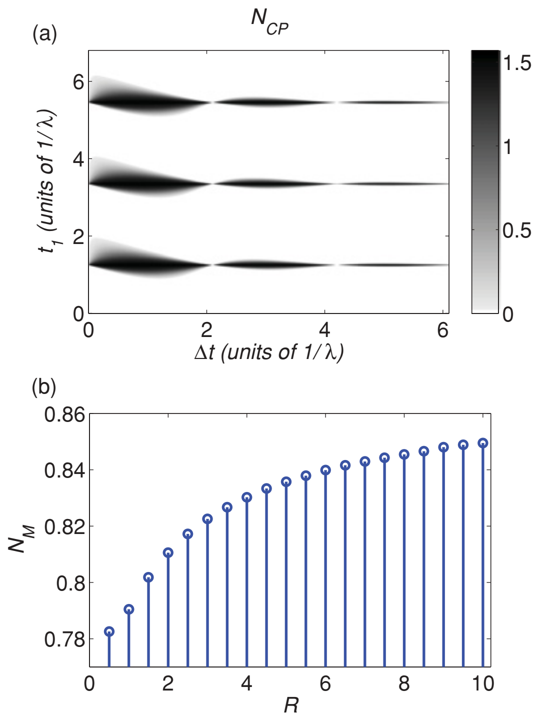

FIG. 1. (Color online) N_{CP} and non-Markovianity in the damping J-C model. (a) N_{CP} versus t_1 and Δt at $R = 5$. Three oscillations are shown. (b) N_M as a function of R. Clearly, N_M monotonically increases with R.

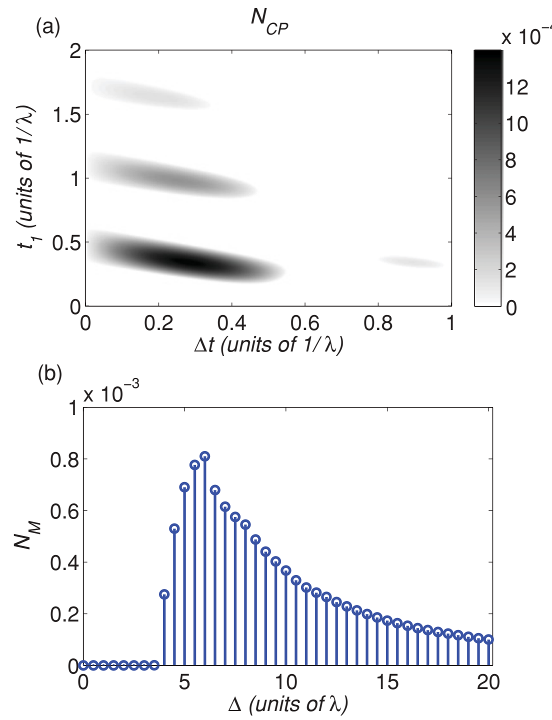

FIG. 2. (Color online) N_{CP} and non-Markovianity in the detuning J-C model. (a) Counter plot of N_{CP} as a function of t_1 and Δt, with $\Delta = 10$ and $\gamma_0 = 0.3$. (b) N_M as a function of Δ. Other parameters chosen are the same as in (a). The maximum non-Markovianity appears at about $\Delta = 6$.

detuned from the system Rabi frequency ω_0. The dynamics in the interaction picture is governed by Eq. (17), but $s(t)$ and $\gamma(t)$ are determined by the Lorenz spectral density

$$J(\omega) = \frac{1}{2\pi}\frac{\gamma_0\lambda^2/2}{(\omega_0 - \Delta - \omega)^2 + \lambda^2}, \tag{20}$$

where Δ denotes the detuning.

We extend Eq. (17) to a compound system by introducing an ancilla as we did in the first example, and then we calculate $N_{CP}(t_1,t_2)$ numerically. We plot N_{CP} as a function of t_1 and Δt in Fig. 2(a) for a typical non-Markovian case. We see that $N_{CP}(t_1,t_2)$ is finite in contrast to infinite values in the same region for the first example. On the other hand, N_{CP} of all $\Lambda(t_2,t_1)$ (with different t_1 and t_2) are far less than $\frac{\pi}{2}$, indicating weaker non-Markovianity in comparison with the first example. Finally, from Eq. (12) or Eq. (13) we have $N_M = 3.66 \times 10^{-4}$ in this case.

Now we discuss the dependence of non-Markovianity on the detuning. We plot N_M in Fig. 2(b) with different Δ. We find that N_M appears nonzero at about $\Delta = 4$, first increasing then decreasing with Δ. This result is similar to that in [7], where the non-Markovianity is measured by the decreases of trace distance.

C. A two-level system coupling to a finite spin bath

In the third example we consider a central spin-$\frac{1}{2}$ coupling to a bath of N spin-$\frac{1}{2}$s. The interaction Hamiltonian is

$$H = \sum_{k=1}^{N} A_k\sigma_z\sigma_z^k, \tag{21}$$

where $A_k = A/\sqrt{N}$ represents the coupling constants. The non-Markovianity of the central spin in this model is discussed in [1]. Assume the initial state of the whole system is $\rho_s(0) \otimes (\frac{1}{2^N}I)$, i.e., all spins in the reservoir are in a maximal mixed state. The density matrix of the central spin at time t takes

$$\rho(t) = \begin{pmatrix} \rho_{11} & \rho_{12}\cos^N\left(\frac{2At}{\sqrt{N}}\right) \\ \rho_{21}\cos^N\left(\frac{2At}{\sqrt{N}}\right) & \rho_{22} \end{pmatrix}. \tag{22}$$

In terms of the dynamical map, the dynamics can be represented as $\Lambda(t,0)\rho = \frac{1}{2}[1 - \cos^N(\frac{2At}{\sqrt{N}})]\sigma_z\rho\sigma_z + \frac{1}{2}[1 + \cos^N(\frac{2At}{\sqrt{N}})]\rho$. This is equivalent to the following master equation,

$$\dot{\rho} = \gamma(t)\mathcal{L}(\rho), \tag{23}$$

where $\mathcal{L}(\rho) = \sigma_z\rho\sigma_z - \rho$ and the time-dependent $\gamma(t) = A\sqrt{N}\tan(\frac{2At}{\sqrt{N}})$. This example is discussed in several papers as a classical example to quantify non-Markovianity, and the non-Markovianity is infinite [7,8]. By our definition of non-Markovianity, it is finite. This allows us to establish a relation between non-Markovianity N_M and the number of spin N in the reservoir.

It is easy to calculate ρ_Λ [defined in Eq. (9)] by Eq. (23),

$$[\Lambda(t_2,t_1)\otimes \mathrm{I}]|\psi\rangle\langle\psi| = \begin{pmatrix} 0.5 & 0 & 0 & 0.5k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5k & 0 & 0 & 0.5 \end{pmatrix} \tag{24}$$

with $k = \frac{\cos^N(\frac{2At_2}{\sqrt{N}})}{\cos^N(\frac{2At_1}{\sqrt{N}})}$ $(t_2 \geqslant t_1 \geqslant 0)$. Then we have

$$N_{CP}(t_1,t_2) = \begin{cases} \arctan \frac{1}{2}(|k|-1), & |k|>1 \\ 0, & -1 \leqslant k \leqslant 1. \end{cases} \tag{25}$$

The incomplete positivity of the map $\Lambda(t_2,t_1)$ can be examined by plotting N_{CP} in (t_1,t_2) plan. Figure 3(a) shows this result, taking only one spin as the environment. (For $N>1$, the results are similar.) We find that N_{CP} is a periodic function of both t_1 and Δt. When $t_1 = (2n+1)\frac{\pi}{4}$ and $\Delta t \neq n\frac{\pi}{2} (n = 0,1,2\ldots)$, $N_{CP} = \frac{\pi}{2}$ reaches its maximum. The non-Markovianity measure in this case is $N_M = 0.505$.

Now we try to find the relation between non-Markovianity and the number of spins N in the environment; the result is plotted in Fig. 3(b). We find that N_M increases with the spin number N. For a very large N, N_M arrives at its maximum. However, the coupling strength $A_k = A/\sqrt{N}$ is very small in large N limit, implying that the characteristic time of the central spin tends to be infinitely long in this limit, i.e., non-Markovian dynamics can be observed only on a long time scale. If we are interested in a limited time interval $(0,t)$, all dynamical maps $\Lambda(t_1,t_2)$ are completely positive; there is no non-Markovian effect. This can be understood as follows. With a fixed T one always can choose N so that $\frac{2AT}{\sqrt{N}}$ is close to zero. The off-diagonal element of the density matrix Eq. (24) then takes $\rho_{12}e^{-2A^2(t_2^2-t_1^2)}$, indicating that the density matrix in Eq. (24) describes a typical Markovian process.

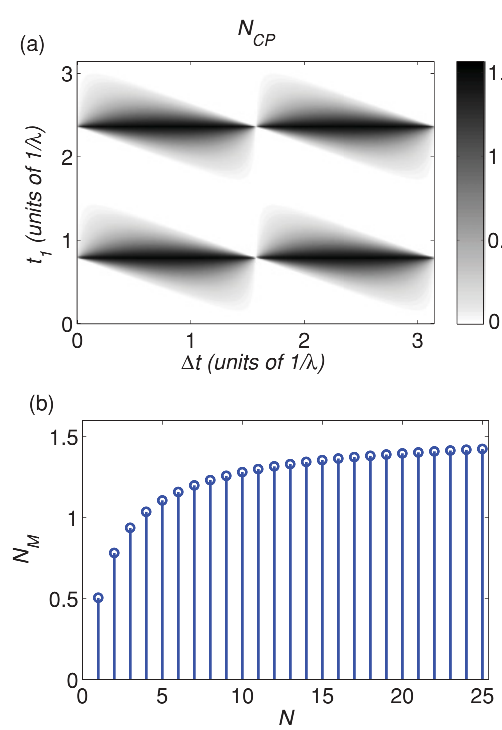

FIG. 3. (Color online) N_{CP} and non-Markovianity in the N-spin bath model. (a) N_{CP} as a function of t_1 and Δt for $N=1$. It shows that N_{CP} is a periodic function of t_1 and t_2; two periods are shown here. (b) N_M with different N. Non-Markovianity increases with N and tends to $\frac{\pi}{2}$ when the limit $N \to +\infty$ is taken after $T \to \infty$.

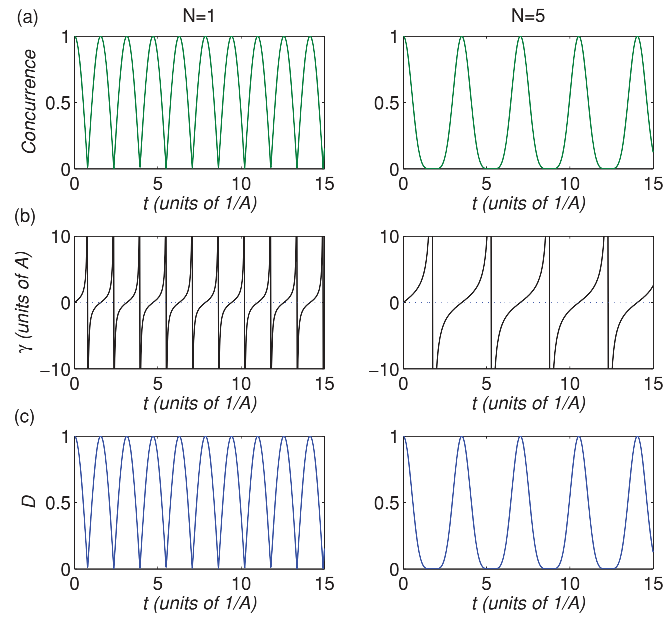

FIG. 4. (Color online) (a) Entanglement as a function of time [8]. (b) Evolution of the decay rate for the second measure in [8]. (c) Evolution of trace distance of the measure in [17]. The left three figures are plotted for $N=1$ and the right three for $N=5$.

Taking this model as an example, we now compare our measure with that in [7] and [8]. The number $N=1$ and 5 for the environmental spins will be chosen for the comparison.

First, we calculate the non-Markovianity defined in [8],

$$\mathcal{I} = \int_{t_0}^{t_{\max}} \left|\frac{d\mathcal{E}[\rho_{SA}(t)]}{dt}\right| dt - \Delta\mathcal{E}, \tag{26}$$

where $\mathcal{E}$ denotes an entanglement measure, and $\Delta\mathcal{E} = \mathcal{E}[\rho_{SA}(t_0)] - \mathcal{E}[\rho_{SA}(t_{\max})]$ with $\rho_{SA}(0) = |\phi\rangle\langle\phi|$. $|\phi\rangle$ is the maximally entangled state defined in Eq. (9). This measure was defined to count the increment of entanglement in time evolution from t_0 to $t_{\max}$. Choosing the concurrence [22] $C(\rho)$ as the entanglement measure, we have $\mathcal{E}(t) = C(\Lambda(t,0) \otimes \mathbb{I}|\psi\rangle\langle\psi|)$. Straightforward calculation yields $\mathcal{E}(t) = |\cos^N(\frac{2At}{\sqrt{N}})|$. We plot $\mathcal{E}(t)$ in Fig. 4(a) for $N=1$ (left) and $N=5$ (right), respectively. If $(t_0,t_{\max}) = (0,\infty)$, we obtain $\mathcal{I}=\infty$ for any N, since $\mathcal{E}(t)$ is an oscillating function of t, and $0 \leqslant \Delta\mathcal{E} \leqslant 1$. The non-Markovianity measure $\mathcal{I}$ depends on $t_{\max}$, when $t_{\max}$ is infinite.

Our measure is based on the same fact as the second non-Markovianity measure proposed in [8]. Now we examine the difference between these two measures. In [8] the measure is given by $\mathcal{D}_{N_M} = \mathcal{I}/(\mathcal{I}+1)$, with $\mathcal{I} = \int_0^\infty g(t)dt$ and $g(t) = \lim_{\epsilon\to 0^+} \frac{\|(\varepsilon_{(t+\epsilon,t)}\otimes \mathbb{I})(|\phi\rangle\langle\phi|)\|-1}{\epsilon}$. For the evolution governed by Eq. (23), $g(t)=0$ for $\gamma(t) \geqslant 0$ and $g(t) = -2\gamma(t)$ for $\gamma(t)<0$ (see also [8]). Then $\mathcal{I} = -2\int_{\gamma(t)<0}\gamma(t)dt$, with $\gamma(t) = A\sqrt{N}\tan(\frac{2At}{\sqrt{N}})$ in this example, which was plotted in Fig. 4(b) for $N=1$ (left) and 5 (right), respectively. It is clear that $\mathcal{I}$ is proportional to the area between the curve of $\gamma(t)<0$ and the t axis. Because $\gamma(t)$ has singular points, the area is infinite, i.e., $\mathcal{I}=\infty$. Then, $\mathcal{D}_{N_M} = 1$ regardless of N.

Next we turn to the non-Markovianity based on the trace distance [7], which was given by

$$\mathcal{N} = \max_{\rho_{1,2}(0)} \sum_i [D(\rho^{(1)}(b_i),\rho^{(2)}(b_i)) - D(\rho^{(1)}(a_i),\rho^{(2)}(a_i))]. \tag{27}$$

The maximum is taken over all pairs of initial states and (a_i,b_i) are all time intervals where D increases. Equation (22) yields $D(t) = \sqrt{a^2 + \cos^{2N}(\frac{2At}{\sqrt{N}})|b|^2}$, where $a = \rho_{11}^{(1)}(0) - \rho_{11}^{(2)}(0)$ and $b = \rho_{12}^{(1)}(0) - \rho_{21}^{(2)}(0)$. The maximal growth of the trace distance occurs for the case where the initial states lie in the antipodal points on the equator of the Bloch sphere [7]. Then $D(t) = |\cos^N(\frac{2At}{\sqrt{N}})|$. (Note that $D(t)$ has the same expression with $E(t)$ above.) We plot $D(t)$ with initial states $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$ for $N = 1$ (left) and $N = 5$ (right), respectively, in Fig. 4(c). Clearly, $D(t)$ is an oscillating function of time, reaching its maximum 1 periodically and giving $\mathcal{N} = \infty$ [7]. For more comparisons between the measures in [7] and [8], see [23].

V. SUMMARY

The divisibility of dynamical maps may be a good starting point to quantify non-Markovianity [8]. In this paper we have presented a measure for non-Markovianity based on the divisibility of the dynamical map. This measure has the advantage that it is easy to calculate and no optimization is required. Three examples are illustrated which show that the measure can nicely manifest the non-Markovianity. We also compare our measure with others in the literature; differences have been found and discussed.

ACKNOWLEDGMENTS

This work is supported by NSF of China under Grants No. 61078011 and No. 10935010, as well as the National Research Foundation and Ministry of Education, Singapore, under academic research Grant No. WBS: R-710-000-008-271.

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